Symmetric cubic graphs as Cayley graphs

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Graph symmetries

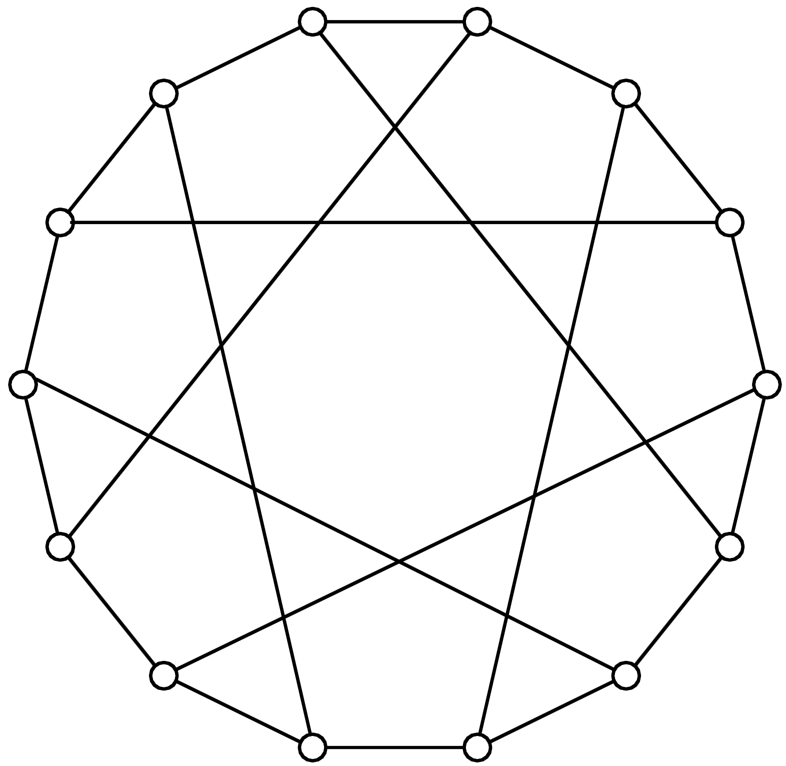
A graph X is called vertex-transitive if its automorphism group $A = \text{Aut}(X)$ has a single orbit on vertices, or edge-transitive if A has a single orbit on edges, or arc-transitive (or symmetric) if A has a single orbit on the set of arcs – where an arc is an ordered pair (v, w) of adjacent vertices.

An s-arc is a path of length s in which any three consecutive vertices are distinct, and a graph is s-arc-transitive if its automorphism group A has a single orbit on s-arcs, and s-arc-regular if its automorphism group A acts regularly (sharply-transitively) on s-arcs.

