Aspects of the use of orthogonal basis functions in the element-free Galerkin method

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SUMMARY

The element-free Galerkin (EFG) method is probably the most widely used meshless method at present. In the EFG method, shape functions are derived from a moving least-squares approximation using a polynomial basis, a calculation involving the inversion of a small matrix. A new implementation of the EFG method was published soon after the original where an alternative approach using an orthogonal basis was proposed to avoid matrix inversion in the formulation of the shape functions. In this paper we revisit this topic and show that the difficulties associated with the use of a polynomial basis remain present in the orthogonal case. We also show that certain terms in the derivative expressions are omitted in the new implementation of the EFG, which can lead to errors. Finally, we propose a new approach that avoids inversion while maintaining accuracy. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Meshless (or meshfree) methods continue to be of interest to researchers in computational solid mechanics since they offer the possibility of high accuracy without the preprocessing overhead of mesh generation. A wide variety of methods have been proposed as outlined in recent surveys [1–3]. The most widely cited papers in meshless methods are devoted to the element-free Galerkin (EFG) method by Belytschko [4] and co-workers, although many other methods have been proposed [5–7]. As with many other meshless methods the EFG method uses a moving least-squares (MLS) approximation for the field variable, e.g. displacement in solid mechanics, to derive shape functions.

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\( \phi_I \) for each node \( I \) supporting the approximation at a given point \( \mathbf{x} \). The field variable \( u^h \) is then approximated as

\[
u^h(\mathbf{x}) = \sum_{I}^{n} \phi_I(\mathbf{x}) u_I \tag{1}\]

where \( u_I \) are the ‘fictitious’ nodal values of the field variable, the adjective referring to the fact that since a MLS scheme is used \( u_I \) will rarely ever equal \( u^h \) at a node, and \( n \) is the number of nodes in support at \( \mathbf{x} \). Once shape functions have been determined, the procedure to turn a governing pde into a system of discrete algebraic equations, via a weighted residual approach for instance, follows much the same path as with finite element (FE) methods, although there are serious issues with imposition of essential boundary conditions that do not appear in the FE method \([8–10]\). In the original EFG method \([4]\), the MLS approximation uses a polynomial basis, which then requires a matrix inversion to be carried out to find the shape functions. A paper published in the same year \([11]\) highlighted the potential problems that could arise with the matrix inversion in certain cases and proposed a ‘new implementation’ of the EFG method where the need for matrix inversion was removed via the use of an orthogonal basis. This paper revisits some aspects of the ‘new implementation’ of \([11]\).

The paper is structured as follows. First, the MLS approximation and the expressions to derive shape functions using a polynomial basis are presented. Next, the alternative use of an orthogonal basis is outlined and the removal of the need for matrix inversion is shown not to remove the degradation in solution accuracy, which arises from the spatial distribution of nodes and not from the basis used. In Section 4 missing terms in the shape function derivatives are shown to lead to errors. Finally, we propose alterations to the new implementation of the EFG method in Section 5 that remedy both of these issues satisfactorily.

2. DERIVATION OF SHAPE FUNCTIONS USING THE MLS APPROXIMATION

Given a set of \( n \) data pairs \( \mathbf{U} = \{u_I, \mathbf{x}_I\}, I = 1, 2, \ldots, n \) to interpolate an unknown field value \( u(\mathbf{x}) \), the MLS approximation can be constructed as

\[
u^h(\mathbf{x}) = \sum_{I}^{n} \phi_I(\mathbf{x}) u_I = \Phi(\mathbf{x}) \mathbf{u} \tag{2}\]

where \( u^h(\mathbf{x}) \) denotes the approximate value of \( u(\mathbf{x}) \), \( n \) is the number of nodes in support at \( \mathbf{x} \) and \( \phi_I(\mathbf{x}) \) is the shape function of node \( I \) at \( \mathbf{x} \). \( \Phi(\mathbf{x}) \) is a \( 1 \times n \) matrix collecting together the shape functions \( \phi_I \) and \( \mathbf{u} \) is a vector containing the fictitious nodal values. As in the FE method if \( u(\mathbf{x}) \) is approximated as a polynomial then

