Gravitational assist in celestial mechanics—a tutorial

James A. Van Allen
Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa 52242-1479

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In planning certain types of trajectories of spacecraft within the solar system, engineers rely on a technique called gravitational assist, or gravity assist. This technique underlies the feasibility of effecting a net change in both the speed and direction of motion of a spacecraft by passage through the gravitational field of a planet or a planetary satellite. The resulting increase, or decrease, in the kinetic energy of the spacecraft appears to contradict the casual expectation that in such an encounter the kinetic energy of the spacecraft after the encounter would be the same as that before the encounter. This paper describes the December 1973 encounter of the Pioneer 10 spacecraft with the planet Jupiter as a real-life example of gravitational assist. It then discusses the physical principles involved in understanding the dynamics of the encounter and concludes with remarks on the important role of gravitational assist in space exploration with artificial spacecraft and in understanding the motion of comets within the solar system.

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I. THE PARADOX OF GRAVITATIONAL ASSIST

A paraphrased NASA announcement of late 1973: ‘‘Following its launch and escape from the Earth's gravitational field in early March 1972, the Pioneer 10 spacecraft has been on a Keplerian elliptical orbit about the Sun with an aphelion distance of abut 6 astronomical units ($1\ \mathrm{AU}=1.50\times 10^{8}$ km, the mean distance of the Earth from the Sun). During the planned 4 December 1973 close encounter of the spacecraft with the planet Jupiter at 5.05 AU, the spacecraft's speed will be increased so that its subsequent path will be a hyperbolic escape orbit from the solar system. This effect of the encounter with Jupiter is called a gravitational assist.’’

A physics student is entitled to be puzzled by such an announcement and may well express this puzzlement as follows, ‘‘I have learned that the total energy, kinetic plus potential, of a test particle is constant as it is moves through the gravitational field of a far more massive body. The speed of the particle will be increased during the encounter and the direction of its velocity vector will be changed; but as the particle recedes from the encounter, its speed will gradually decrease to the same value as that during approach. This expectation is in clear conflict with the NASA announcement. What am I missing?’’

The purpose of this paper is to address this paradox from a physicist's point of view.

II. THE ENCOUNTER OF PIONEER 10 WITH JUPITER

An important example of gravitational assist will now be described using actual data for the December 1973 encounter of the Pioneer 10 spacecraft with the plant Jupiter.[1] The description of the kinetics of this encounter has been derived from the detailed ephemerides prepared by NASA's Jet Propulsion Laboratory and Ames Research Center. Minor approximations have been made in the interest of clarity but essential validity has been preserved. One such minor approximation is treating the encounter as having occurred in a plane parallel to that of the Earth's orbit about the Sun (the ecliptic plane), i.e., in two dimensions.

Two coordinate systems, or frames of reference, will be employed. The respective axes of the two systems are parallel to each other and the systems have a fixed orientation with respect to distant stars. The primary axis of each is parallel to the vernal equinox (γ), the ascending node of the ecliptic on the Earth's equator. The planetocentric system has its origin at the center of Jupiter and the heliocentric system, at the center of the Sun. Neither is a truly inertial coordinate system but both are adequately accurate approximations thereto for the present purpose.

The following vector symbols are adopted:

$\mathbf{v}_0$ = pre-enounter velocity of the spacecraft in the planetocentric system;

$\mathbf{v}_1$ = post-encounter velocity of the spacecraft in the planetocentric system;

$\mathbf{v}_0'$ = pre-encounter velocity of the spacecraft in the heliocentric system;

$\mathbf{v}_1'$ = post-encounter velocity of the spacecraft in the heliocentric system;

$\mathbf{W}'$ = velocity of Jupiter in the heliocentric system, nearly constant throughout the encounter.

As Pioneer 10 approached Jupiter but was at such a distance from it (about 33 Jovian radii) that the gravitational force of the Sun was less than 1% of that of the planet, its velocity vector $\mathbf{v}_0'$ in the heliocentric coordinate system was of magnitude $9.8\ \mathrm{km\,s^{-1}}$ at an angle of 49° counterclockwise (as viewed from the north ecliptic pole) of the line from the Sun to the planet. During the encounter, the heliocentric velocity of Jupiter $\mathbf{W}'$ was eastward perpendicular to the Sun-planet line with a magnitude of $13.5\ \mathrm{km\,s^{-1}}$.

