Problems and complexity in influence and social, simple games

Maria Serna

Computer Science Dept., U. Politècnica de Catalunya
Barcelona Graduate School of Mathematics, BGSMath

Liverpool, September 20, 2018

1. Simple games
2. Influence games
3. Social simple games
4. Conclusions and open problems

Framework

- Topics
  - Coalitional Game Theory
  - Decision/Voting/Social Choice Theory
  - Social Network Analysis
  - Algorithms and Complexity
- Models
  - Simple Games
  - Graphs
- Focus
  - Subfamilies of simple games
  - Complexity study of some properties of simple games.

Simple Games

- [Taylor, Zwicker, 1999]
  A simple game is a pair $(N, \mathcal{W})$:
  - N is a set of players,
  - $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotone set of *winning coalitions*.
  - $\mathcal{L} = \mathcal{P}(N) \backslash \mathcal{W}$ is the set of *losing coalitions*.
- Members of $N = \{1, \ldots, n\}$ are called players or voters.
  Any set of voters is called a coalition
  - N is the grand coalition
  - $\emptyset$ is the null coalition
  - The sets in $\mathcal{W}$ are the winning coalitions
  - A set that is not in $\mathcal{W}$ is called a losing coalition.

Simple games: Representation

Due to monotonicity, any one of the following families of coalitions define a simple game on a set of players N:

- winning $\mathcal{W}$.
- losing $\mathcal{L}$.
- minimal winning $\mathcal{W}^m = \{X \in \mathcal{W}; \forall Z \in \mathcal{W}, Z \not\subseteq X\}$
- maximal losing $\mathcal{L}^M = \{X \in \mathcal{L}; \forall Z \in \mathcal{L}, X \not\subseteq L\}$

This leads to three explicit forms of representing a game

$$(N, \mathcal{W}),\ (N, \mathcal{W}^m),\ (N, \mathcal{L}),\ (N, \mathcal{L}^M)$$

Problems on simple games

In general we state a property P (or parameter) for simple games, and consider the associated decision (function) problem which has the form:

Name: IsP
Input: A simple game Γ
Question: Does Γ *satisfy property* P*?*

Two properties

A simple game (N, W) is

- strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$.
  It never happens that a coalition and its complement lose.
- proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.
  It never happens that a coalition and its complement win.

IsStrong: Simple Games losing forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The IsStrong *problem, when* Γ *is given by* $(N, \mathcal{L})$ *or* $(N, \mathcal{L}^M)$ *can be solved in polynomial time.*

IsStrong: Simple Games losing forms

Γ is strong if $S \notin W$ implies $N \setminus S \in W$

- Γ is not strong iff

$$\exists S \subseteq N : S, N \setminus S \in L$$

$$\exists S \subseteq N : \exists L_1, L_2 \in L^M : S \subseteq L_1 \;\wedge\; N \setminus S \subseteq L_2$$

- iff there are two maximal losing coalitions L_1 and L_2 such that $L_1 \cup L_2 = N$.
- This can be checked in polynomial time, given $\mathcal{L}$ or $\mathcal{L}^M$.

IsStrong: minimal winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The IsStrong *problem is coNP-complete when the input game is given by* $(N, \mathcal{W}^m)$.

- The property can be expressed as

$$\forall S\ [(S \in \mathcal{W}) \text{ or } (S \notin \mathcal{W} \text{ and } N \setminus S \in \mathcal{W})]$$

- Observe that the property can be checked in polynomial time given S and $\mathcal{W}^m$.
- Thus the problem belongs to coNP.

IsStrong: minimal winning forms

Name: SET SPLITTING
Input: Given a collection C of subsets of a finite set N
Question: It is possible to partition N into two subsets P and $N \setminus P$ such that no subset in C is entirely contained in either P or $N \setminus P$?

- We have to decide whether $P \subseteq N$ exists such that

$$\forall S \in C : S \not\subseteq P \wedge S \not\subseteq N \setminus P$$

We associate to a set splitting instance (N, C) the simple game in explicit minimal winning form (N, C^m) and show that this is a valid reduction.

