Temporal Vertex Cover with a Sliding Time Window

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Durham University

These results have been presented in **ICALP 2018**

Joint work with:
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AlgoUK, University of Liverpool
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Static and Temporal Graphs

Modern networks are highly dynamic:

- Social networks: friendships are added/removed, individuals leave, new ones enter
- Transportation networks: transportation units change with time their position in the network
- Physical systems: e.g. systems of interacting particles

The common characteristic in all these applications:

- the graph topology is subject to discrete changes over time

$\Rightarrow$ the notion of vertex adjacency must be appropriately re-defined (by introducing the time dimension in the graph definition)

Various graph concepts (e.g. reachability, connectivity):

- crucially depend on the exact temporal ordering of the edges

Temporal graphs

Formally:

Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

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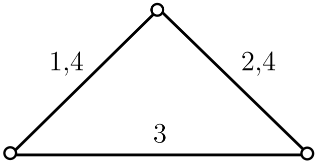

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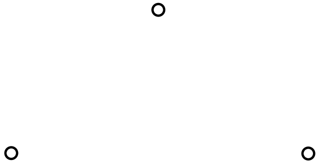

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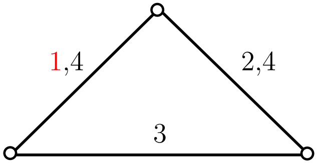

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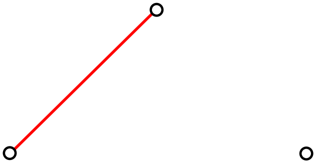

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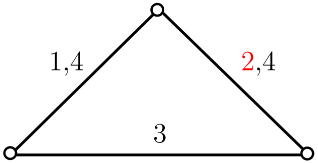

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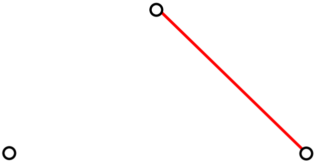

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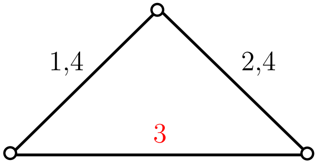

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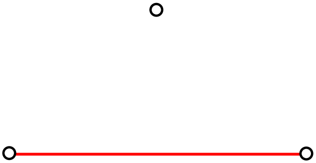

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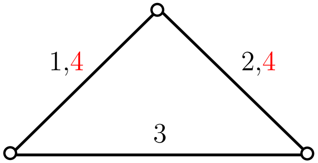

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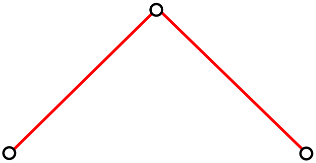

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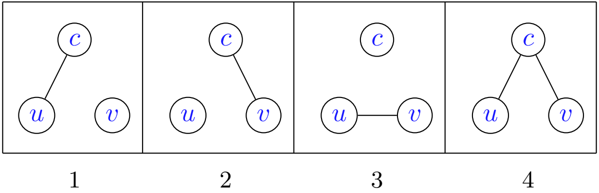
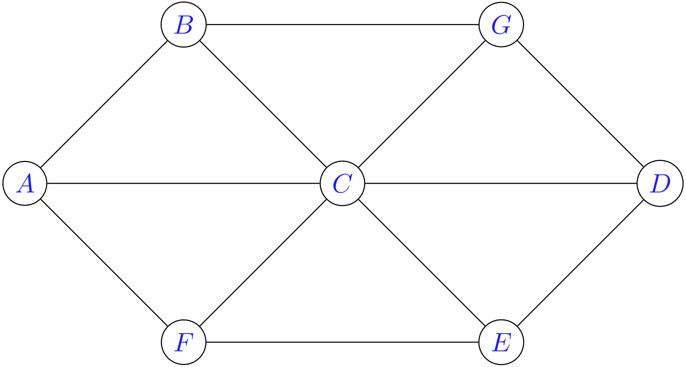
Alternatively, we can view it as a sequence of static graphs, the snapshots:



Example: static vs temporal graphs

From paths to temporal paths:



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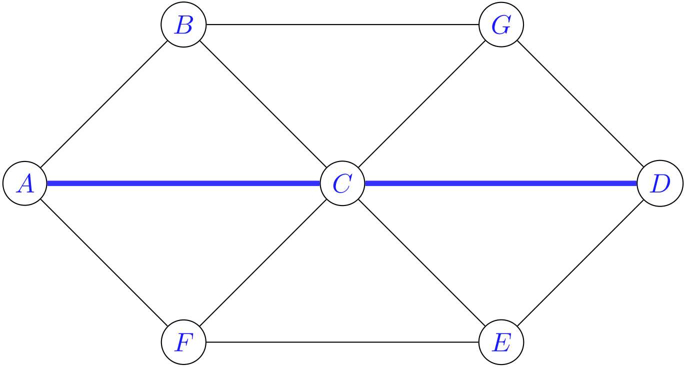

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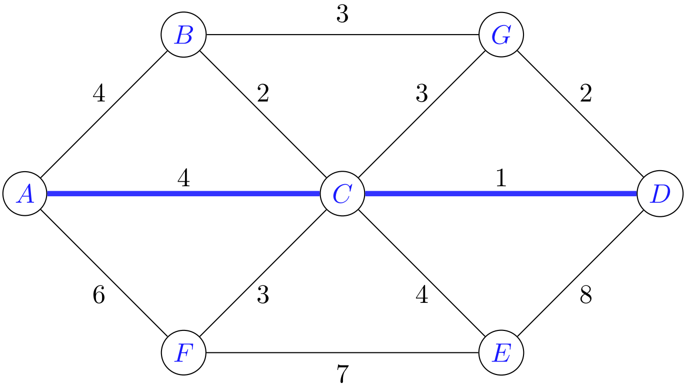