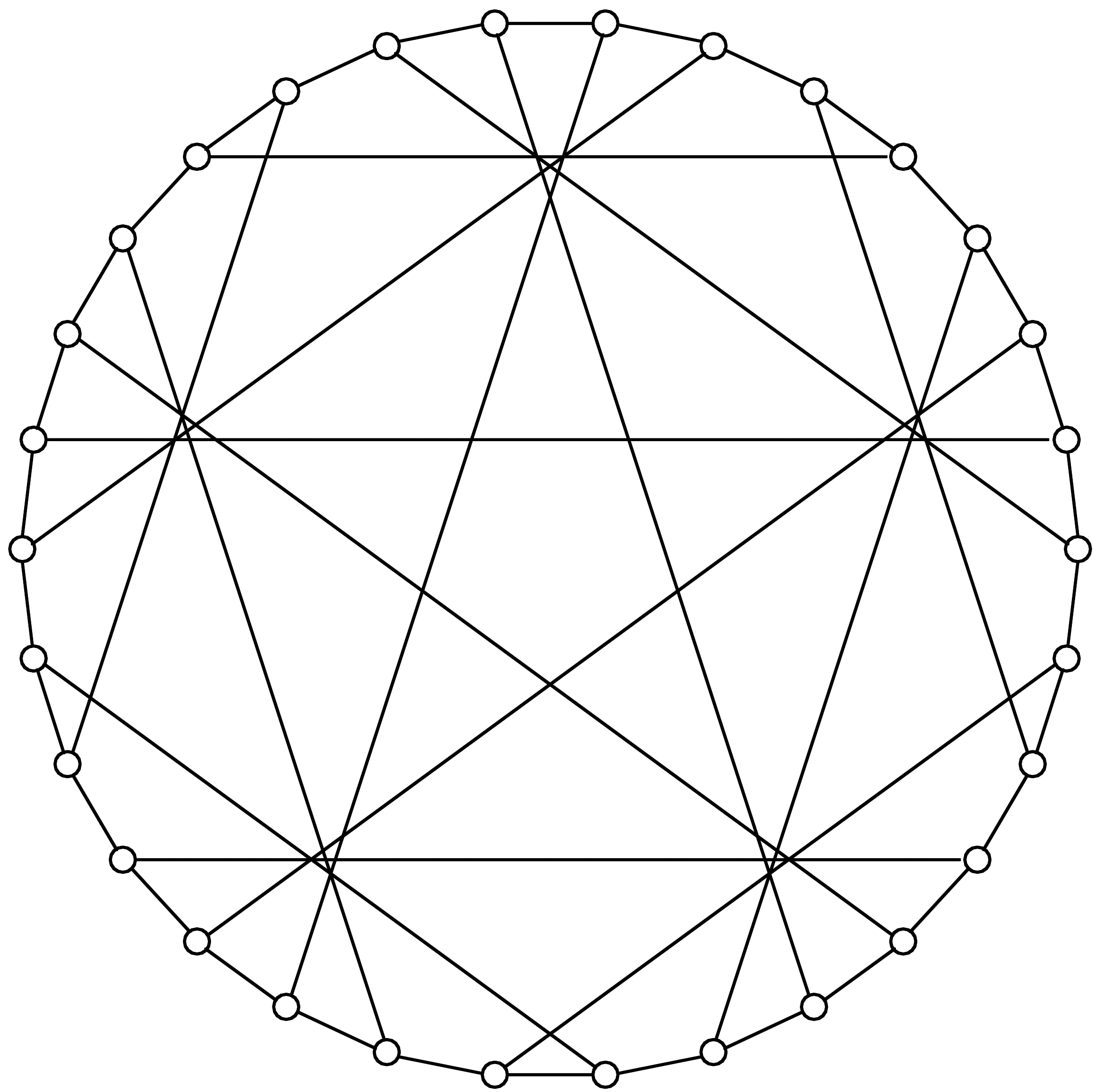
Examples

- The cycle graph C_n is vertex-, edge- and arc-transitive ... and s-arc-transitive for all $s \geq 2$
- The complete graph K_n is VT, ET, AT and 2-AT ... but not 3-AT when $n \geq 4$
- The complete bipartite graph $K_{n,n}$ is VT, ET, AT, 2-AT and 3-AT , but not 4-AT when $n \geq 3$
- The Heawood graph (the incidence graph of the Fano plane) is 4-AT, but not 5-AT
- Tutte's 8-cage (with vertices the 30 odd involutions in S_6 and with (a, b) adjacent to $(a, b)(c, d)(e, f)$) is 5-AT.

Example: Heawood graph (4-arc-regular)



Example: Tutte's 8-cage (5-arc-regular)



Cayley graphs

A Cayley graph is a kind of graphical representation of part of a multiplication table for a group: for any subset S of a group G, let $X = \text{Cay}(G, S)$ be the graph with vertex-set G and with edges of the form $\{g, xg\}$ for all $g \in G$ and all $x \in S$.

Then G acts by right mutiplication as a vertex regular group of automorphisms of X.

Alternatively, a Cayley graph is any graph X that admits a group G of automorphisms that acts regularly on $V(X)$.

In that case we can fix any particular vertex v, and then let $S = \{x \in G \mid \{v, v^x\} \in E(X)\}$, the subset of G consisting of automorphisms that take v to one of its neighbours.