IsProper: winning forms

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The IsProper *problem, when the game is given by* $(N, \mathcal{W})$ *or* $(N, \mathcal{W}^m)$*, can be solved in polynomial time.*

IsProper: winning forms

- Γ is not proper iff

$$\exists S \subseteq N : S, N \setminus S \in W$$

$$\exists S \subseteq N : \exists W_1, W_2 \in W^m : W_1 \subseteq S \ \wedge \ W_2 \subseteq N \setminus S.$$

- iff there are two minimal winning coalitions W_1 and W_2 such that $W_1 \cap W_2 = \emptyset$.
- Which can be checked in polynomial time when $\mathcal{W}^m$ or $\mathcal{W}$ are given.

IsProper: maximal losing form

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The ISPROPER *problem is coNP-complete when the input game is given in extensive maximal losing form.*

- Working with the definitions it easily seen to belong to coNP.

IsProper: maximal losing form

To show that the problem is also coNP-hard we provide a reduction from the hard case of the IsStrong problem, $(N, \mathcal{W}^m)$.

- If a family C of subsets of N is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- The dual of a game $\Gamma = (N, \mathcal{W})$ is the game $\Gamma^d = (N, \mathcal{W}^d)$ where $\mathcal{W}^d = \{S \mid N \setminus S \notin \mathcal{W}\}$.
- It is well known that a game is strong iff its dual is proper
- Given $\Gamma = (N, W^m)$, the maximal losing coalitions of Γ^d can be obtained in polynomial time.

Summary

Input $\rightarrow$	(N, W)	(N, W^m)	(N, L)	(N, L^M)
IsStrong	P	co-NPC	P	P
IsProper	P	P	P	co-NPC

[J. Freixas, X. Molinero, M. Olsen, MS, 2011]

Weighted Voting Games

- Weighted voting games (WVG)
  A simple game for which there exists a quota q and it is possible to assign to each $i \in N$ a weight w_i, so that
  $$X \in \mathcal{W} \text{ iff } \sum_{i \in X} w_i \geq q.$$
- WVG can be represented by a tuple of integers $(q; w_1, \ldots, w_n)$ [Carreras and Freixas, Math. Soc.Sci., 1996].

Decision is taken without interplay of the participants

Influence Games: influence spreading model

- An influence graph is a tuple (G, f), where:
  - $G = (V, E)$ is a labeled and directed graph, and
  - $f : V \rightarrow \mathbb{N}$ is a labeling function that quantify how influenceable each node, player or agent is.

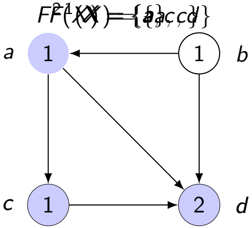
Influence Games: influence spreading model

- Linear threshold model [Chen, 2009].
  From an *initial activation* $X \subseteq V$, activate every node u having at least $f(u)$ predecessors in X.

  Repeat until no more nodes are activated.

  The final set of activated nodes $F(X)$ is the spread of influence from X.

  $F(X)$ is polynomial time computable.



Influence Games

- An influence game is a tuple (G, f, q, N), where:
  - (G, f) is an influence graph,
  - $N \subseteq V(G)$ is the set of players, and
  - $q > 0$ is an integer, the *quota*.
  - $S \subseteq N$ is winning iff $|F(S)| \geq q$.
- F is monotonic, for any $X \subseteq Y \subseteq N$, if $F(X) \subseteq F(Y)$.
- Influence games are simple games.

Participants can being influenced to adopt a new trend but have negative "initial" disposition.

A particular influence game: $\Gamma(G)$

Definition

Given an undirected graph $G = (V, E)$,
$\Gamma(G)$ is the influence game $(G, f, |V|, V)$ where,
for any $v \in V$, $f(v) = d_G(v)$.

Influence game $\Gamma(G)$

Recall that a set $S \subseteq V$ is a vertex cover of a graph G if and only if, for any edge $(u, v) \in E$, u or v (or both) belong to S. From the definitions we get the following result.

