

# An optimal algorithm for the $k$ -fixed-endpoint path cover on proper interval graphs

George B. Mertzios and Walter Unger

Department of Computer Science  
RWTH Aachen, Germany  
{mertzios, quax}@cs.rwth-aachen.de

**Abstract.** In this paper we consider the  $k$ -fixed-endpoint path cover problem on proper interval graphs, which is a generalization of the path cover problem. Given a graph  $G$  and a set  $T$  of  $k$  vertices, a  $k$ -fixed-endpoint path cover of  $G$  with respect to  $T$  is a set of vertex-disjoint paths that covers the vertices of  $G$ , such that the vertices of  $T$  are all endpoints of these paths. The goal is to compute a  $k$ -fixed-endpoint path cover of  $G$  with minimum cardinality. We propose an optimal algorithm for this problem with runtime  $O(n)$ , where  $n$  is the number of intervals in  $G$ . This algorithm is based on the *Stair Normal Interval Representation (SNIR) matrix* that characterizes proper interval graphs. In this characterization, every maximal clique of the graph is represented by one matrix element; the proposed algorithm uses this structural property, in order to determine directly the paths in an optimal solution.

Keywords: proper interval graph, perfect graph, path cover, SNIR matrix, linear-time algorithm.

## 1 Introduction

A graph  $G$  is called an *interval graph*, if its vertices can be assigned to intervals on the real line, such that two vertices of  $G$  are adjacent if and only if the corresponding intervals intersect. The set of intervals assigned to the vertices of  $G$  is called a *realization* of  $G$ . If  $G$  has a realization, in which no interval contains another one properly, then  $G$  is called a *proper interval graph*. Proper interval graphs arise naturally in biological applications such as the physical mapping of DNA [1]. Several linear-time recognition algorithms have been presented for both graph classes in the literature [2,3,4,5]. These classes of graphs have numerous applications to scheduling problems, biology, VLSI circuit design, as well as to psychology and social sciences [6,7].

Several difficult optimization problems, which are NP-hard for general graphs [8], are solvable in polynomial time on interval and proper interval graphs. Some of them are the maximum clique, the maximum independent

set [9,10], the Hamiltonian cycle (HC) and the Hamiltonian path (HP) problem [11]. A generalization of the HP problem is the path cover (PC) problem. That is, given a graph  $G$ , the goal is to find the minimum number of vertex-disjoint simple paths that cover all vertices of  $G$ . Except graph theory, the PC problem finds many applications in the area of database design, networks, code optimization and mapping parallel programs to parallel architectures [12,13,14,15].

The PC problem is known to be NP-complete even on the classes of planar graphs [16], bipartite graphs, chordal graphs [17], chordal bipartite graphs, strongly chordal graphs [18], as well as in several classes of intersection graphs [19]. On the other hand, it is solvable in linear  $O(n + m)$  time on interval graphs with  $n$  vertices and  $m$  edges [12]. For the greater class of circular-arc graphs there is an optimal  $O(n)$ -time approximation algorithm, given a set of  $n$  arcs with endpoints sorted [20]. The cardinality of the path cover found by this approximation algorithm is at most one more than the optimal one. Several variants of the HP and the PC problems are of great interest. The simplest of them are the 1HP and 2HP problems, where the goal is to decide whether  $G$  has a Hamiltonian path with one, or two fixed endpoints respectively. Both problems are NP-hard for general graphs, as a generalization of the HP problem, while their complexity status remains open for interval graphs [21,22,23].

In this paper, we consider the  $k$ -fixed-endpoint path cover ( $k$ PC) problem, which generalizes the PC problem in the following way. Given a graph  $G$  and a set  $T$  of  $k$  vertices, the goal is to find a path cover of  $G$  with minimum cardinality, such that the elements of  $T$  are endpoints of these paths. Note that the vertices of  $V \setminus T$  are allowed to be endpoints of these paths as well. For  $k = 1, 2$ , the  $k$ PC problem constitutes a direct generalization of the 1HP and 2HP problems respectively. For the case, where the input graph is a cograph on  $n$  vertices and  $m$  edges, a linear  $O(n + m)$  time algorithm has been recently presented, while its complexity status remains open for interval graphs [22].

We propose a runtime-optimal algorithm for the  $k$ PC problem on proper interval graphs with runtime  $O(n)$ , based on the zero-one *Stair Normal Interval Representation (SNIR) matrix*  $H_G$  that characterizes a proper interval graph  $G$  on  $n$  vertices [24]. In this characterization, every maximal clique is represented by one matrix element. It provides insight and may be useful for the efficient formulation and solution of several optimization problems. In particular, it has been used to solve in polynomial time the  $k$ -cluster problem on interval graphs [25], while this problem is NP-hard for chordal graphs [26]. In most of the practical

applications, the intervals of the input are sorted by their right or left endpoints. Given such an interval realization of  $G$ , the proposed algorithm constructs at first in  $O(n)$  time a particular perfect ordering of the vertices of  $G$ , which complies with the ordering of the vertices in the SNIR matrix  $H_G$  [24].