Examples

- The cycle graph C_n is Cayley
- The complete graph K_n is Cayley:
  $K_n = \text{Cay}(G, G \setminus \{1\})$ for every group G of order n.
- The complete bipartite graph $K_{n,n}$ is $\text{Cay}(G, G \setminus H)$ whenever $H \leq G$ with $|G| = 2n = 2|H|$

while

- The Petersen graph, the Gray graph and the Hoffman-Singleton graph are vertex-transitive and edge-transitive but are not Cayley graphs.

Question: When is a Cayley graph arc-transitive?

Let X be the Cayley graph Cay(G,S) for some group G and subset $S \subset G$. If S is an orbit of some subgroup H of the automorphism group of G then H induces a group of automorphisms of X acting transitively on the arcs incident with the identity vertex 1, namely the arcs $(1,x)$ with $x \in S$. This condition, however, is sufficient but not necessary.

Exercise: Find an example of an arc-transitive finite Cayley graph X for which there is no group G, subset $S \subset G$ and group $H \leq$ Aut(G) acting transitively on S, such that X is isomorphic to the Cayley graph Cay(G,S).

More challenging exercise: Is there a 3-valent example?

A better approach for cubic (3-valent) graphs

Instead of taking a Cayley graph and asking whether it's symmetric (arc-transitive), in the 3-valent case we can take a symmetric graph and decide if it's Cayley!

To see how this works, we need to know about symmetric cubic graphs . . .

Background on symmetric cubic graphs

Let X be a connected finite symmetric cubic (3-valent) graph, and let $A = \text{Aut}(X)$. Then the stabiliser A_v of every vertex v induces a 3-cycle on the neighbourhood of v, so 3 divides $|A_v|$. Similarly, the stabiliser A_v of every arc (v, w) either fixes or swaps the two 2-arcs of the form (v, w, z).

By connectedness and induction on the length of a chain of stabilisers, it follows that $|A_v| = 3 \cdot 2^k$ for some k.

Tutte's theorem (1947 & 1959): $k \leq 4$. And furthermore, $\text{Aut}(X)$ acts regularly on the s-arcs of X for some $s \leq 5$. (In particular, there are no finite 6-arc-transitive cubic graphs.)

[NB: This has a nice proof, using relatively easy group and graph theory – e.g. see *Algebraic Graph Theory* by N. Biggs]

Background on symmetric cubic graphs (cont.)

Djoković & Miller (1980) proved there are up to seven classes of connected finite symmetric cubic graphs, which we now call 1, 2^1, 2^2, 3, 4^1, 4^2 and 5, where for $1 \leq s \leq 5$ the class s or s^1 consists of all s-arc-regular examples that have an arc-reversing automorphism of order 2, while for $s \in \{2,4\}$ the class s^2 consists of all s-arc-regular examples that have NO arc-reversing automorphism of order 2.

Conder & Lorimer (1989) showed that all seven classes are non-empty, by producing the first known examples of graphs in the classes 2^2 and 4^2.

Conder & Nedela (2009) refined the Djoković-Miller classification, according to the types of arc-transitive subgroups contained in the automorphism group of the graph.

The classification by Conder & Nedela (2009):

s	Action type	Bipartite?	Smallest example	Minimal?
1	(1)	Sometimes	F026 in the 'Foster census'	No
2	$(1, 2^1)$	Sometimes	F004 (K_4)	No
2	(2^1)	Sometimes	F084	No
2	(2^2)	Sometimes	F448C [Conder-Dobcsányi]	No
3	$(1, 2^1, 2^2, 3)$	Always	F006 ($K_{3,3}$)	No
3	$(2^1, 2^2, 3)$	Always	F020B (GP(10,3))	No
3	$(2^1, 3)$	Never	F010 (Petersen)	No
3	$(2^2, 3)$	Never	F028 (Coxeter)	No
3	(3)	Sometimes	F110	No
4	$(1, 4^1)$	Always	F014 (Heawood)	Yes
4	(4^1)	Sometimes	F102 (Sextet(17))	No
4	(4^2)	Sometimes	3^{10}-fold cover of F468?	No
5	$(1, 4^1, 4^2, 5)$	Always	Biggs-Conway graph	Yes
5	$(4^1, 4^2, 5)$	Always	F030 (Tutte's 8-cage)	No
5	$(4^1, 5)$	Never	S_{10} graph [Conder (1987)]	No
5	$(4^2, 5)$	Never	F234B (Wong's graph)	No
5	(5)	Sometimes	$M_{24} \wr C_2$ graph?	No