\[
u^h(\mathbf{x}) = \sum_{j}^{m} p_j(\mathbf{x}) a_j(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \mathbf{a}(\mathbf{x}) \tag{3}\]

where \( m \) is the number of monomials in the basis matrix \( \mathbf{p}(\mathbf{x}) \), e.g. \( m = 3 \) for a linear basis in 2D or a quadratic basis in 1D, and \( \mathbf{a}(\mathbf{x}) \) is a vector of coefficients. \( \mathbf{p}^T(\mathbf{x}) = [p_1(\mathbf{x}), \ldots, p_m(\mathbf{x})] \) is built using Pascal’s triangle in 2D and Pascal’s pyramid in 3D, and for convenience we call it the

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Pascal basis here. In the MLS approximation, the shape functions are obtained by minimizing a weighted residual $J$ to determine the coefficients $a(x)$ where

$$J(x) = \sum_{I}^{n} w(x)|p^T(x_I)a(x) - u_I|^2$$

(4)

where $w_I(x) \equiv w(x-x_I)$ is the weight function for point $x$. Equation (4) leads to the following

$$A(x)a(x) = B(x)u$$

(5)

where the elements of matrix $A(x)_{m \times m}$ are given by

$$A_{jk} = \sum_{I}^{n} w_I(x)p_j(x_I)p_k(x_I) \quad j, k = 1, \ldots, m$$

(6)

and the elements of matrix $B(x)_{m \times n}$ by

$$B_{jl} = w_I(x)p_j(x_I) \quad j = 1, \ldots, m, \quad I = 1, \ldots, n$$

(7)

The coefficients $a(x)$ can be found from (5) by inverting $A(x)$

$$a(x) = A(x)^{-1}B(x)u$$

so (3) becomes

$$u^h(x) = p(x)^T A(x)^{-1} B(x)u$$

(8)

and the shape functions are found by comparison with (2) as

$$\Phi = p^T A^{-1} B$$

(9)

where the dependence on $x$ for all terms has been removed for clarity. The derivatives of the shape functions can be found as

$$\Phi_{,k} = p_{,k}^T A^{-1} B + p^T (A_{,k}^{-1} B + A^{-1} B_{,k})$$

(10)

where $k$ denotes the coordinate index and

$$A_{,k}^{-1} = -A^{-1} A_{,k} A^{-1}$$

(11)

$A$ and $B$ can be written in matrix form as

$$A = PWP^T$$

(12a)

$$B = PW$$

(12b)

where $P$ is an $m \times n$ matrix defined by

$$P = [p(x_1), \ldots, p(x_n)]$$

(13)

and $W$ is an $n \times n$ diagonal matrix

$$W = [\text{diag}(w_1(x), \ldots, w_n(x))]_{n \times n}$$

(14)
3. USE OF ORTHOGONALIZATION

For certain arrangements of supporting nodes, \( A \) can become ill-conditioned, as will be highlighted later. If \( A \) is ill-conditioned, the solution for \( a(x) \) will be prone to error, and hence also the shape functions \( \Phi \). Lu et al. [11] propose overcoming this problem by using an orthogonal basis \( \tilde{p} \) that leads to a diagonal \( A \) and hence a trivial inversion process. While apparently a reasonable solution to this problem (a claim also repeated in later references, e.g. [12]) ill-conditioning is not removed using the orthogonal basis, as we will show below. The Schmidt orthogonalization procedure to obtain \( \tilde{p} \) is described in [11] and is not repeated here.

To make comparisons \( \tilde{A}(x) \) and \( \tilde{a}(x) \) denote the equivalent expressions constructed from the orthogonal basis \( \tilde{p}(x) \). If the basis used is orthogonal then using (6)

\[
\tilde{A}_{jk} = \sum_{l} w_{l}(x) \tilde{p}_{j}(x_{l}) \tilde{p}_{k}(x_{l}) = 0, \quad j \neq k
\]  

From (15) \( \tilde{A}(x) \) now becomes a diagonal matrix

\[
\tilde{A}(x) = \begin{bmatrix}
\sum_{l} w_{l}(x) \tilde{p}_{1}^{2}(x_{l}) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sum_{l} w_{l}(x) \tilde{p}_{m}^{2}(x_{l})
\end{bmatrix}
\]  

and the elements of \( \tilde{B} \) are

\[
\tilde{B}_{jI}(x) = w_{I}(x) \tilde{p}_{j}(x_{I})
\]  