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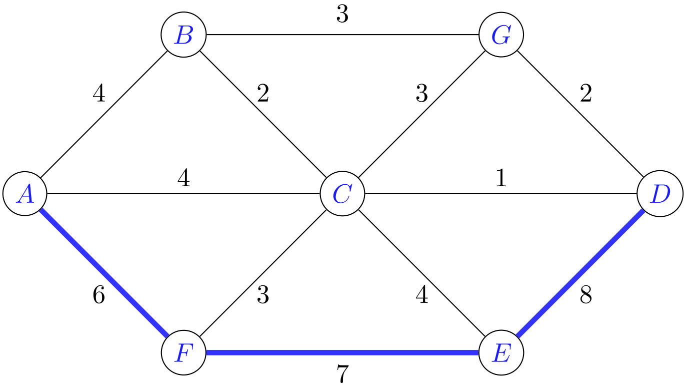

Overview

- Basic definitions
- Temporal vertex cover
- Temporal vertex cover with a sliding time window
- Open problems

Basic definitions I

To specify a temporal graph class, we can:

- either restrict the underlying graph G,
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For a class $\mathcal{X}$ of static graphs we say that a temporal graph (G, λ) is

- $\mathcal{X}$ temporal, if $G \in \mathcal{X}$;
- always $\mathcal{X}$ temporal, if $G_i \in \mathcal{X}$ for every $i \in [T] = \{1, 2, \ldots, T\}$.

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Definition (Temporal Vertex Subset)

A pair $(u, t) \in V \times [T]$ is called the appearance of vertex u at time t.
A temporal vertex subset of (G, λ) is a set $\mathcal{S} \subseteq V \times [T]$ of vertex appearances in (G, λ).

Basic definitions II

Definition (Edge is Temporally Covered)

A vertex appearance (w, t) temporally covers an edge e if:

(i) w covers e, i.e. $w \in e$, and

(ii) $t \in \lambda(e)$, i.e. the edge e is active during the time slot t.

Basic definitions II

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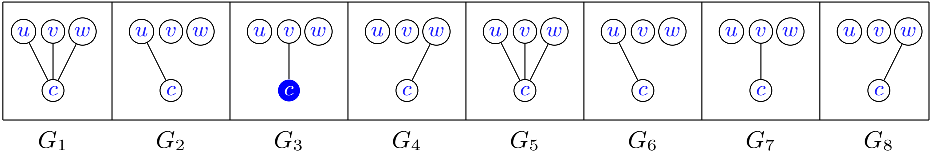

– $(c, 3)$ temporally covers edge cv, but

– $(c, 3)$ temporally covers neither cu, nor cw.

Basic definitions: Temporal Vertex Cover

Definition (Temporal Vertex Cover)

A temporal vertex cover of (G, λ) is a temporal vertex subset $\mathcal{S}$ of (G, λ) such that every edge $e \in E(G)$ is temporally covered by at least one vertex appearance in $\mathcal{S}$.

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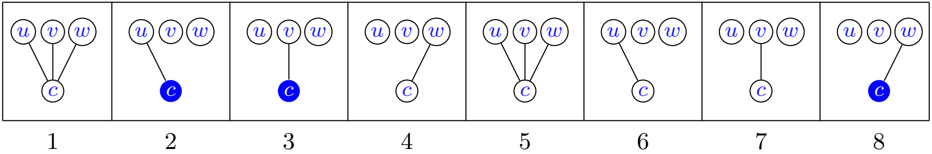

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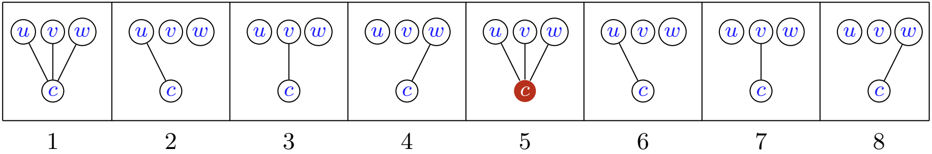

- $\{(c,2),(c,3),(c,8)\}$ is a Temporal Vertex Cover
- $\{(c,5)\}$ is a minimum Temporal Vertex Cover

Basic definitions: Temporal Vertex Cover

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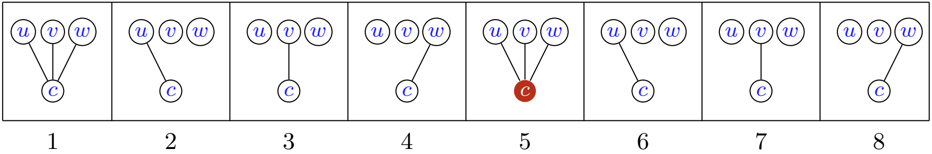

TEMPORAL VERTEX COVER (TVC)

Input: A temporal graph (G, λ).

Output: A temporal vertex cover $\mathcal{S}$ of (G, λ) with the minimum $|\mathcal{S}|$.