Seven universal groups for finite symmetric cubic graphs

Fundamental to the classifications by Djoković & Miller and Conder & Nedela are seven finitely-presented groups G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5, with the property that a connected finite symmetric cubic graph X lies in class s, s^1 or s^2 if and only Aut(X) is a quotient of G_s, G_s^1 or G_s^2 (respectively).

Each of these groups is an amalgamated free product $V *_R E$ of subgroups V and E mapping to the stabilisers in Aut(X) of a vertex v and edge $\{v, w\}$, with amalgamated subgroup $R = V \cap E$ mapping to the stabiliser of the arc (v, w).

The seven groups (amalgamated free products)

$G_1 \;= \langle\, h, a \mid h^3 = a^2 = 1 \,\rangle \cong \mathrm{PSL}_2(\mathbb{Z})$, the modular group
[1-arc-regular graphs]

${G_2}^1 = \langle\, h, p, a \mid h^3 = p^2 = a^2 = 1,\; php = h^{-1},\; a^{-1}pa = p \,\rangle$
[2-arc-regular, with arc-reversing automs of order 2]

${G_2}^2 = \langle\, h, p, a \mid h^3 = p^2 = 1,\; a^2 = p,\; php = h^{-1},\; a^{-1}pa = p \,\rangle$
[2-arc-regular, without arc-reversing automs of order 2]

$$G_3 \;= \langle\, h, p, q, a \mid h^3 = p^2 = q^2 = a^2 = 1,\; pq = qp,\; php = h,\; qhq = h^{-1},\; a^{-1}pa = q \,\rangle$$
[3-arc-regular graphs]

$$G_4{}^1 = \langle\, h, p, q, r, a \mid h^3 = p^2 = q^2 = r^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ (qr)^2 = p,\ h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle$$

[4-arc-regular, with arc-reversing automs of order 2]

$$G_4{}^2 = \langle\, h, p, q, r, a \mid h^3 = p^2 = q^2 = r^2 = 1,\ a^2 = p,\ pq = qp,\ pr = rp,\ (qr)^2 = p,\ h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle$$

[4-arc-regular, without arc-reversing automs of order 2]

$$G_5 = \langle\, h, p, q, r, s, a \mid h^3 = p^2 = q^2 = r^2 = s^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ ps = sp,\ qr = rq,\ qs = sq,\ (rs)^2 = pq,\ h^{-1}ph = p,\ h^{-1}qh = r,\ h^{-1}rh = pqr,\ shs = h^{-1},\ a^{-1}pa = q,\ a^{-1}ra = s \,\rangle$$

[5-arc-regular graphs]

Back to cubic Cayley graphs ...

The classification of finite symmetric cubic graphs by 17 different 'action types' was obtained by considering the pre-images of arc-transitive subgroups of Aut(X).

We can do the same to decide when X is Cayley, by considering the pre-images of vertex-regular subgroups. A key observation is that a vertex-regular subgroup is complementary to the stabiliser of a vertex.

It turns out that in five of the 17 classes, there is no Cayley graph, while in two others, every example is a Cayley graph. In eight of the remaining ten cases there are necessary conditions on the order of the graph, and in the remaining two cases (viz. those where the action type of X is (1) or $(1, 2^1)$), there is no such condition.

Theorem [MC (2016)]

Let X be a connected symmetric cubic graph on n vertices.

(a) If X has action type (2^1), (2^2), $(2^1, 3)$, $(2^2, 3)$ or (4^2), then X is not a Cayley graph.