The pre-encounter velocity $\mathbf{v}_0$ of the spacecraft in the planetocentric coordinate system was given by the vector equation

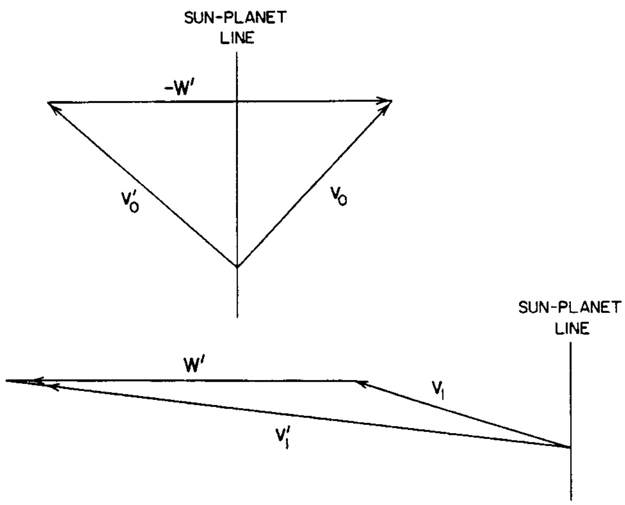

Fig. 1. The upper vector diagram represents Eq. (1) of the text. It shows the transformation of the heliocentric velocity $\mathbf{v}_0'$ of the spacecraft before encounter to its planetocentric velocity $\mathbf{v}_0$. $\mathbf{W}'$ is the heliocentric velocity of Jupiter. The lower vector diagram represents Eq. (2) in the text. It shows the transformation of the planetocentric velocity of the spacecraft $\mathbf{v}_1$ after encounter to its heliocentric velocity $\mathbf{v}_1'$.

$$\mathbf{v}_0 = \mathbf{v}_0' - \mathbf{W}'. \tag{1}$$

The vector diagram corresponding to Eq. (1) is shown in the upper panel of Fig. 1. The vector $\mathbf{v}_0$ had a magnitude of 8.9 km s^{-1} and an angle of 43° clockwise of the Sun-planet line. At the spacecraft's closest approach to the planet it was at a radial distance of 2.84 planetary radii or 2.027×10^5 km and its speed in the planetocentric coordinate system was 37 km s^{-1}. In the planetocentric system the path of the spacecraft was a hyperbola with constant total energy and with the focus of the hyperbola at the center of the planet.[2,3] As the spacecraft receded beyond the planet's gravitational influence, its velocity vector $\mathbf{v}_1$ had been rotated counterclockwise form $\mathbf{v}_0$ by 116° and its magnitude had returned to its pre-encounter value of 8.9 km s^{-1}. A portion of the hyperbolic encounter trajectory is shown in Fig. 2.

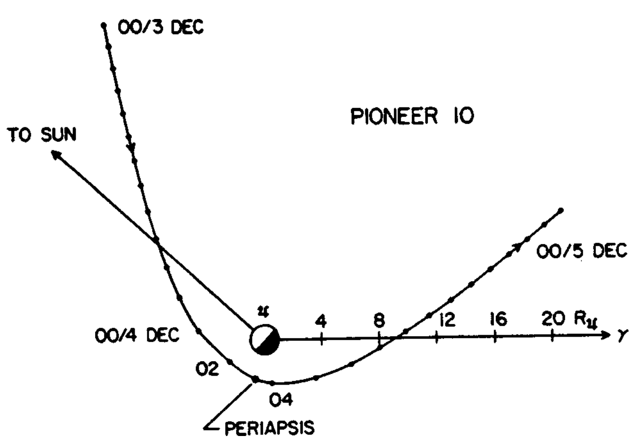

Fig. 2. The ecliptic plane projection of the December 1973 hyperbolic encounter trajectory of Pioneer 10 with Jupiter as viewed in the planetocentric coordinate system. The small dots along the trajectory are at 2 hour intervals and the large dot at the moment of closest approach is labeled periapsis. The symbol γ represents Earth's vernal equinox.