Basic definitions: Sliding Window Temporal Vertex Cover

Definition (Time Windows)

1. For every time slot $t \in [1, T - \Delta + 1]$:
   the time window $W_t = [t, t + \Delta - 1]$ is the sequence of the Δ consecutive time slots $t, t+1, \ldots, t + \Delta - 1$.

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3. $\mathcal{S}[W_t] = \{(w, t) \in \mathcal{S} : t \in W_t\}$ is the restriction of the temporal vertex subset $\mathcal{S}$ to the window W_t.

Basic definitions: Sliding Window Temporal Vertex Cover

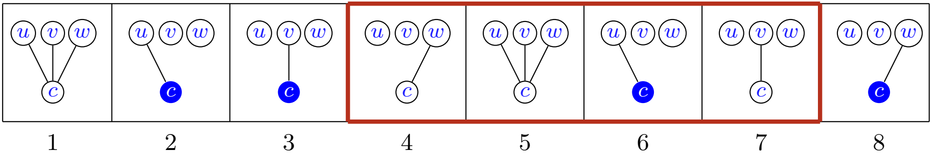
Definition (Sliding Δ-Window Temporal Vertex Cover)

A sliding Δ-window temporal vertex cover of (G, λ) is a temporal vertex subset $\mathcal{S}$ of (G, λ) such that:

- for every time window W_t and for every edge $e \in E[W_t]$,
- e is temporally covered by at least one vertex appearance $(w, t) \in \mathcal{S}[W_t]$.

Basic definitions: Sliding Window Temporal Vertex Cover

Example ($\Delta = 4$)



- $\{(c,2),(c,3),(c,6),(c,8)\}$ is not a sliding Δ-window temporal vertex cover, as edges $cv, cw \in E[W_4]$ are not temporally covered in window W_4.

Basic definitions: Sliding Window Temporal Vertex Cover

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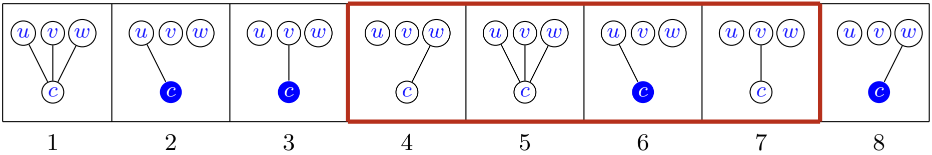

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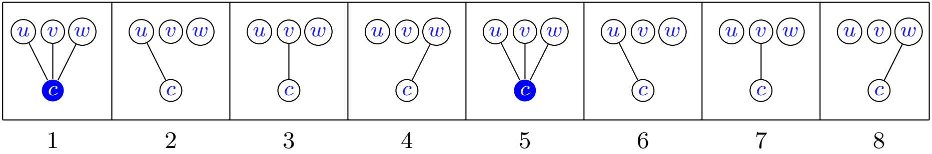

– $\{(c, 1), (c, 5)\}$ is a sliding Δ-window temporal vertex cover.

Basic definitions: Sliding Window Temporal Vertex Cover

Sliding Window Temporal Vertex Cover (SW-TVC)

Input: A temporal graph (G, λ) with lifetime T, and an integer $\Delta \leq T$.
Output: A sliding Δ-window temporal vertex cover $\mathcal{S}$ of (G, λ) with the minimum $|\mathcal{S}|$.

Overview

- Basic definitions
- Temporal vertex cover
- Temporal vertex cover with a sliding time window
- Open problems

Temporal Vertex Cover: the star temporal case

Lemma

TVC on star temporal graphs is equivalent to SET COVER.

- leafs of the underlying star $\leftrightarrow$ ground set of the SET COVER instance
- each snapshot graph $\leftrightarrow$ a set in the SET COVER instance

Goal: Choose sets (snapshots) to cover all elements (leafs' edges)

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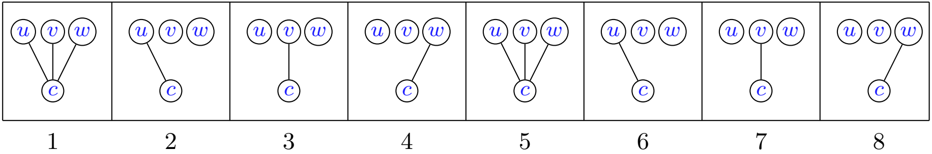

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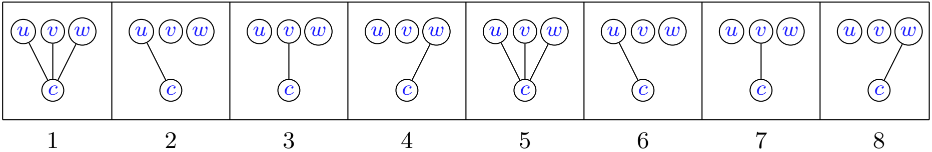

1. Universe: $\{u, v, w\}$
2. Sets: $S_1 = \{u, v, w\}$, $S_2 = \{u\}$, $S_3 = \{v\}$, $S_4 = \{w\}$, $\ldots$

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4. TVC on star temporal graphs can be $\ln n$-approximated in polynomial time.
5. For general graphs: $2\ln n$-approximation algorithm by a similar reduction from TVC to SET COVER