(b) If X has action type $(1, 4^1)$ or $(1, 4^1, 4^2, 5)$, then X is a Cayley graph.

(c) If X is a Cayley graph and has one of the eight action types $(1, 2^1, 2^2, 3)$, $(2^1, 2^2, 3)$, (3), (4^1), $(4^1, 4^2, 5)$, $(4^1, 5)$, $(4^2, 5)$ or (5), then n must satisfy a divisibility condition.

(d) If X has action type (1) or $(1, 2^1)$, then there are no analogous restrictions on n for X to be a Cayley graph.

Also there are infinitely many Cayley graphs for each of the action types in (b) to (d), and infinitely many non-Cayley graphs for each action type not in (b).

Detail in part (c) **for a symmetric cubic Cayley graph**:

- Action type $(1, 2^1, 2^2, 3)$: n must be divisible by 6 or 440
- Action type $(2^1, 2^2, 3)$: n must be divisible by 220
- Action type (3): n must be divisible by 110
- Action type (4^1): n must be divisible by 506
- Action type $(4^1, 4^2, 5)$: n must be divisible by 3072432
- Action type $(4^1, 5)$: n must be divisible by 47!
- Action type $(4^2, 5)$: n must be divisible by 47!
- Action type (5): n must be divisible by $47!/2$.

Example: Action type (1)

In this case, Aut(X) is a quotient of the modular group $G_1 = \langle\, h, a \mid h^3 = a^2 = 1\,\rangle$, with $V = \langle h\rangle$ and $E = \langle a\rangle$.

In G_1 there are two conjugacy classes of subgroups of index 3 complementary to $\langle h\rangle$ which could be the pre-image of a Cayley group for a 1-arc-regular cubic graph X. These two classes have representatives $\langle a, hah^{-1}, h^{-1}ah\rangle$ and $\langle a, hah\rangle$.

If the image of $\langle a, hah^{-1}, h^{-1}ah\rangle$ has index 3 in Aut(X), then X is an arc-transitive Cayley graph (and the Cayley set S can be taken as the set of images of a, hah^{-1} and $h^{-1}ah$).

This occurs for F026 and other 1-arc-regular graphs in the Foster census, as well as elementary abelian p-covers of these graphs for large p.

The other representative $\langle a, hah \rangle$ is not normal in G_1, but its core is the subgroup $\langle (ha)^2, (ha)^2 \rangle$, of index 6 in G_1. For X to be a Cayley graph, the order of the image of ha in $A = \mathrm{Aut}\, X$ has to be even, but greater than 4 (since otherwise we get K_4, which is 2-arc-regular).

This occurs for the graphs F144A, F432E* and F576B*, as well as elementary abelian p-covers of these for large p.

There are also examples for which the image of neither $\langle a, hah^{-1}, h^{-1}ah \rangle$ nor $\langle a, hah \rangle$ has index 3 in $\mathrm{Aut}\, X$, and these are not Cayley graphs. Examples include F448A, F720D and F720F (plus elementary abelian p-covers of these for large primes p), and the underling graphs of chiral 3-valent regular maps with non-abelian simple automorphism groups.

Why do we get the abelian covers?

Here we can use the 'Macbeath trick', introduced by Murray Macbeath in the 1960s for Riemann surfaces.

Let U be the universal group ($= G_s$, G_s^1 or G_s^2) for the action of $A = \mathrm{Aut}(X)$ on X, and let K be the kernel of the epimorphism from U to A. This kernel acts on the cubic tree X_3 with trivial vertex-stabilisers, and hence is a free group, of finite rank k. Now for any prime p greater than $16|A|$, the subgroup $N_p = K'K^{(p)}$ generated by the derived subgroup K' and all pth powers of elements of K is characteristic in K and hence normal in U. The quotient K/N_p is isomorphic to the elementary abelian p-group $\mathbb{Z}_p^k$ of rank k and exponent p, and U/N_p is isomorphic to an extension of $\mathbb{Z}_k$ by $U/K \cong A$.