Therefore, (5) can be solved without matrix inversion by

\[
\tilde{a}_{j}(x) = \frac{\sum_{I} w_{I} \tilde{B}_{jI} u_{I}}{\tilde{A}_{jj}}
\]  

and the shape functions \( \tilde{\phi}(x) \) are given as

\[
\tilde{\phi}_{I}(x) = \frac{\sum_{j} \tilde{B}_{jI} \tilde{p}_{j}(x)}{\tilde{A}_{jj}} = w(x_{I}) \sum_{j} \tilde{p}_{j}(x_{I}) \tilde{p}_{j}(x) \frac{\tilde{p}_{j}(x)}{\tilde{A}_{jj}}
\]  

The aim of the orthogonalization process is to remove inaccuracies in the MLS approximation by avoiding the inversion of an ill-conditioned matrix. Here, we show that this process does not result in better accuracy as expected as the source of inaccuracy (causing the ill-conditioning) is not removed by this procedure. As both the orthogonal basis \( \tilde{p}(x) \) and Pascal basis \( p(x) \) are complete bases and span the entire finite spaces (2D or 3D), a linear relation exists between them, i.e. each \( \tilde{p}_{j}(x) \) can be written as a linear combination of \( p_{i}(x) \) as

\[
\tilde{p}_{j}(x) = \sum_{l} t_{jl} p_{i}(x), \quad j = 1, \ldots, m
\]
and in matrix form as

\[ \tilde{p}(x) = Tp(x) \]  

(21)

where \( T \) is a transformation matrix of coefficients \( t_{ji} \). The completeness condition requires that \( \det(T) \neq 0 \) and the Schmidt orthogonalization makes

\[ t_{ji} = \begin{cases} 0, & i > j \\ 1, & i = j \end{cases} \]  

(22)

Thus, \( T \) is a lower triangular matrix mapping \( p(x) \) to \( \tilde{p}(x) \) with all diagonal entries of unit value

\[
T = \begin{bmatrix}
1 & 0 \\
t_{21} & 1 \\
\vdots & \vdots & \ddots \\
t_{m1} & t_{m2} & \cdots & 1 \\
\end{bmatrix}_{m \times m}
\]  

(23)

Maintaining the convention that quantities derived from the orthogonal basis are written as \( \tilde{A}(x) \), from (13) and (21) we have

\[ \tilde{P} = TP \]  

(24)

By substituting (24) into

\[ \tilde{A}(x) = \tilde{P}(x)W(x)\tilde{P}^T \]  

(25)

we have

\[ \tilde{A}(x) = TA(x)T^T \]  

(26)

When \( A \) is ill-conditioned its determinant is close to zero so

\[ |\det(A)| = |\det(PWP^T)| \leq |\det(PIP^T)| \approx 0 \]  

(27)

since the weight function is non-negative and bounded to value \( w_I(x) \leq 1 \). Noting that \( \det(T) = 1 \) from (23), we have

\[ \det(\tilde{A}) = \det(A) \approx 0 \]  

(28)

so the use of the orthogonal basis leads to a similarly ill-conditioned system. \( \det(\tilde{A}) = 0 \) implies one or more of \( \tilde{A}_{ij} \) is close to zero and hence a small perturbation in \( B(x) \) will be amplified into a large error in \( a(x) \) and the shape function \( \Phi_j(x) \). The inaccuracy associated with an ill-conditioned \( A(x) \) is an inherent round-off error, which can never be ameliorated via the orthogonalization process. Its source is a poor spatial arrangement of nodes in support at \( x \). With a linear basis in 2D, this occurs when three nodes lie on a line. Another way of viewing this is to note that the area enclosed by a simplex formed from the nodes in support must be non-zero. In this case \( \det(P) \equiv 0 \), which leads to (27). The only possible solution is to improve the nodal arrangement to make non-zero areas (2D) or volumes (3D). This has not been highlighted before to our knowledge. It is important to note that this source of ill-conditioning is not the same as mentioned in some other references.
In [13] a technique is used where the origin is shifted to the point of interest when calculating the basis matrix. This avoids accuracy problems due to the size of the arguments in the basis but is not related to the problem cited above, which is due to node topology. Reference [14] mentions the importance of nodal arrangements but only insofar as they affect the integration necessary for the weak form, again not the problem that we highlight here.