Overview

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- Alternative models
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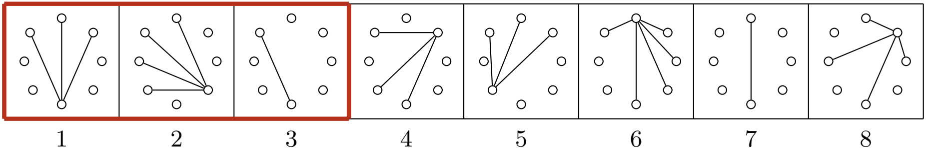

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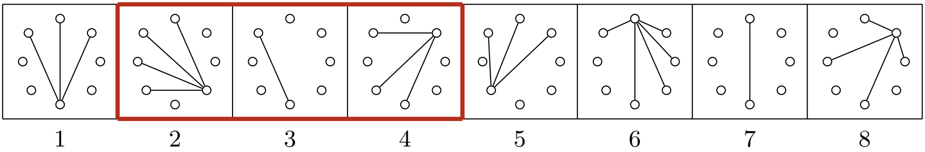

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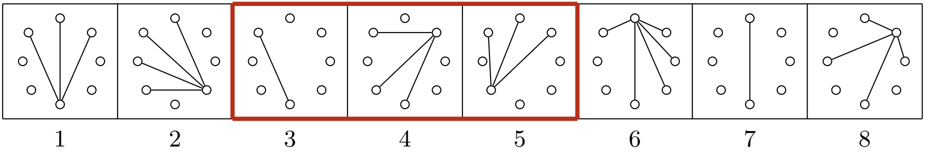

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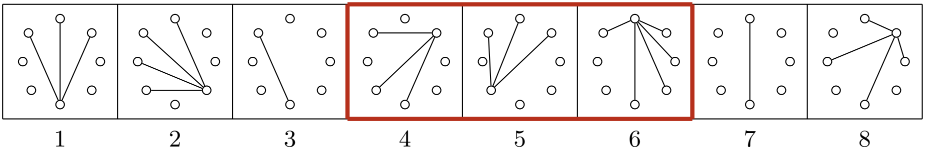

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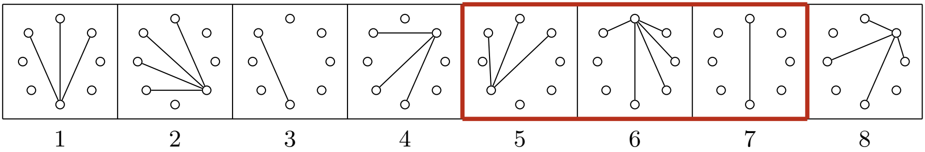

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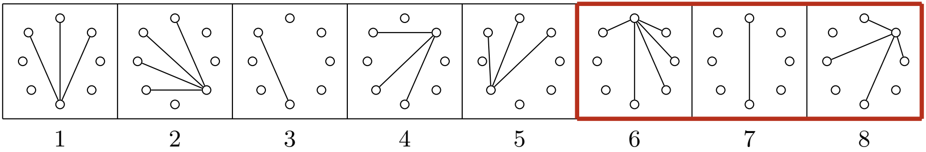

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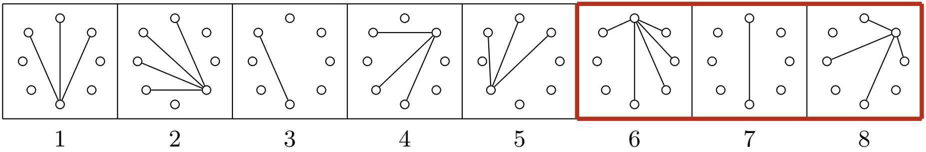

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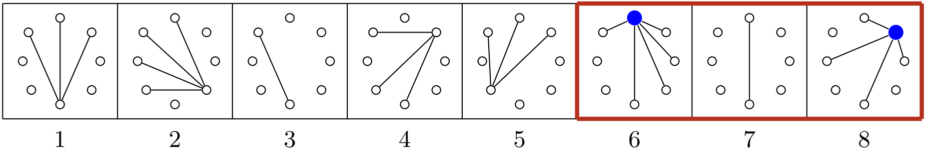
- For variables $x_1, x_2, \ldots, x_\Delta$ we define $f(t; x_1, x_2, \ldots, x_\Delta)$ to be the smallest cardinality of a sliding Δ-window temporal vertex cover $\mathcal{S}$ of $(G, \lambda)|_{[1,t+\Delta-1]}$, such that the solution at times $t, t+1, \ldots, t+\Delta-1$ is defined by the variables $x_1, x_2, \ldots, x_\Delta$.

SW-TVC: **always star** temporal graphs

$f(6; 1, 0, 1)$



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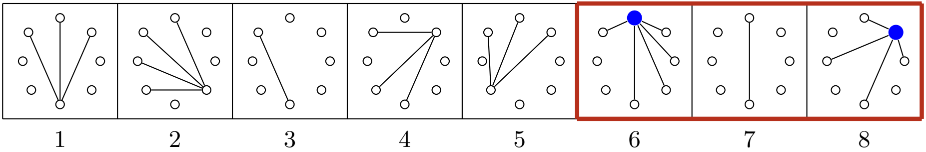

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Lemma (dynamic programming)

$$f(t; x_1, x_2, \ldots, x_\Delta) = x_\Delta + \min_{y\in\{0,1\}} \{f(t-1; y, x_1, x_2, \ldots, x_{\Delta-1})\}$$

SW-TVC: **always star** temporal graphs

Theorem

SW-TVC *on always star temporal graphs can be solved in* $O(T\Delta(n+m) \cdot 2^{\Delta})$ *time.*

Theorem

SW-TVC on always star temporal graphs can be solved in $O(T\Delta(n+m)\cdot 2^{\Delta})$ time.