The natural homomorphism from U to U/N_p makes U/N_p an s-arc-regular group of automorphisms of a cubic graph X_p that is a cover of X, with abelian covering group $K/N_p \cong \mathbb{Z}_p^k$.

The condition that $p > 16|A|$ ensures that if X_p is t-arc-regular with automorphism group A_p, then K/N_p is a normal Sylow p-subgroup of A_p, and the quotient $A_p/(K/N_p)$ acts as a group of automorphisms of X, so $t = s$. Thus X_p is s-arc-regular, with full automorphism group U/N_p.

Moreover, X_p has the same action type as X, and similarly, X_p is a Cayley graph if and only if X is a Cayley graph.

Hence for given X, we get infinitely many covers of X having the same action type as X, and being Cayley or non-Cayley according to whether X is Cayley or non-Cayley.

‘Anti’-examples: Action types (2^2) and (4^2)

These cases are easy to deal with, because in any cubic Cayley graph, the connection set S contains at least one involution, and then clearly this acts on the graph as an arc-reversing automorphism of order 2. Hence no symmetric cubic graph of type 2^2 or 4^2 is a Cayley graph.

Another clear example: Action type $(1, 4^1)$

In this case the pre-image of a 1-arc-regular subgroup of Aut(X) has index 8 in the universal group G_4^1, and the permutation group induced on the eight cosets is isomorphic to PGL$(2, 7)$, of order 336. Thus X is a regular cover of the Heawood graph, and then because the latter is a Cayley graph, it follows that also X is a Cayley graph. Hence we have infinitely many Cayley graphs with this action type, and no non-Cayley graphs.

A similar example: Action type $(1, 4^1, 4^2, 5^1)$

In this case the pre-image of a 1-arc-regular subgroup of Aut(X) has index 16 in the universal group G_5, and the permutation group induced on the eight cosets is isomorphic to the automorphism group of the Biggs-Conway graph of order 2352. Hence in particular, X is a cover of the Biggs-Conway graph. Then because the latter is a Cayley graph, also X is a Cayley graph. So for this action type we have infinitely many Cayley graphs and no non-Cayley graphs.

Interesting cases: Suppose X is 3-arc-regular

In the group G_3 there are five conjugacy classes of subgroups of index 12 that are complementary to the preimage of a vertex-stabiliser in $A = \mathrm{Aut}\,X$, viz. the classes with representatives $\langle\, a, hah \,\rangle$, $\langle\, a, hah^{-1}, h^{-1}ah \,\rangle$, $\langle\, a, hapqh \,\rangle$, $\langle\, a, hah^{-1}, h^{-1}apqh \,\rangle$ and $\langle\, a, haph \,\rangle$.

In these cases, the permutations induced by the generators of G_3 on the cosets of the index 12 representative subgroup generate a group Q of order 72, 72, 1152, 1152 or 1320, isomorphic to the wreath product $S_3 \wr C_2$ in the first two cases, to $S_4 \wr C_2$ in the third and fourth cases, and to $\mathrm{PGL}(2,11)$ in the fifth case. Hence the order of any 3-arc-regular finite cubic Cayley graph is divisible by $72/12 = 6$ (or $1152/12 = 96$) or $1320/12 = 110$.

Also by considering the pre-images of subgroups of A that could act regularly on the 1- or 2-arcs of a 3-arc-regular Cayley graph X, we find that

- if A has an index 4 subgroup acting on X with type 1, then $|V(X)|$ is divisible by 6 or by $5280/12 = 440$, while
- if A does not have an index 4 subgroup acting on X with type 1, then $|V(X)|$ is divisible by 110, and
- if A has an index 2 subgroup acting with type 2^1 or type 2^2, then it also has one that acts with the other of those two types, and $|V(X)|$ is divisible by $2640/12 = 220$.

In particular, the action type of X cannot be $(2^1, 3)$ or $(2^2, 3)$, and if the action type is $(2^1, 2^2, 3)$, then the order of X is divisible by 220.