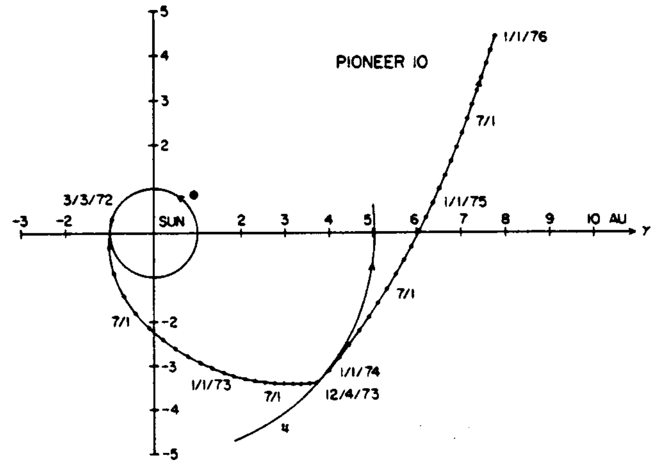

Fig. 3. The projection on the ecliptic plane of the heliocentric path of Pioneer 10 before and after its close encounter with Jupiter on 4 December 1973. The dots along the trajectory are at one-month intervals. Also shown are the orbit of the Earth (⊗) and a portion of the orbit of Jupiter (♃).

In the heliocentric coordinate system, the post-encounter velocity $\mathbf{v}_1'$ of the spacecraft was given by

$$\mathbf{v}_1' = \mathbf{v}_1 + \mathbf{W}', \tag{2}$$

as shown in the lower panel of Fig. 1. The magnitude of $\mathbf{v}_1'$ was 22.4 km s^{-1} and it was directed at 83° counterclockwise of the Sun-planet line.

Hence, by virtue of gravitational assist the heliocentric speed of the spacecraft had increased from 9.8 km s^{-1} to 22.4 km s^{-1} and its kinetic energy had increased by a factor of 5.2.

At 5.05 AU the necessary heliocentric speed for escape from the Sun's gravitational field is 18.7 km s^{-1}. Therefore, the overall effect of the encounter was to transform the (captive) elliptical orbit of Pioneer 10 into a hyperbolic escape orbit from the solar system (Fig. 3).

The enormous efficacy of gravitational assist is emphasized by estimating the necessary magnitude of a multistage combination of chemical rockets for providing the same increase in speed and kinetic energy and change of direction of the spacecraft as did its encounter with Jupiter. Such a combination comparable to the huge Atlas/Centaur/Upper Stage vehicle that launched the 260 kg spacecraft from Cape Canaveral would have been needed to provide the same effect if applied to the spacecraft in free space.

If Pioneer 10 had passed ahead of Jupiter rather than behind it, the heliocentric kinetic energy of the spacecraft would have been decreased rather than increased. Indeed, it is possible to decrease the heliocentric velocity of a spacecraft to zero so that it falls radially inward toward the Sun if an initial speed of about 41 km s^{-1} can be achieved at 1 AU. The analysis of energy-decreasing cases employs the same type of considerations as presented above.

III. THE PHYSICS OF GRAVITATIONAL ASSIST

The foregoing description of the encounter of Pioneer 10 with Jupiter is a faithful account from the engineering point of view. However, it tends to support rather than dispel the paradox of Sec. I.

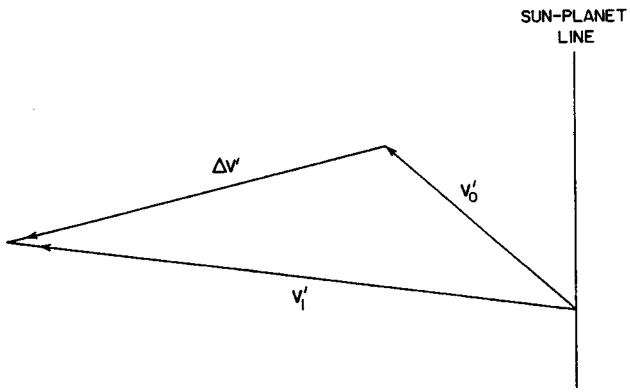

Fig. 4. This vector diagram represents Eq. (3) of the text $\mathbf{v}_1'=\mathbf{v}_0'+\Delta\mathbf{v}'$ wherein $\mathbf{v}_0'$ and $\mathbf{v}_1'$ are the heliocentric velocities of the spacecraft before and after the encounter, respectively, and $\Delta\mathbf{v}'$ is the net change of velocity.

The essence of the paradox is ignoring the effect of the encounter on the velocity of the rapidly moving planet. This effect is essential in principle, though miniscule in relative magnitude.

In the language of Newtonian physics, the sum of the linear momenta of the spacecraft and the planet after the encounter was the same as the sum of their momenta before the encounter. Also the encounter was perfectly elastic so that the sum of the kinetic energies of the two bodies after the encounter was the same as the sum of their kinetic energies before the encounter.