Algorithm SW-TVC on always star temporal graphs

Input: An always star temporal graph (G,λ) with lifetime T and a natural $\Delta \leq T$.
Output: The cardinality of a minimum sliding Δ-window temporal vertex cover in (G,λ).

```
for t = 1 to T − Δ + 1 do
    for all x_1, x_2, ..., x_Δ ∈ {0,1} do
        if {(c_{t+i−1}, t+i−1) | x_i = 1} is a TVC of (G,λ)|_[t,t+Δ−1] then
            if t = 1 then
                f(t; x_1, ..., x_Δ) ← Σ_{i=1}^{Δ} x_i
            else
                f(t; x_1, ..., x_Δ) ← x_Δ + min_{y∈{0,1}} {f(t−1; y, x_1, ..., x_{Δ−1})}
        else
            f(t; x_1, ..., x_Δ) ← ∞
return min_{x_1,...,x_Δ∈{0,1}} {f(T−Δ+1; x_1, ..., x_Δ)}
```

SW-TVC: the general case

Theorem

SW-TVC *on general temporal graphs can be solved in* $O(T\Delta(n+m) \cdot 2^{n(\Delta+1)})$ *time.*

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SW-TVC *on general temporal graphs can be solved in* $O(T\Delta(n+m) \cdot 2^{n(\Delta+1)})$ *time.*

Main idea:

- for each of the Δ snapshots in the (currently) last Δ-window, we enumerate all 2^n vertex subsets,
- instead of just enumerating over the truth values of Δ Boolean variables ("always star" case)

SW-TVC: the general case

Theorem

SW-TVC *on general temporal graphs can be solved in* $O(T\Delta(n+m)\cdot 2^{n(\Delta+1)})$ *time.*

Algorithm SW-TVC on general temporal graphs

Input: A temporal graph (G,λ) with lifetime T and a natural $\Delta \leq T$.
Output: The smallest cardinality of a sliding Δ-window temporal vertex cover in (G,λ).

for $t = 1$ to $T - \Delta + 1$ **do**
  for all $A_1, A_2, \ldots, A_\Delta \subseteq V$ **do**
    if $\bigcup_{i=1}^{\Delta}(A_i, t+i-1)$ is a TVC of $(G,\lambda)|_{[t,t+\Delta-1]}$ **then**
      if $t = 1$ **then**
        $f(t; A_1, \ldots, A_\Delta) \leftarrow \sum_{i=1}^{\Delta} |A_i|$
      else
        $f(t; A_1, \ldots, A_\Delta) \leftarrow |A_\Delta| + \min_{X \subseteq V}\{f(t-1; X, A_1, \ldots, A_{\Delta-1})\}$
    else
      $f(t; A_1, \ldots, A_\Delta) \leftarrow \infty$
return $\min_{A_1,\ldots,A_\Delta \subseteq V}\{f(T-\Delta+1; A_1, \ldots, A_\Delta)\}$

SW-TVC: Optimality under ETH

Theorem

For any two (arbitrarily growing) functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$, there exists a constant $\varepsilon \in (0, 1)$ such that $\mathrm{SW\text{-}TVC}$ cannot be solved in $f(T) \cdot 2^{\varepsilon n \cdot g(\Delta)}$ time assuming ETH.

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Proof (idea):

- reduction from VERTEX COVER
- $T = \Delta = 2$
- $G_1 = G$; G_2 is an independent set
- given f and g, choose appropriate ε

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Corollary

Our $O(T\Delta(n + m) \cdot 2^{n(\Delta+1)})$-time algorithm is asymptotically almost optimal.

SW-TVC: always bounded vertex cover number temporal graphs

Let $\mathcal{C}_k$ be the class of graphs with the vertex cover number at most k.

Theorem

$\mathrm{SW\text{-}TVC}$ *on always $\mathcal{C}_k$ temporal graphs can be solved in $O(T\Delta(n+m) \cdot n^{k(\Delta+1)})$ time.*

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Theorem

$\mathrm{SW\text{-}TVC}$ *on always* $\mathcal{C}_k$ *temporal graphs can be solved in* $O(T\Delta(n+m) \cdot n^{k(\Delta+1)})$ *time.*

Main idea:

- in the optimal solution, the choice at step i is a subset of a minimum vertex cover at this snapshot

$\Rightarrow$ for each of the Δ last snapshots, enumerate all n^k vertex subsets (candidates for vertex cover at snapshot i)

Δ-TVC

If the parameter Δ (the size of a sliding window) is fixed, we refer to SW-TVC as $\Delta\text{-TVC}$ (i.e. Δ is a part of the problem name).

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Observation

$(\Delta + 1)\text{-}\mathrm{TVC}$ *is at least as hard as* $\Delta\text{-}\mathrm{TVC}$.

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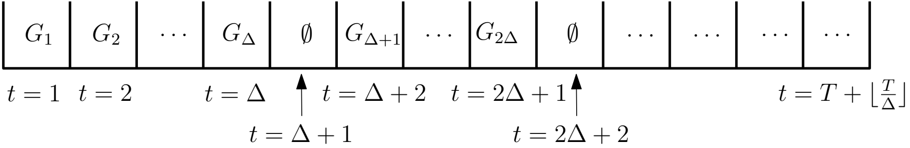
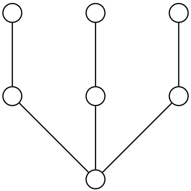


2-TVC is hard to approximate

Let $\mathcal{X}$ be the class of graphs whose connected components are induced subgraphs of graph Ψ.