Another important feature should be noted that the shape functions derived using either basis are identical. Substituting (24) into
\[ \tilde{B}(x) = \tilde{P}(x)W(x) \]
we can show that
\[ \tilde{B}(x) = TB(x) \]
Then substituting (30) and (25) into
\[ \tilde{\phi}(x) = \tilde{p}(x)^T \tilde{A}(x)^{-1} \tilde{B}(x) \]
we can see that
\[ \tilde{\phi}(x) \equiv \phi(x) \]

3.1. Example: simple degradation cases

An example is now presented to demonstrate the degradation in accuracy due to nodal arrangement. Results are compared using the standard Pascal basis and the bases derived from the orthogonalization process. Figure 1 shows three nodes (labelled 1–3) used to provide an MLS approximation at \( x \), a point located at (1.5, 1.5). First, node 2 is located on the line between nodes 1 and 3, which is the degradation case in 2D mentioned above. Node 2 is then moved off the line to produce several close-to-degradation cases. The support radius is set at 2.0 units for all nodes. This value is not of significance but means that all three nodes are in support at \( x \). Table I shows the results from calculations varying the location of node 2. Results are provided for cases where node 2 is located 0.001, 0.01 and 0.1 units above the line joining nodes 1 and 3 (cases \( b, c, d \)) in addition to the inline case \( \text{(a)} \). Results are also given where node 2 is perturbed by 0.0001 units \( (a_{x}, b_{x}, c_{x}, d_{x}) \) to examine the effect of inaccuracy in nodal coordinates. Estimates of the reciprocal condition number of \( A \) and the minimum values of \( \tilde{A}_{jj} \) are given along with the values of shape functions, which are identical for both approaches as proved above. (Note that the MLS approximation leads to negative shape functions, which nevertheless form a partition of unity.)

![Figure 1. The nodal arrangement for the study of degradation cases.](image-url)
Case $a$ shows that no shape functions can be derived for the degradation case as $A$ is singular or there is a zero on the diagonal of $\tilde{A}$. As node 2 is moved away from the line, ill-conditioning decreases rapidly and the range of the shape functions decreases by two orders of magnitude. The effect of slight perturbations in the coordinates of node 2 is also seen to be of much greater significance when close to the degradation case regardless of the basis used. If we define the relative error in the shape functions caused by the small perturbation in cases $b_e, c_e, d_e$ as

$$e_\Phi = \frac{\|\Phi(x) - \Phi_e(x)\|}{\|\Phi(x)\|}$$

(33)

and plot the variation in error with condition of $A$ and $\tilde{A}_{jj}$ (Figure 2), the power law dependence between condition and error is clear. For practical problems, the significance of this effect could be considerable as errors in coordinate values may be encountered where nodes are fitted to a
X. ZHUANG AND C. AUGARDE

curve or a surface for the discretization of a domain, and in retrieving stress fields based on nodal coordinates.

4. THE EFFECT OF MISSING TERMS IN THE DERIVATIVES

Recalling a general way to obtain shape functions is by solving Equation (9), where $\Phi(x)$ is a result of all the terms in $A(x)$ and $B(x)$, which can be expressed by

$$\phi_I(x) = F(p_j(x), p_i(x_I)p_j(x_I), w_I(x)) \quad i, j = 1, \ldots, m$$

(34)

$F$ can be decomposed into $F_1$ and $F_2$

$$\phi_I(x) = F_1(\tilde{p}_j(x), \tilde{p}_j^2(x_I), w_I(x)) + F_2(\tilde{p}_j(x), \tilde{p}_i(x_I), \tilde{p}_j(x_I), w_I(x)) \quad i \neq j$$

(35)

The use of an orthogonal basis then makes $F_2$ zero as all the off-diagonal terms in $\tilde{A}$ become zero. In this special case, the shape functions $\phi_I(x)$ become

$$\phi_I(x) = F_1(\tilde{p}_j(x), \tilde{p}_j^2(x_I), w_I(x))$$

(36)