The following discussion employs several symbols in addition to those defined in Sec. II:

$\Delta\mathbf{v}'=$ change of heliocentric velocity of the spacecraft as a result of the encounter;

$\Delta\mathbf{W}'=$ change of heliocentric velocity of the planet as a result of the encounter;

$\Delta E_s'=$ change in kinetic energy of the spacecraft;

$\Delta E_p'=$ change in kinetic energy of the planet;

$\theta'=$ angle between $\mathbf{v}_0'$ and $\Delta\mathbf{v}'$, 55°;

$\Theta'=$ angle between $\mathbf{W}'$ and $\Delta\mathbf{W}'$, 166°;

$m=$ mass of the spacecraft, 260 kg;

$M=$ mass of Jupiter, 1.90×10^{27} kg.

The overall ballistic effect on Pioneer 10 by the encounter was represented by the following equation:

$$\mathbf{v}_1'=\mathbf{v}_0'+\Delta\mathbf{v}'. \tag{3}$$

This vector equation is shown graphically in Fig. 4. The magnitude of $\Delta\mathbf{v}'$ was $15.3\ \mathrm{km\,s^{-1}}$.

By Newton's second and third laws of motion the impulse applied to the spacecraft by the planet's gravitational force caused a change in its linear momentum $m\Delta\mathbf{v}'$ and the impulse applied to the planet by the spacecraft's gravitational force caused a change in its linear momentum $M\Delta\mathbf{W}'$. The two changes were of equal magnitude and opposite in sense, i.e.,

$$m\Delta\mathbf{v}'=-M\Delta\mathbf{W}' \tag{4}$$

and the ratio of the magnitudes of the two vectors

$$\frac{\Delta W'}{\Delta v'}=\frac{m}{M}=1.4\times 10^{-25}. \tag{5}$$

Hence, using the observed value of $\Delta\mathbf{v}'$,

$$\Delta W'=-2.1\times 10^{-24}\ \ \mathrm{km\,s^{-1}}. \tag{6}$$

Figure 4 shows that $\Delta\mathbf{W}'$ was in such a direction (opposite to that of $\Delta\mathbf{v}'$) as to reduce the orbital speed of the planet and also change its direction of motion. It is clear from (6) that both effects were exceedingly small.

The nature of the encounter assures that the sum of the kinetic energies of the spacecraft and the planet after the encounter was the same as the sum of their kinetic energies before the encounter; i.e., the observed gain in kinetic energy of the spacecraft must be equal to the loss of kinetic energy of the planet. Nevertheless, it is instructive to verify this by a calculation using actual data. The gain in kinetic energy of the spacecraft is

$$\Delta E_S'=\frac{m}{2}[(\mathbf{v}_1'\cdot\mathbf{v}_1')-(\mathbf{v}_0'\cdot\mathbf{v}_0')] \tag{7}$$

and using Eq. (3),

$$\Delta E_S'=m[\mathbf{v}_0'\cdot\Delta\mathbf{v}'+(\Delta v')^2/2]. \tag{8}$$

In Eq. (8) the two terms within the brackets are of comparable magnitude and

$$\mathbf{v}_0'\cdot\Delta\mathbf{v}'=v_0'\Delta v'\cos\theta', \tag{9}$$

where θ' is the angle between the vectors $\mathbf{v}_0'$ and $\Delta\mathbf{v}'$ (Fig. 4). Using the actual values of v_0', $\Delta v'$, and θ'

$$\Delta E_s'=203\ \ m\ \ \mathrm{kg\,km^2\,s^{-2}}. \tag{10}$$

Similarly, for the planet

$$\Delta E_p'=M[\mathbf{W}'\cdot\Delta\mathbf{W}'+(\Delta W')^2/2]. \tag{11}$$

In this case, the second term within the brackets is negligible in magnitude relative to the first. Then using Eq. (4),

$$\Delta E_p'=m[W'\Delta v'\cos\Theta']. \tag{12}$$

Using actual values of W', $\Delta v'$, and Θ',

$$\Delta E_p'=-200\ \ m\ \ \mathrm{kg\,km^2\,s^{-2}}, \tag{13}$$

which is equal to the negative of $\Delta E_s'$ within rounding errors in the data.

If W' had been zero in Eq. (12), there would have been, to high accuracy, no loss of kinetic energy of the planet and hence no gain of kinetic energy by the spacecraft, as was indeed the case in the planetocentric frame of reference. *Thus, it is clear that the observed gain in heliocentric kinetic energy of Pioneer 10 in its close encounter with Jupiter was entirely dependent on the fact that the planet itself was in motion*.