Ψ

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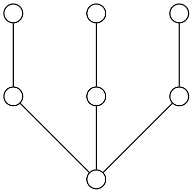

Ψ

Clearly, Vertex Cover is linearly solvable on graphs from $\mathcal{X}$.

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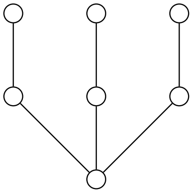

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Clearly, VERTEX COVER is linearly solvable on graphs from $\mathcal{X}$.

Theorem

There is no PTAS for 2-TVC on always $\mathcal{X}$ temporal graphs.

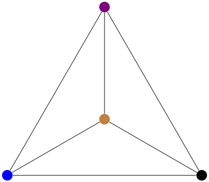
2-TVC is hard to approximate

Theorem

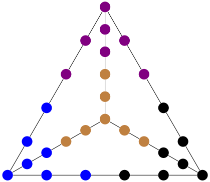
There is no PTAS for 2-TVC on always $\mathcal{X}$ temporal graphs.

Proof (sketch):

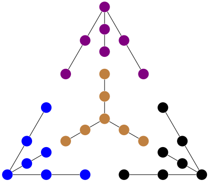
1. Let $\mathcal{Y}$ be the class of graphs which can be obtained from cubic graphs by subdividing every edge exactly 4 times.
2. There is no PTAS for Vertex Cover on $\mathcal{Y}$.
3. Reduce Vertex Cover on $\mathcal{Y}$ to 2-TVC on always $\mathcal{X}$ temporal graphs such that optimal solutions of both problems have same size.



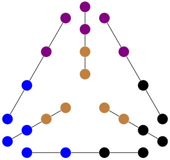
K_4



The 4-subdivision of K_4



Snapshot G_1



Snapshot G_2

SW-TVC: approximation algorithms I

Reduction from SW-TVC to SET COVER.

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1. The universe: the set of all pairs $(e, t) \in E \times [T - \Delta + 1]$ such that e appears (and so must be temporally covered) within window W_t.

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2. The sets: for every vertex appearance (v, s) we define $C_{v,s}$ to be the set of elements (e, t) in the universe, such that (v, s) temporally covers e in window W_t.

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Consequences:

1. $O(\ln n + \ln \Delta)$-approximation
(every set $C_{v,s}$ has at most $n\Delta$ elements
$\Rightarrow$ approximation factor $H_{n\Delta} - \frac{1}{2} \approx \ln n + \ln \Delta$)

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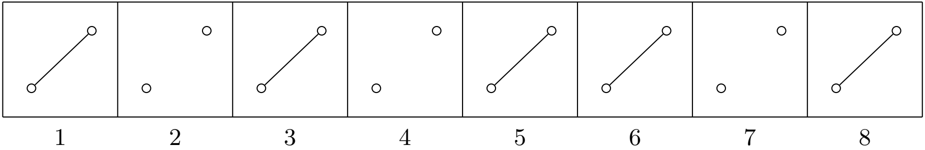
$\Rightarrow$ 2Δ-approximation

SW-TVC: approximation algorithms II

Single-edge temporal graph: **exact algorithm**



SW-TVC: approximation algorithms II

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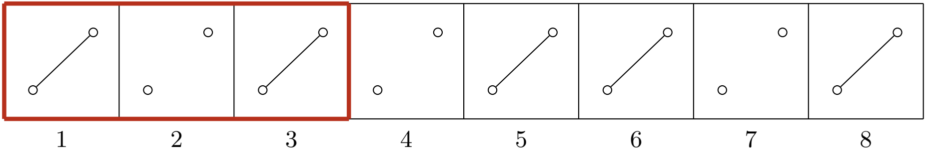

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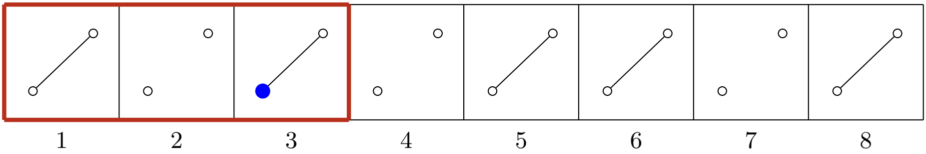

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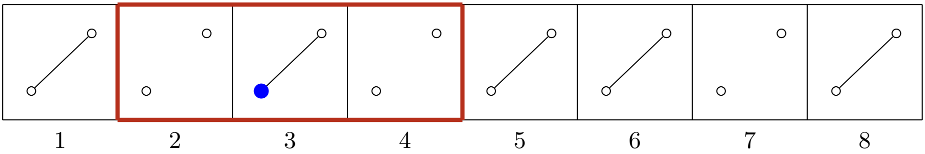

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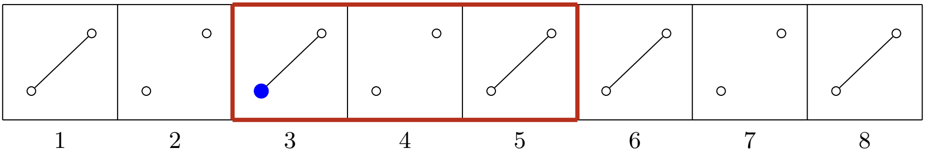