However, it does not follow that the matrix of derivatives of $\tilde{A}(x)$ is also diagonal. This appears to have been assumed in [11] since the expression given there (Equation (17)) is

$$\phi_I(x) = w_I(x) \sum_j C_{jI}(x)$$

(37)

$$C_{jI}(x) = \frac{\tilde{p}_j(x, x_I) \tilde{p}_j(x_I, x)}{\tilde{A}_{jj}}$$

(38)

Note that here $\tilde{A}_{jj}$, which is defined by Equation (15) is equivalent to $b_j(x)$ in [11]. Then it is obvious to see that Equation (37) has a zero-value term hidden behind

$$\phi_I(x) = w_I(x) \sum_j C_{jI}(x) + \underbrace{F_2(\tilde{p}_j(x), \tilde{p}_i(x_I), \tilde{p}_j(x_I), w_I(x))}_{\text{zero}} \quad i \neq j$$

(39)

In [11], the last term on the l.h.s is omitted when determining the shape function derivatives. It should be noted that the omission here is not the same as the diffuse derivatives of the diffuse element method (DEM) discussed in [15]. This term vanishes in solving shape functions, but not necessarily in terms of its derivatives, i.e.

$$\phi_{I,k}(x) = \left( w_I(x) \sum_j C_{jI}(x) \right)_{,k} + \underbrace{F_2(\tilde{p}_j(x), \tilde{p}_i(x_I), \tilde{p}_j(x_I), w_I(x))}_{\text{non-zero}}_{,k}$$

(40)

To investigate if this omission is indeed significant a standard patch test was carried out. Figure 3 shows the patch size and nodal distribution. The background integration cells are eight right-angled triangles using a 4-point Gauss integration scheme in each. Displacements producing constant strain are prescribed along the four boundaries. Three formulations using MLS approximations are compared: the original Pascal basis approach including off-diagonal terms (termed method A),
USE OF ORTHOGONAL BASIS IN THE EFG METHOD

the DEM [16] and the new EFG formulation in [11]. The shape function derivatives, in any basis, are given by Equations (10) and (11). In the DEM, the derivatives of shape functions (known as diffuse derivatives) are given by

$$\phi_{,k} = p_{,k}^T A^{-1} B$$

which is the first term in Equation (11). While the derivatives used in [11] ignore the off-diagonal terms in $A(x)$, the derivatives of $A_{,k}^{-1}$ used there can be expressed as

$$A_{,k}^{-1} = -A^{-1} \text{diag}(\text{diag}(A_{,k})) A^{-1}$$

The weight function used is the exponential function from [4]

$$w_I(d_I^2) = \begin{cases} e^{-(d_I/c)^2} - e^{-(d_{nl}/c)^2}, & d_I \leq d_{nl} \\ 0, & d_I > d_{nl} \end{cases}$$

where $d_I$ is the distance between the point of interest and the node, $d_{nl}$ is the nodal support radius and $c = c_I$ is the maximum distance between supporting nodes. Here, the relative error norm of displacement is given by

$$r_u = \frac{u_{\text{num}} - u_{\text{exact}}}{u_{\text{exact}}}$$

Figure 4 plots $r_u$ with varying $d_I/c$. In all the tests essential boundary conditions are imposed directly.

Method A and the DEM pass the patch test for all values of $d_I/c$, while the formulation from [11] shows a loss of accuracy and sensitivity to $d_I/c$. This formulation does not pass the patch test and shows an error norm of 0.2 with node 9 located in the centre (compare with the first row of Table II in [11]). This error is clearly caused by the omission of off-diagonal elements. The DEM formulation performs much better than [11] and it is interesting to see the extra term in the derivatives of $A^{-1}B$ makes the method more unstable than not adding it at all (DEM).
5. AN ALTERNATIVE APPROACH