Moreover, if X is any 3-arc-regular cubic Cayley graph, then

- X is a cover of F006 or F096B or F440C if its action type is $(1, 2^1, 2^2, 3)$,
- X is a cover of F220C if its action type is $(2^1, 2^2, 3)$, while
- X is a cover of F110 if its action type is (3).

Each of these 'minimal' examples has infinitely many elementary abelian p-covers that are Cayley graphs with the same action type.

Also there are infinitely many non-Cayley graphs with a given one of the five possible action types: F040, the Desargues graph, the Petersen graph, the Coxeter graph and F182D are non-Cayley graphs with action types $(1, 2^1, 2^2, 3)$, $(2^1, 2^2, 3)$, $(2^1, 3)$, $(2^3, 3)$ and (3), and then the Macbeath trick applies.

Final remarks (on the remaining 8 cases)

Action type $(1, 2^1)$ is similar to action type (1), with no obvious restrictions on $n = |V(X)|$ for X to be Cayley.

Action type (2^1) is impossible, because in the permutation representation of G_2^1 on cosets of the pre-image of a potential Cayley vertex-regular subgroup of Aut(X), there is always a subgroup of index 2 that will act regularly on arcs.

Action type (4^1) involves four classes of subgroups of index 24 in G_4^1 that do not give X as a cover of the Heawood graph, and these give X as a cover of two minimal examples of orders 506 and 23! when X is Cayley. [NB: 506 | 23!]

The cases where X is 5-arc-regular are similar to those where X is 3- or 4-arc-regular, but involve larger examples (with the Biggs-Conway graph of order 2352 playing a major role).

Thank You!

Although not quite finished ...

An advertisement: 42nd Australasian Conference on Combinatorial Mathematics and Combinatorial Computing in early December 2018, hopefully at this nice place in New Zealand:

Answers to exercises

(1) *Find an example of an arc-transitive finite Cayley graph X for which there is no group G, subset $S \subset G$ and group $H \leq$ Aut(G) acting transitively on S, such that X is isomorphic to the Cayley graph* Cay(G, S).

Easy: The complete graph K_6. This is arc-transitive, and is a Cayley graph for each of the two groups of order 6 (namely C_6 and D_3), but neither Aut(C_6) nor Aut(D_3) has a subgroup acting transitively on the 5 non-trivial elements.

(2) *Is there a 3-valent example?*

Harder, but yes. The 3-arc-regular graph F110 and the 1-arc-regular graph F144A (mentioned earlier) are Cayley graphs, but neither is isomorphic to Cay(G, S) for a group G and subset S of G that is an orbit of some subgroup of Aut(G).

Title: ‘Symmetric Cubic Graphs as Cayley Graphs’

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Abstract:

A graph X is *symmetric* if its automorphism group acts transitively on the arcs of X, and s*-arc-transitive* if its automorphism group acts transitively on the set of s-arcs of X. Furthermore, if the latter action is sharply-transitive on s-arcs, then X is s*-arc-regular.*

It was shown by Tutte (1947, 1959) that every finite symmetric cubic graph is s-arc-regular for some $s \leq 5$. Djoković

and Miller (1980) took this further by showing that there are seven types of arc-transitive group action on finite cubic graphs, characterised by the stabilisers of a vertex and an edge. The latter classification was refined by Conder and Nedela (2009), in terms of what types of arc-transitive subgroup can occur in the automorphism group of X.

In this talk we consider the question of when a finite symmetric cubic graph can be a Cayley graph. We show that in five of the 17 Conder-Nedela classes, there is no Cayley graph, while in two others, every graph is a Cayley graph. In eight of the remaining ten classes, we give necessary conditions on the order of the graph for it to be Cayley; there is no such condition in the other two. Also we use covers (and the ‘Macbeath trick’) to show that in each of those last ten classes, there are infinitely many Cayley graphs, and infinitely many non-Cayley graphs.

This research grew out of some recent discussions with Klavdija Kutnar and Dragan Marušič.