Prophet Inequalities Made Easy:
Stochastic Optimization by Pricing Non-Stochastic Inputs

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Joint work with Michal Feldman (Tel Aviv), Thomas Kesselheim (Bonn), and Brendan Lucier (Microsoft)

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An Online Assignment Problem

- Fixed set of $m$ seats to assign
- Sequence of $n$ requests arrive online
- Each request assigns a value to each set of seats
- Goal: assign seats online to maximize total value
An Online Assignment Problem
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Need 4 adjacent seats.
Value: 100/seat for front row, 80/seat otherwise.
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An Online Assignment Problem

Need a single seat.
Value: 1000,
must be a corner.
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Need a single seat.
Value: 1000,
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Value 1 for any single seat.
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Compare $v(\text{Online})$ to $v(\text{Offline}-\text{Opt})$:
An Online Assignment Problem

Compare $v(\text{Online})$ to $v(\text{Offline-Opt})$:

- Worst-case competitive analysis:
  Pointless, even with only a single seat
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- Random arrival order:
  Generalizes secretary problem
An Online Assignment Problem

Compare $v(\text{Online})$ to $v(\text{Offline-Opt})$:

- Worst-case competitive analysis:
  Pointless, even with only a single seat

- Random arrival order:
  Generalizes secretary problem

- Known distributions:
  Generalizes prophet inequality
The Known Distributions Model

- Valuations are drawn from known independent distributions
  \[ \sim D_1 \quad \sim D_2 \quad \sim D_3 \quad \cdots \]

- Compare value of assignment to optimal offline solution

  \[
  \text{competitive ratio} = \min_{\text{distribution } D} \frac{\mathbb{E}_{I \sim D}[\text{ALG}(I)]}{\mathbb{E}_{I \sim D}[\text{OPT}(I)]}
  \]
The Known Distributions Model

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\textbf{Thm.} [Krengel, Sucheston, Garling 1977]
There is a \( \frac{1}{2} \)-competitive algorithm for the assignment problem with one seat/item.
One Approach: Posted Pricing

Use distributions, to assign a price to each seat.
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Each request receives whichever seats maximize $utility = value - price$ among all remaining seats.
One Approach: Posted Pricing

Need 4 adjacent seats.
Value: $100/seat in first row, $80/seat otherwise.
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Questions

For a combinatorial allocation problem, given distributions over valuations:

- Is there a **pricing rule** that approximates the expected optimal (offline) total value? 
  I.e., that establishes a **prophet inequality**?

- Can we compute these prices **efficiently**?

- Further desiderata: Anonymous, static, item prices
Our Contribution

A framework for prophet inequalities based on posted prices

Key idea: Stochastic Optimization by Pricing Non-Stochastic Inputs

To use framework: Reason only about the (easier) full-information case, where values are known in advance
Our Contribution

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Running Example: General Combinatorial Auctions

- Each item can be allocated at most once
- Arbitrary values for sets of up to $k$ items

**Thm.** [Hazan et al. 2006]
Even offline, lower bound of $\Omega(k / \log k)$, assuming $P = NP$.

The greedy algorithm is an $O(k)$ approximation (for offline problem)

**Our goal:** An $O(k)$ approximation, via posted prices
Pricing via Dual LP: Does not Work

\[
\text{max } 4x_{1,BC} + 100x_{2,AB} + 100x_{2,AC}
\]  
\[
\text{s.t. } x_{2,AB} + x_{2,AC} \leq 1
\]
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& x_{1,BC} + x_{2,AB} \leq 1 \\
& x_{1,BC} + x_{2,AC} \leq 1
\end{align*}
\]

- Optimal dual solution sets \( p_A = p_B = p_C = 2 \)
- So buyer 1 buys items B and C
- Value of allocation is 4, where we could have gotten 100
Single-Item Case, Full Information

\[ v_1 = 10 \quad v_2 = 30 \quad v_3 = 15 \quad v_4 = 80 \quad v_5 = 5 \]
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Let \( v_i = \max_k v_k \)