Here, we propose a new approach that maintains some of the advantages of the method in [11] using orthogonal bases, but avoids the errors associated with omission of off-diagonal terms in the derivatives. First, we focus on the relations between the derivatives of the shape functions using the two types of polynomial basis. The shape function derivatives using the orthogonal basis can be written generally as

$$\tilde{\Phi}_k = \tilde{p}_k^T \tilde{A}^{-1} \tilde{B} + \tilde{p}_k^T (\tilde{A}_k^{-1} \tilde{B} + \tilde{A}^{-1} \tilde{B}_k^T)$$  \hspace{1cm} (45)

where $\tilde{A}_k^{-1}$ is computed by

$$\tilde{A}_k^{-1} = -\tilde{A}^{-1} \tilde{A}_k \tilde{A}^{-1}$$  \hspace{1cm} (46)

From Equations (21), (25), (30), we can write the derivatives of $\tilde{p}(x)$, $\tilde{A}(x)$ and $\tilde{B}(x)$ as

$$\tilde{p}_k = (T \tilde{p}(x))_k$$  \hspace{1cm} (47a)

$$\tilde{A}_k = (\tilde{P}(x) \tilde{W}(x) \tilde{P}(x)^T)_k$$  \hspace{1cm} (47b)

$$\tilde{B}_k = (\tilde{P}(x) \tilde{W}(x))_k$$  \hspace{1cm} (47c)
USE OF ORTHOGONAL BASIS IN THE EFG METHOD

It appears from the above that we need to differentiate \( T \) and \( \tilde{P} \) with respect to \( x \). However this is not necessary. Firstly,

\[
\tilde{P}(x_I, x)_k = (TP)_k = T_k P + TP_k
\]

(48)

Now, it is clear that \( P_k = 0 \) and it can also be shown that the derivatives of \( T \) are zero. From (32), the relation between the derivatives of the shape functions from two bases is

\[
\tilde{\phi}_k = \phi_k
\]

(49)

Substituting (21), (26), (30) and (46) into (45) gives

\[
T_k = 0
\]

(50)

to satisfy (49). Thus, there is no need to obtain derivatives of \( T \) and \( \tilde{P} \) for shape function evaluation.

The derivatives of \( \tilde{p} \), \( \tilde{A} \) and \( \tilde{B} \) can be computed as those from Pascal basis

\[
\tilde{p}_k = T p(x)_k
\]

(51a)

\[
\tilde{A}_k = \tilde{P} W_k \tilde{P}^T
\]

(51b)

\[
\tilde{B}_k = \tilde{P} W_k
\]

(51c)

where \( W_k \) is computed from the nodal data. The above analysis shows that \( T(x, x_I) \) is only needed for finding the orthogonal basis as \( T = \text{const.} \) The shape function derivatives, like the shape functions themselves, can be constructed from the Pascal basis or from an orthogonal basis. Another way to understand this is that the MLS approximation should exactly reproduce any polynomial field and we are free to choose any polynomial basis as long as it is complete. Equation (18) in [11] expresses the derivatives of shape functions as

\[
\phi_{I,k}(x) = w_{I,k} \sum J C_{j I} + w_I \sum J C_{j I,k}(x)
\]

where

\[
C_{j I,k}(x) = [q_{j,k}(x, x)q_j(x_I, x) + q_j(x, x)q_{j,k}(x_I, x) - b_{j,k}(x)C_{j I}(x)]/b_j(x)
\]

\[
b_{j,k}(x) = \sum J [w_{J,k}(x) + q_j^2(x_I, x) + 2w_j(x)q_j(x_I, x)q_{j,k}(x_I, x)]
\]

(52)

The term \( q_{j,k}(x_I, x) \) comes from the nodal coordinates of supporting nodes once a polynomial basis is defined, \( q_{j,k}(x_I, x) \) should be regarded as constant and therefore there is no need to differentiate.

5.1. Implementation and examples

Implementation of this new form of the EFG method takes the following steps:

1. Obtain the orthogonal basis using nodal data \( U = \{u_I, x_I\} \)
2. Compute the shape functions \( \phi_I \)
   (a) Compute \( P(x) \) using the orthogonal basis
Figure 5. The model for the Timoshenko cantilever beam problem.