Recognition of this fact dispels the paradox of Sec. I and answers the student's question, "What am I missing?" Further explanation is provided in the Appendix.

IV. REMARKS

It is, of course, possible to determine the heliocentric trajectories of both the spacecraft and the planet by stepwise numerical calculation without any explicit mention of gravitational assist. But such an approach abandons the conceptual clarity of using the well-understood hyperbolic trajectory of the spacecraft in the planetocentric frame of reference and hinders the recognition of the basic principles of physics involved in the encounter.

Gravitational assist is an example of the restricted three-body problem of celestial mechanics, long known to astronomers following the seminal work of Lagrante (circa 1772).[4,5] In this problem, as in the example in the present paper, there are two massive bodies (the Sun and Jupiter) and a third body (Pioneer 10) of far lesser mass, each moving in the gravitational field of the others. The third body is referred to as of infinitesimal mass, subject to gravitational forces by the other two bodies but exerting no force on them, a satisfactory approximation though defective in principle. The motion of comets within the solar system has been a classical subject of the restricted three-body problem. Specific solutions show how a heliocentric parabolic orbit can be transformed into a hyperbolic one or into an elliptical one during encounter with a planet. The essential parameters in each case are the ''impact parameter'' (the perpendicular distance from the center of the planet to the planetocentric asymptotic approach vector of the comet); the gravitational escape speed of an object from the surface of the planet, namely $\sqrt{2GM/r}$, where G is Newton's gravitational constant, M is the mass of the planet and r is its radius; the heliocentric orbital speed of the planet; and the planetocentric approach speed of the comet. Because Jupiter's mass M is much greater than that of any other planet, it has the most prominent role in perturbing the orbits of comets.

Another astronomical example of the three-body problem arises in considering the potential capture of a passing asteroid into an elliptical orbit about the Earth (or another planet). In the two-body problem, involving only the Earth and the asteroid, capture is impossible. The asteroid either strikes the Earth or it flies by in an Earth-centered hyperbola. But if a second massive body, the Moon, is added to the problem, the capture of the asteroid becomes possible under very specialized circumstances.

The gravitational assist technique has been of central importance in many space missions during the past 30 years: Pioneer 10's encounter with Jupiter; Pioneer 11's encounters with Jupiter and Saturn; Voyager 1's encounters with Jupiter and Saturn; Voyager 2's encounters with Jupiter, Saturn, Uranus, and Neptune; Mariner 10's encounters with Venus and Mercury, Cassini's encounters with Venus and the Earth; Ulysses' encounter with Jupiter to achieve an out-of-ecliptic orbit; and the successive encounters of the Galileo spacecraft with satellites of Jupiter as it has been in orbit about that planet.[6,7] The most straightforward planning of future space missions inward toward the Sun and outward to Pluto depends on gravitational assist by Jupiter fly-bys.

ACKNOWLEDGMENTS

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APPENDIX

Here is a succinct way to see why a small spacecraft can extract substantial energy from a massive, moving planet.

Let the momentum imparted by Jupiter to the spacecraft be $\Delta\mathbf{p}'$. After the spacecraft has left the planet's vicinity, Jupiter's momentum is $M\,\mathbf{W}' - \Delta\mathbf{p}'$, and so its velocity is $\mathbf{W}' - \Delta\mathbf{p}'/M$. Thus Jupiter's kinetic energy has become

$$\tfrac{1}{2}M(\mathbf{W}' - \Delta\mathbf{p}'/M)\cdot(\mathbf{W}' - \Delta\mathbf{p}'/M)$$
$$= \frac{1}{2}MW'^2 - \mathbf{W}'\cdot\Delta\mathbf{p}' + \frac{(\Delta p')^2}{2M}.$$

To first approximation, Jupiter has lost an amount of energy equal to $\mathbf{W}'\cdot\Delta\mathbf{p}'$, *independent of how massive the planet is*. Thus even a tiny spacecraft can extract a finite amount of energy from an overwhelmingly massive planet—provided the planet is moving initially. Here ''finite'' is contrasted with ''infinitesimal.''

Why is the initial motion of the planet so crucial? To extract energy, the spacecraft must exert a force on the planet and do a negative amount of work on it. Thus the force must act over a distance. If the force must set the massive planet into motion, then the distance traveled in a fixed time interval will be inversely proportional to the planet's mass M and hence will be insignificantly small. If, however, the planet is already moving, then the distance traveled will be independent of M (to good approximation) and hence the work done may be significant, as we have seen.

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