Lemma

Let G be an undirected graph without isolated vertices. X is winning in $\Gamma(G)$ if and only if X is a vertex cover of G. Furthermore, the influence game $\Gamma(G)$ can be obtained in polynomial time, given a description of G.

Isproper and IsStrong

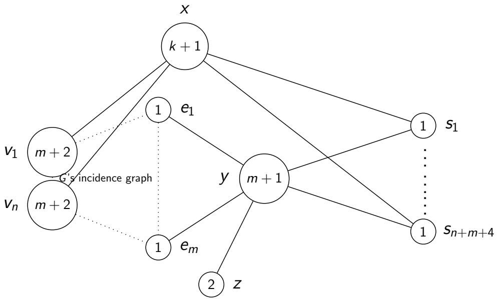
Theorem

For influence games IsProper *and* IsStrong *are* coNP*-complete.*

- Membership in coNP follows from the definitions.
- To get the hardness results, we provide reductions from problems related to Vertex Cover.
- Assume that a graph G has n vertices and m edges.

$\Delta_1(G, k)$

Let $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$. Assume $n = 2k + 1$, set $\alpha = m + n + 4$ and consider the influence graph (G_1, f_1):



$\Delta_1(G, k) = (G_1, f_1, q_1, N_1)$ where $q_1 = \alpha$ and $N_1 = \{v_1, \dots, v_n, z\}$.

$\Delta_1(G, k)$ is proper iff G has a vertex cover with size $\leq k = (n-1)/2$.
As

> *Name:* HALF VERTEX COVER
> *Input: Given a graph with an odd number of vertices n.*
> *Question: Is there a vertex cover with size* $\leq (n-1)/2$*?*

is NP-complete.

Theorem

The ISPROPER *problem for influence games is coNP-complete.*

IsStrong

Theorem

The IsProper *problem for influence games is coNP-complete.*

The result follows from another graph gadget from the complement of the

> *Name:* Half independent set
> *Input: Given a graph with an even number of vertices n.*
> *Question: Is there an independent set with size* $\geq n/2$*?*

which is NP-complete.

Subfamilies of Influence Games

Let G be an undirected graph (without isolated vertices).

Maximum Influence Game

$\Gamma = (G, f, |V|, V)$ where, for $v \in V$, $f(v) = d_G(v)$.

Minimum Influence Game

$\Gamma = (G, 1_V, q, N)$ where $1_V(v) = 1$, for $v \in V$.

IsProper: characterization

- Assume that G is connected.
- If G is a singleton $\Gamma(G)$ is proper. Otherwise,
- The winning coalitions of $\Gamma = \Gamma(G)$ coincide with the vertex covers of G. But, the complement of a vertex cover is an independent set.
- If $G = (V, E)$ is bipartite, let (V_1, V_2) be a partition of V so that V_1 and V_2 are independent sets.
  Now V_1 and $V_2 = N \setminus V_1$ are winning and Γ is not proper.
- if Γ is not proper, then the game admits two disjoint winning coalitions i.e, two disjoint vertex covers of G, and
  hence both of them must be independent sets.
  Thus G is bipartite.

IsStrong: characterization

- If G has at least two non-incident edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$, $\{u_1, v_1\}$ and $N \setminus \{u_1, v_1\}$ are both losing and Γ is not strong.
- When the game is not strong, there is X such that both X and $N \setminus X$ are losing.
  For this to happen there must be an edge uncovered by X and another edge uncovered by $N \setminus X$. Thus G must have two non-incident edges.
- A graph where all edges are incident is either a triangle or a star.

Maximum Influence games

Lemma

In a maximum influence game Γ on a connected graph G the following properties hold.

- *Γ is proper if and only if G is either not bipartite or a singleton.*
- *Γ is strong if and only if G is either a star or a triangle.*

As a consequence of those characterizations the IsProper and IsStrong can be solved in polynomial time.

Minimum influence games

$\Gamma = (G, 1_V, q, N)$ where $1_V(v) = 1$ for any $v \in V$.