- **Case 1:** Somebody \( i' < i \) buys item
  \[ \Rightarrow \text{revenue} \geq \frac{1}{2} v_i \]
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- **Case 1:** Somebody \( i' < i \) buys item
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- **Case 2:** Nobody \( i' < i \) buys item
  \[ \Rightarrow u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i \]
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Let $v_i = \max_k v_k$

- **Case 1:** Somebody $i' < i$ buys item
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- **Case 2:** Nobody $i' < i$ buys item
  \[ \Rightarrow u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i \]

**In either case:** welfare = revenue + utilities $\geq \frac{1}{2} v_i$
Extension to Bayesian Setting

Let \( p = \frac{1}{2} \mathbb{E}_{\tilde{\nu} \sim \mathcal{D}}[\max_k \tilde{\nu}_k] \)
Extension to Bayesian Setting

Let \( p = \frac{1}{2} \mathbb{E}_{\tilde{v} \sim \mathcal{D}}[\max_k \tilde{v}_k] \)

Revenue argument:

\[
\mathbb{E}[\text{revenue}] = p \cdot \Pr[\text{somebody buys item}]
\]
Extension to Bayesian Setting

Let \( p = \frac{1}{2} \mathbb{E}_{\tilde{v} \sim \mathcal{D}}[\max_k \tilde{v}_k] \)

Revenue argument:

\[
\mathbb{E}[\text{revenue}] = p \cdot \mathbb{P}[\text{somebody buys item}]
\]

Utility argument:

\[
\mathbb{E}[\text{utilities}] = \sum_i \mathbb{E} \left[ (v_i - p)^+ \mathbb{1}_{\text{no } i' < i \text{ buys item}} \right]
\]

\[
= \sum_i \mathbb{E} \left[ (v_i - p)^+ \right] \cdot \mathbb{P} \left[ \text{no } i' < i \text{ buys item} \right]
\]

\[
\geq \mathbb{E} \left[ \max_i (v_i - p)^+ \right] \cdot \mathbb{P} \left[ \text{nobody buys item} \right]
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Extension to Bayesian Setting

Let $p = \frac{1}{2} E_{\tilde{v} \sim D}[\max_k \tilde{v}_k]$

Revenue argument:

$$E[\text{revenue}] = p \cdot Pr[\text{somebody buys item}]$$

Utility argument:

$$E[\text{utilities}] = \sum_i E[(v_i - p)^+ 1_{\text{no } i' < i \text{ buys item}}]$$

$$= \sum_i E[(v_i - p)^+] \cdot Pr[\text{no } i' < i \text{ buys item}]$$

$$\geq E[\max_i (v_i - p)^+] \cdot Pr[\text{nobody buys item}]$$

By choice of $p$:

$$E[\text{welfare}] = E[\text{revenue}] + E[\text{utilities}] \geq \frac{1}{2} E[\max_i v_i]$$
Economic Intuition

- Price is **high enough** so that the revenue will offset any sold item’s contribution to the expected optimal value
- Price is **low enough** so that the buyers could have extracted high utility from any items left unsold at the end
Def. [Dütting, Feldman, Kesselheim, Lucier 2017]

A pricing rule $p^v$ is $(\alpha, \beta)$-balanced with respect to valuation profile $v = (v_1, \ldots, v_n)$ if for all feasible $x$ and all $x'$ that are feasible “after” $x$

\[
\begin{align*}
(a) \quad & \sum_i p^v_i(x_i) \geq \frac{1}{\alpha} \left( v(\text{OPT}(v)) - v(\text{OPT}(v | x)) \right) \\
(b) \quad & \sum_i p^v_i(x'_i) \leq \beta v(\text{OPT}(v | x))
\end{align*}
\]

- $v(\text{OPT}(v | x))$: Value that remains after allocating to $x$
- $v(\text{OPT}(v)) - v(\text{OPT}(v | x))$: Value lost due to allocating to $x$
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\begin{align*}
\text{(a)} \quad \sum_i p^v_i(x_i \mid x_{[i-1]}) & \geq \frac{1}{\alpha} \left( v(\text{OPT}(v)) - v(\text{OPT}(v \mid x)) \right) \\
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(a) $\sum_i p_i^v(x_i | x_{[i-1]}) \geq \frac{1}{\alpha} \left( v(\text{OPT}(v)) - v(\text{OPT}(v | x)) \right)$