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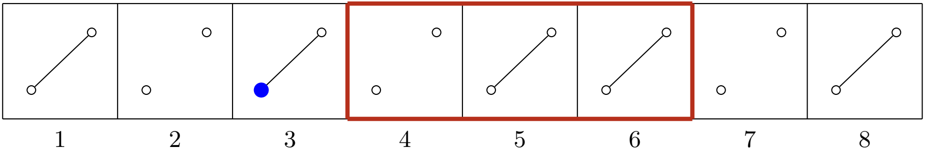

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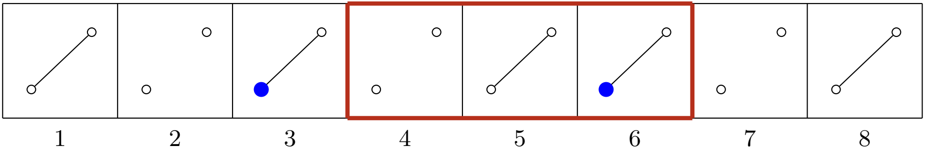

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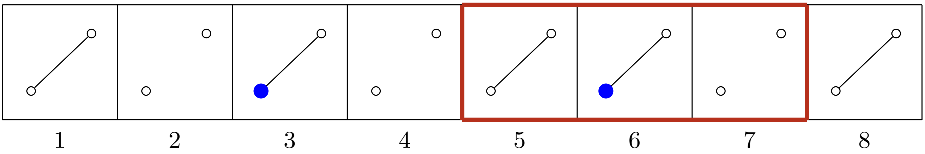

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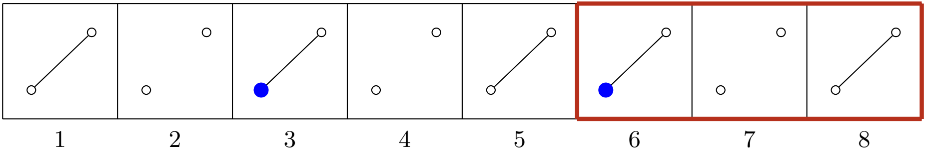


SW-TVC: approximation algorithms II

Single-edge temporal graph: exact algorithm

Algorithm SW-TVC on single-edge temporal graphs

Input: A temporal graph (G, λ) of lifetime T with $V(G) = \{u, v\}$; and $\Delta \leq T$.
Output: A minimum-cardinality sliding Δ-window temporal vertex cover $\mathcal{S}$ of (G, λ).

```
S ← ∅
t = 1
while t ≤ T − Δ + 1 do
    if ∃ r ∈ [t, t + Δ − 1] such that uv ∈ E_r then
        choose maximum such r and add (u, r) to S
        t ← r + 1
    else
        t ← t + 1
return S
```

- greedy algorithm
- linear time

SW-TVC: approximation algorithms II

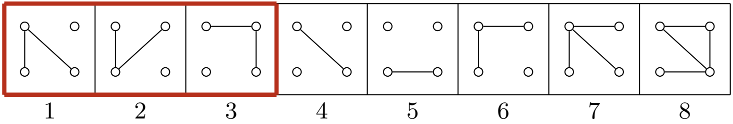
Always degree at most d temporal graphs: d-approx. algorithm

Main idea:

- solve independently each single-edge subgraph of G
- take the union of the solutions

SW-TVC: approximation algorithms II

Always degree at most d temporal graphs: **d-approx. algorithm**



SW-TVC: approximation algorithms II

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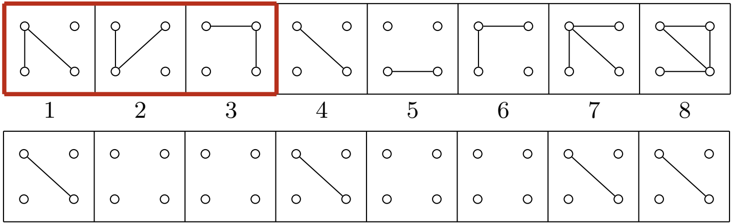

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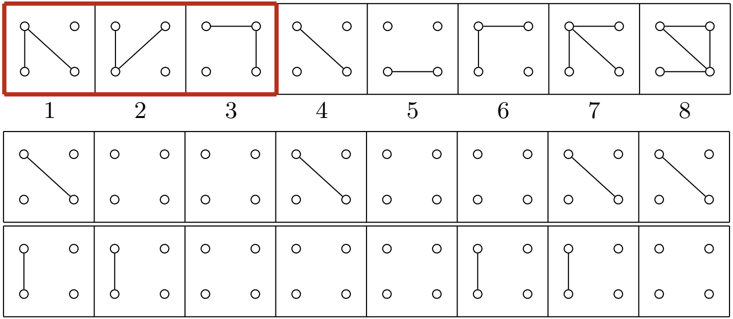