- Observe that, if G is connected, any non-empty vertex subset of N wins.
- For the disconnected case we can analyze the game with respect to a suitable weighted game.
- Assume that G has k connected components, $C_1, \ldots, C_k$. Without loss of generality, we assume that all the connected components of G have non-empty intersection with N. For $1 \leq i \leq k$, let $w_i = |V(C_i)|$ and $n_i = |V(C_i) \cap N|$.

Minimum Influence

Lemma

If a winning coalition is minimal then it has at most one node in each connected component. Minimal winning coalitions are in a many-to-one correspondence with the minimal winning coalitions of the weighted game $[q; w_1, \ldots, w_k]$.

Minimum Influence

Theorem

For unweighted influence games with minimum influence, the problems IsProper *and* IsStrong *belong to* P.

Minimum Influence: IsProper

Let $\Gamma = (G, 1_V, q, N)$ be an unweighted influence game with minimum influence.

- For the IsProper problem it is enough to check whether there is a winning coalition whose complement is also winning and answer accordingly.
- We separate the connected components in two sets: $A = \{i \mid n_i = 1\}$ and $B = \{i \mid n_i > 1\}$.
  Let $N_A = \cup_{i \in A}(N \cap V(C_i))$ and $N_B = N \setminus N_A$.
  Let $w_A = \sum_{i \in A} w_i$ and $w_B = w_N - w_A$.

- All the components in B have at least two vertices. We can find a set $X \subseteq N_B$ such that $|F(X)| = |F(N_B \setminus X)| = w_B$.
- If $w_B \geq q$, Γ is not proper.
- If $w_B < q$, Γ is proper iff the influence game Γ' played on the graph formed by the connected components belonging to A and quota $q' = q - w_B$ is proper.
  But Γ' is equivalent to the weighted game with a player for each component in $i \in A$ with associated weight w_i and quota q'.
- Let α_{min} be the minimum $\alpha \in \{q', \ldots, w_A\}$ for which there is a set $S \subseteq A$ with $\sum_{i \in S} w_i = \alpha$.
  Γ' is proper if and only if $w_A - \alpha_{min} < q'$.
- The value α_{min} can be computed by solving several instances of the 0-1-Knapsack with weights polynomial in n.

Summary of results

Input →	(N,W)	(N,W^m)	(N,L)	(N,L^M)
IsStrong	P	co-NPC	P	P
IsProper	P	P	P	co-NPC

Input →	(G,f,q)	G **MaxInf**	(G,q) **MinInf**
IsStrong	co-NPC	P	P
IsProper	co-NPC	P	P

[X. Molinero, F. Riquelme, MS, 2015]

A social requirement

- It is natural to think that in a networked society a coalition can be formed only if the participants can communicate.
- We explore the simplest scenario in which the players in a simple game $(N, \mathcal{W})$ belong to a social network, modeled by an undirected graph H.
- We require that a set $S \subset N$ can win if and only if $H[S]$ is connected.
- But, the additional condition breaks the monotonicity required to $\mathcal{W}$.

Social simple games

- A social simple game is a tuple $\Gamma = (N, \mathcal{W}, H)$, where $(N, \mathcal{W})$ is a simple game and $H = (N, E)$ is an undirected graph (the social graph).

  $X \subseteq N$ is a winning coalition iff $X \in \mathcal{W}$ and the subgraph of H induced by X, $H[X]$, is connected. A set of players that is not a winning coalition will be called losing coalition.
- We can consider also social weighthed voting games or social influence games.
- Observe that $(N, \mathcal{W}, K_{|N|}) = (N, \mathcal{W})$. So, any problem that is hard for a family of simple games remains hard for the corresponding social version.