(b) $\sum_i p_i^v(x_i' | x_{[i-1]}) \leq \beta v(\text{OPT}(v | x))$

**Variant:** Weakly balanced prices, replace RHS of second condition with $\beta v(\text{OPT}(v))$
**Thm.** [Dütting, Feldman, Kesselheim, Lucier 2017]

If \( p^v \) is \((\alpha, \beta)\)-balanced for every \( v \), then setting

\[
p_i(x_i \mid y) = \frac{\alpha}{1 + \alpha \beta} \mathbb{E}_{\tilde{v} \sim \mathcal{D}} \left[ p^\tilde{v}_i(x_i \mid y) \right]
\]

achieves welfare at least \( \frac{1}{1 + \alpha \beta} \mathbb{E}[v \ (\text{OPT}(v))] \).
Extension Theorem

**Thm.** [Dütting, Feldman, Kesselheim, Lucier 2017]

If $p^v$ is $(\alpha, \beta)$-balanced for every $v$, then setting

$$p_i(x_i \mid y) = \frac{\alpha}{1 + \alpha \beta} E_{\tilde{v} \sim D} \left[ p_i^\tilde{v}(x_i \mid y) \right]$$

achieves welfare at least $\frac{1}{1 + \alpha \beta} E[v(OPT(v))]$.

**Variant:** Weakly balanced, then welfare $\geq \frac{1}{4\alpha \beta} \cdot E[v(OPT(v))]$.
Back to Running Example

- Each item can be allocated at most once
- Arbitrary values for sets of up to $k$ items

- Let $x^* = \text{OPT allocation for valuation profile } v$
- Consider prices

$$p_j = \begin{cases} 
\frac{1}{|x^*_i|} v_i(x^*_i) & j \text{ is in } x^*_i, \text{ and} \\
0 & \text{otherwise}
\end{cases}$$
Back to Running Example

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Verifying Balancedness

**Claim:** These prices are weakly \((k, 1)\)-balanced.
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**Proof Sketch:**

[Prices low enough]:

All item prices sum up to \(v(OPT(v))\).
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[Prices high enough]:
\[ v(OPT(\{v \mid x\})) \geq \text{value of sets in } x^* \text{ not touched by } x \]
\[ v(OPT(v)) - v(OPT(\{v \mid x\})) \leq \text{value of sets in } x^* \text{ touched by } x \]

To touch $x^*$: Price at least \[ \frac{1}{|x_i^*|} v_i(x_i^*) \geq \frac{1}{k} v_i(x_i^*) \].
What about Computation?

- If we can compute balanced prices for any fixed input, then we can estimate the average prices by sampling.

  ⇒ Additive $\epsilon$ loss, with $POLY(n, m, \frac{1}{\epsilon})$ samples.
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- To compute prices for any fixed input:
  Use *fractional* solution $x^*$ instead of integral one.
What about Computation?

- If we can compute balanced prices for any fixed input, then we can estimate the average prices by sampling.
  \[ \Rightarrow \text{Additive } \epsilon \text{ loss, with } \text{POLY}(n, m, \frac{1}{\epsilon}) \text{ samples} \]

- To compute prices for any fixed input:
  Use \textit{fractional} solution \( x^* \) instead of integral one.
  \[ \Rightarrow \text{polytime } O(k)\text{-approximation} \]
## Additional Applications

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Further Results

- Composition Theorem:
  Results for simple settings imply results for more complex ones

- Connection to Auction Theory:
  Often Price of Anarchy analyses imply posted price mechanisms with comparable performance
Conclusion

- General framework for pricing and prophet inequalities
- Not all prophet inequalities fit into our framework
  - Can one extend the framework?
  - Do all prophet inequalities have a counterpart in prices?
- What can you do with fewer samples, random order, ...?
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Thank you! Questions?