SW-TVC: approximation algorithms II

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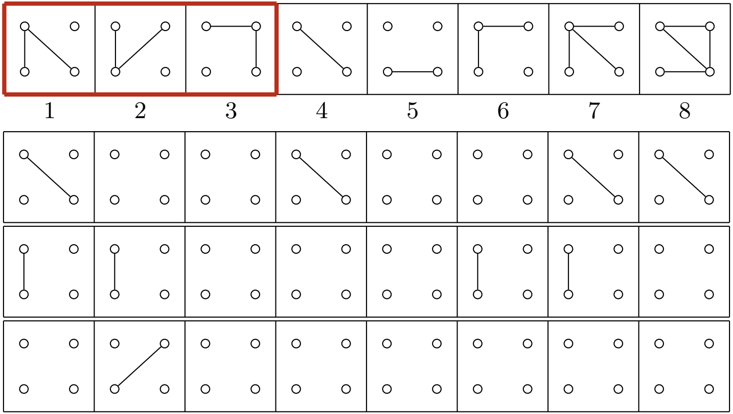

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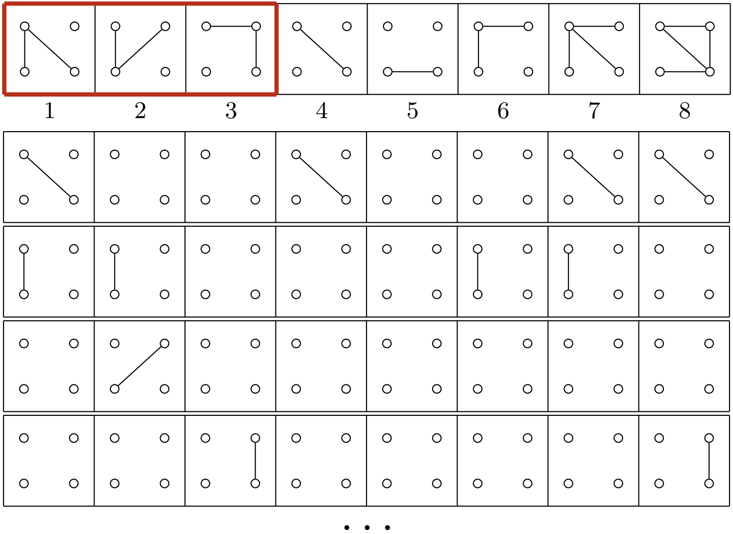
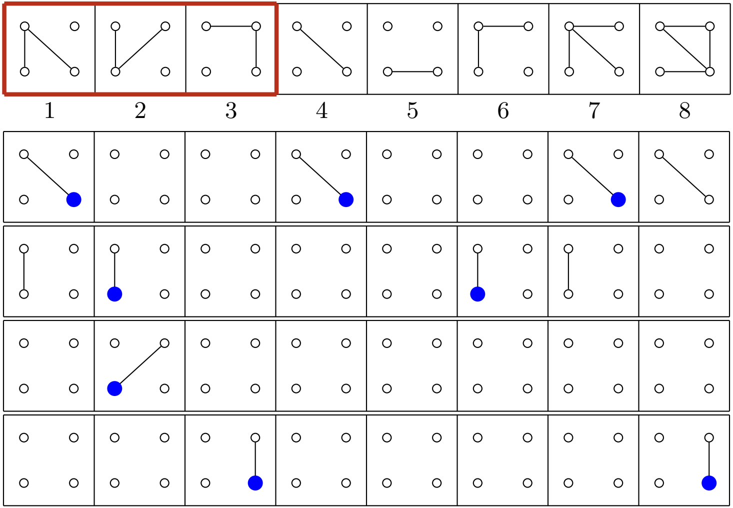


SW-TVC: approximation algorithms II

Always degree at most d temporal graphs: d-approx. algorithm



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SW-TVC: approximation algorithms II

Always degree at most d temporal graphs: d-approx. algorithm

Algorithm d-approximation of $\mathrm{SW\text{-}TVC}$ on always degree at most d temporal graphs

Input: An always degree at most d temporal graph (G, λ) of lifetime T, and $\Delta \leq T$.
Output: A sliding Δ-window temporal vertex cover $\mathcal{S}$ of (G, λ).

1: **for** every edge $uv \in E(G)$ **do**
2: Compute an optimal solution $\mathcal{S}_{uv}$ of the problem for $(G[\{u, v\}], \lambda)]$
3: $\mathcal{S} \leftarrow \mathcal{S} \cup \mathcal{S}_{uv}$
return $\mathcal{S}$

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Lemma

The above algorithm is a $O(mT)$-time d-approximation algorithm for SW-TVC on always degree at most d temporal graphs.

SW-TVC: approximation algorithms II

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Lemma

The above algorithm is a $O(mT)$-time d-approximation algorithm for SW-TVC on always degree at most d temporal graphs.

Corollary

SW-TVC can be optimally solved in $O(mT)$ time on the class of always degree at most 1 (matching) temporal graphs.

Overview

- Basic definitions
- Alternative models
- Temporal vertex cover
- Temporal vertex cover with a sliding time window
- Open problems

Open Problems

Problem 1

Determine the complexity status of $\Delta\text{-TVC}$ on degree at most 2 temporal graphs.

1. $\Delta\text{-TVC}$ on always degree at most 1 can be solved in linear time.

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2. $\Delta\text{-TVC}$ on always degree at most 3: no PTAS

Open Problems

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1. Δ-TVC on always degree at most 1 can be solved in linear time.
2. Δ-TVC on always degree at most 3: no PTAS, even when:
    1. the underlying graph has degree at most 3; and
    2. connected components of snapshots have at most 7 vertices.

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Can Δ-TVC on general graphs be approximated within a factor better than 2Δ?

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Problem 2

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Problem 3

Can Δ-TVC on always degree at most d temporal graphs be approximated within a factor better than d?

Thank you!