Social simple games: cases of interest

Input →	(N, W)	(N, W^m)	(N, L)	(N, L^M)
IsStrong	P	co-NPC	P	P
IsProper	P	P	P	co-NPC

Input →	(G, f, q)	G **MaxInf**	G **MinInf**
IsStrong	co-NPC	P	P
IsProper	co-NPC	P	P

IsProper

Theorem

The IsProper *coNP-complete for social simple games given by $(N, \mathcal{W}^m, H)$, social maximum influence games and social minimum influence games on connected social graphs.*

By reductions from the complement of the

Name: 2-Disjoint Connected Subgraphs
Input: A connected graph $G = (V, E)$ and two disjoint subsets of terminal vertices $Z_1, Z_2 \subseteq V$.
Question: Does there exist a partition A_1, A_2 of V, with $Z_1 \subseteq A_1$, $Z_2 \subseteq A_2$ and G_{A_1}, G_{A_2} both connected?

which is NP-complete [Hof, Paulusuma, Woeginger, 2009]

Theorem

The IsProper *is polynomial time solvable for social simple games given by* $(N, \mathcal{W}, H)$ *and social maximum influence games and social minimum influence games on disconnected social graphs.*

Input: $(N, \mathcal{W}, H)$

1. Check whether $(N, \mathcal{W})$ is proper. If it is proper return 'yes'.
2. Create a set of tuples $\hat{S} = \{(S, N \backslash S) \mid S, N \backslash S \in W\}$.
3. for all tuples $(S, N \backslash S) \in \hat{S}$
4. If both $H[S]$ and $H[N \backslash S]$ are connected return 'no'.
5. return 'yes'.

Theorem

The IsProper *is polynomial time solvable for social simple games given by* $(N, \mathcal{W}, H)$ *and social maximum influence games and social minimum influence games on disconnected social graphs.*

- When H is disconnected if it has more than two connected components, the complementary of a winning coalition will always be losing.
- When H has only two connected components we only have to check that those connected components are winning.

IsStrong: social simple maximal losing

Theorem

The IsStrong *is polynomial time solvable for social simple games given by* $(N, \mathcal{L}^m, H)$, $(N, \mathcal{L}, H)$ *and* $(N, \mathcal{W}, H)$.

Input: $(N, \mathcal{L}^M, G)$

1. If $(N, \mathcal{L}^M)$ is not strong return 'no'.
2. for all $S \in \mathcal{L}^M$
3. if $H[N \backslash S]$ is not connected, return 'no'.
4. for all $S \in \mathcal{L}^M$ and $s \in S$
5. If there is $t \in N \backslash S$ such that $(s, t) \in E$ return 'no'.
6. return 'yes'.

Results on social games

In addition the input has the social graph H.

Input $\rightarrow$	(N, W)	(N, W^m)	(N, L)	(N, L^M)
IsStrong	P	co-NPC	P	P
IsProper	P	co-NPC	?	co-NPC

Input $\rightarrow$	(G, f, q)	G **MaxInf**	(G, q) **MinInf**
IsStrong	co-NPC	?	?
IsProper	co-NPC	co-NPC (P if H discon.)	co-NPC (P if H discon.)

[J. Castellví, X. Molinero, MS, E. van Hove, 2018]

Conclusions

- We have analyzed simple and influence and their social counterpart.
- We have concentrated on the study of few computational problems.
- In the social version some problems are harder.
- We indirectly studied some new families of simple games defined through graphs
  - $\mathcal{W}^m$ is formed by vertex covers.
  - $\mathcal{W}^m$ is formed by connected dominating sets.

Further directions

- IsMonotone, length and width.
- Banzhaf values, Satisfaction and Power.
- Decision systems and centrality measures for networks through influence spread.
- What pseudo games are we getting in the social setting?
- Duality?
- Other ways to impose connectivity?
  - $H[S]$ contains a connected component that wins.
  - $H[S]$ does not contain isolated vertices.
- Majority influence games? Random influence games?

Acknowledgments

Coauthors

J. Castellví (UPC), J. Freixas (UPC), X. Molinero (UPC) ,
M.Olsen (Aarhus), F. Riquelme (Valparaíso), E.van Hove (Twente)

Funding

GOBIERNO DE ESPAÑA MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES

FEDER Fondo Europeo de Desarrollo Regional

BGSMath BARCELONA GRADUATE SCHOOL OF MATHEMATICS

AGAUR Agència de Gestió d'Ajuts Universitaris i de Recerca

Thank you for your attention