Algorithms for Out-Branchings in Digraphs

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Outline

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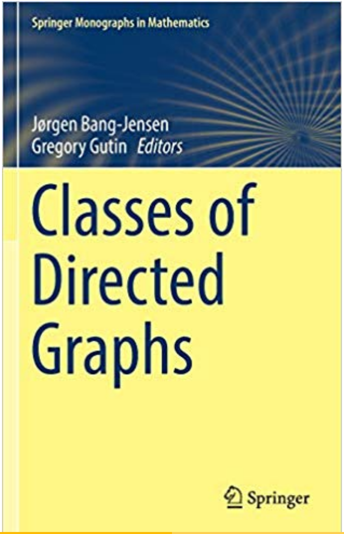
Very Recent Book on Digraphs



Out/In-Trees and Out/In-Branchings

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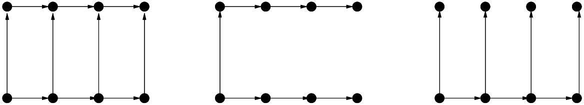
- out-branching = spanning out-tree; in-branching = spanning in-tree
- A digraph D has an out-branching (in-branching) iff D has only one initial (terminal) strongly connected component.

Example



(a) D (b) Q (c) R

Figure 1: A digraph D and its out-branchings with minimum and maximum number of leaves (Q and R, respectively).

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- A min weight out-branching in polynomial time: intersection of two matroids, an $O(n(n+m))$-time algorithm (Edmonds, 1967).

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- Daligault, Gutin, Kim and Yeo (2010): an $O^*(3.72^k)$-time algorithm (currently fastest).

Kernels

- Binkele-Raible, Fernau, Fomin, Lokshtanov, Saurabh and Villanger (2012): no polynomial kernel for k-Leaf-Out-Branching (for arbitrary digraphs) unless $coNP \subseteq NP/poly$, which is highly unlikely.

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- Still open: Is there an $O(k)$-vertex kernel for Rooted k-Leaf-Out-Branching?

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- Gutin, Razgon and Kim, 2009: a polytime algorithm for MinLeaf on DAGs.

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A kernelization is a polytime reduction $(I, k) \mapsto (I', k')$ from a parameterized problem Π to itself such that $(I, k) \in \Pi$ iff $(I', k') \in \Pi$ with $k' + |I'| \leq h(k)$ for a fixed function h; $h(k)$ is the size of the kernel. A kernel is polynomial if $h(k)$ is a polynomial.

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- Still open: Is there an $O(k)$-vertex kernel? There is a $O(k)$-vertex kernel for acyclic [Gutin, Razgon and Kim, 2009] and symmetric [Fomin et al., 2013] digraphs.

Faster Deterministic and Randomized Algorithms

- $O^*(55.8^k)$ [det, Cohen et al. 2010], $O^*(4^k)$ [random, Daligault, 2011],
- $O^*(16^{k(1+o(1))})$ [det, Fomin et al., 2012], $O^*(6.855^k)$ [det, Shachnai and Zehavi, 2015],
- $O^*(5.139^k)$ [det, Zehavi, 2016], $O^*(3.617^k)$ [random, Zehavi, 2015],
- $O^*(2^k)$ [random, Björklund, Kaski and Koutis, 2017], $O^*(3.41^k)$ [det, Gutin, Reidl, Wahlström and Zehavi, 2018].

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- Fix r. Set $x_{ij} = x_i$. Let $\mathcal{B}_{r,k} = \{B \in \mathcal{B}_r : \ iv(B) \geq k\}$. $\exists B \in \mathcal{B}_{r,k}$ iff $\det(K_{\bar{r}})$ has a monomial with at least k distinct x_i's.

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- To check it efficiently, we use efficient color coding and monomial sieving. k-coloring: $\{x_1, \ldots, x_n\} \rightarrow \{y_1, \ldots, y_k\}$.

Algorithm 2

- M-Lemma: (i) Let T be an out-tree s.t. $iv(T) \geq k$. Then T has a matching of size $\geq k/2$; (ii) Let M be a matching in D. In $O^*(1)$ time, we can find an out-branching B of D s.t. every arc of M has at least one vertex as an internal vertex in B.

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- For every $c \in \{0, 1, \ldots, k\}$ consider all sets M' of c arcs in M in which both vertices are leaves in some $B \in \mathcal{B}_{r,k}$. For every such vertex i, x_i gets its own y_j.

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- For $ip \in M \setminus M'$, x_i, x_p get one y_j.

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- $f(k)=O^*(3.857^k)$.

Questions

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