(b) Compute the $A_{jj}$ using the obtained orthogonal basis
(c) Compute $\phi_I$ according to (19)

3. Compute $\phi_{I,k}$

(a) Compute derivatives of $A$ according to (51b)
(b) Compute derivatives of $A^{-1}$ according to (11) with the $A^{-1}$ as diagonal matrix
(c) Compute derivatives of $B$ according to (51c)
(d) Get the value of $\phi_{I,k}$ using (51).

The key difference from the implementation in [11] is that the shape functions are derived using the orthogonal basis approach while the shape function derivatives are obtained from the Pascal basis. A standard patch test was carried out to compare the accuracy of the numerical results from the derivatives missing the off-diagonal elements with the present form.

Example: Timoshenko’s cantilever problem. Timoshenko’s cantilever beam problem [17] (as shown in Figure 5) is widely used to validate the meshless methods, although there are many cases of misuse particularly for demonstration of adaptive analysis. The essential boundary conditions are more complex than usually assumed. The analytical displacement field $\{u_x, u_y\}$ is cubic and is given as follows:

$$u_x = \frac{Py}{6EI}\left[(6L-3x)x + (2+v)y^2 - \frac{3D^2}{2}(1+v)\right]$$

$$u_y = -\frac{P}{6EI}[3vy^2(L-x)+(3L-x)x^2]$$

The stress field $\{\sigma_x, \sigma_y, \sigma_{xy}\}$ is given by

$$\sigma_{xx} = \frac{P(L-x)y}{l}$$

$$\sigma_{yy} = 0$$

$$\sigma_{xy} = -\frac{P}{2l}\left(\frac{D^2}{4} - y^2\right)$$

The performance of the proposed new EFG formulation is compared with the results from DEM. The key difference between these two methods is the extra terms in the shape function derivatives for the former. To study the performance of the two methods, the polynomial basis is varied between linear and quadric, and coarse and fine background integration meshes are used (see Figure 6).
USE OF ORTHOGONAL BASIS IN THE EFG METHOD

![Nodal arrangement and background integration cells for the cantilever beam problem: (a) nodal distribution; (b) coarse background mesh; and (c) fine background mesh.](image)

In all cases the number of nodes is kept constant and elastic material parameters $E = 1 \times 10^5$ and $v = 0.25$ are used.

Deflections along the centreline of the cantilever (normalized with respect to its length) are plotted in Figures 7(a)–(d). The relative error is defined as

$$r_u = \frac{u_{\text{num}} - u_{\text{exact}}}{u_{\text{exact}}}$$

(56)

Errors using this measure of the deflections along the $x$ axis are listed in Table II. The results in Table II show the following:

1. There is an evident improvement of accuracy using DEM as a direct result of more integration points or a higher basis.
2. The present EFG formulation is less sensitive to the choice of polynomial basis and the number of integration points. Given the same polynomial basis and the same distribution of integration points, DEM is less accurate than the new EFG method.
3. Owing to the high degree of overlapping of the MLS approximation, the new EFG method, which includes all the terms for the derivatives, can simulate the cubic displacement field for this problem even with a linear basis. Incomplete or truncated derivatives of the displacement approximation requires finer integration cells or a higher-order basis to achieve the same accuracy.
4. The accuracy of the new EFG formulation and DEM can be improved by using more integration points or using a higher polynomial basis. This does not conflict the first point although the new formulation of EFG is less sensitive to the order of basis and integration cells.
6. CONCLUDING REMARKS

This paper has dealt with a number of issues affecting the ‘new’ formulation of the EFG method proposed in [11]. The Schimdt orthogonalization process in the MLS approximation removes the need for matrix inversion, which is desirable for efficient computational algorithms. However, the source of difficulties in inverting is not dealt with by adopting the orthogonal basis and inaccuracies remain. Terms in the shape function derivatives appear to have been omitted in the formulation in [11] that has been shown to lead to inaccuracies and failure to pass patch tests. It has been shown that the shape functions and shape function derivatives can be determined using different bases, a procedure that maintains the positive features of the use of orthogonalization while avoiding the inaccuracies of the original approach. The results also indicate that adaptive analysis can be performed without increasing the number of unknowns (i.e. nodes) either by (a) increasing the density of background integration cells or (b) increasing the basis order.

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USE OF ORTHOGONAL BASIS IN THE EFG METHOD

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