We introduce the notion of a *singular point* in a proper interval graph  $G$  on  $n$  vertices. An arbitrary vertex of  $G$  is called singular point, if it is the unique common vertex of two consecutive maximal cliques. Due to the special structure of  $H_G$ , we need to compute only  $O(n)$  of its entries, in order to capture the complete information of this matrix. Based on this structure, the algorithm detects the singular points of  $G$  in time  $O(n)$  and then it determines *directly* the paths in an optimal solution, using only the positions of the singular points. Namely, it turns out that every such path is a Hamiltonian path of a particular subgraph  $G_{i,j}$  of  $G$  with two specific endpoints. Here,  $G_{i,j}$  denotes the induced subgraph of the vertices  $\{i, \dots, j\}$  in the ordering of  $H_G$ . Since any algorithm for this problem has to visit at least all  $n$  vertices of  $G$ , this runtime is optimal.

Recently, while writing this paper, it has been drawn to our attention that another algorithm has been independently presented for the  $k$ PC problem on proper interval graphs with runtime  $O(n + m)$  [23], where  $m$  is the number of edges of the input graph. This algorithm uses a greedy approach to augment the already constructed paths with connect/insert operations, by distinguishing whether these paths have already none, one, or two endpoints in  $T$ . The main advantage of the here proposed algorithm, besides its runtime optimality, is that an optimal solution is constructed directly by the positions of the singular points, which is a structural property of the investigated graph. Given an interval realization of the input graph  $G$ , we do not need to visit all its edges, exploiting the special structure of the SNIR matrix. Additionally, the representation of proper interval (resp. interval) graphs by the SNIR (resp. NIR) matrix [24] may lead to efficient algorithms for other open optimization problems, such as the 1HP, 2HP, or even  $k$ PC problem on interval graphs [21,22].

The paper is organized as follows. In Section 2 we recall the SNIR matrix of a proper interval graph. Furthermore, in Section 3 we present an algorithm for the 2HP, based on the SNIR matrix. This algorithm is used in Section 4, in order to derive an algorithm for the  $k$ PC problem on proper interval graphs with runtime  $O(n)$ . Finally, we discuss some conclusions and open questions for further research in Section 5.

## 2 The SNIR matrix

An arbitrary proper interval graph  $G$  on  $n$  vertices can be characterized by the *SNIR matrix*  $H_G$ , which has been introduced in [24]. This is the lower portion of the adjacency matrix of  $G$ , which uses a particular ordering of its vertices. In this ordering, the vertex with index  $i$  corresponds to the  $i^{\text{th}}$  diagonal element of  $H_G$ . Every diagonal element has a chain of consecutive ones immediately below it, while the remaining entries of this column are zero. These chains are ordered in such a way that  $H_G$  has a stair-shape, as it is illustrated in Figure 2(a). We recall now the definitions of a stair and a pick of the SNIR matrix  $H_G$  [24].

**Definition 1.** Consider the SNIR matrix  $H_G$  of the proper interval graph  $G$ . The matrix element  $H_G(i, j)$  is called a pick of  $H_G$ , iff:

1.  $i \geq j$ ,
2. if  $i > j$  then  $H_G(i, j) = 1$ ,
3.  $H_G(i, k) = 0$ , for every  $k \in \{1, 2, \dots, j - 1\}$  and
4.  $H_G(l, j) = 0$ , for every  $l \in \{i + 1, i + 2, \dots, n\}$ .

**Definition 2.** Given the pick  $H_G(i, j)$  of  $H_G$ , the set

$$\mathcal{S} = \{H_G(k, \ell) : j \leq \ell \leq k \leq i\} \quad (1)$$

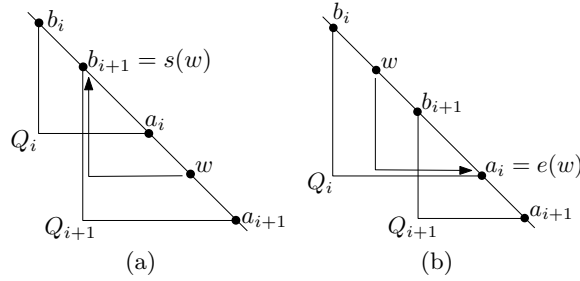
of matrix entries is called the stair of  $H_G$ , which corresponds to this pick.

**Lemma 1** ([24]). Any stair of  $H_G$  corresponds bijectively to a maximal clique of  $G$ .

A stair of  $H_G$  can be recognized in Figure 2(a), where the corresponding pick is marked with a circle. Given an interval realization of  $G$ , the ordering of vertices in  $H_G$  can be computed in time  $O(n)$  [24]. Furthermore, the picks of  $H_G$  can be also computed in time  $O(n)$  during the construction of the ordering of the vertices, since every pick corresponds to the right endpoint of an interval in  $G$  [24]. Due to its stair-shape, the matrix  $H_G$  is uniquely determined by its  $O(n)$  picks.

For an arbitrary vertex  $w$  of  $G$ , denote by  $s(w)$  and  $e(w)$  the adjacent vertices of  $w$  with the smallest and greatest index in this ordering respectively. Due to the stair-shape of  $H_G$ , the vertices  $s(w)$  and  $e(w)$  are the uppermost and lowermost diagonal elements of  $H_G$ , which belong to a common stair with  $w$ . Denote now the maximal cliques of  $G$  by  $Q_1, Q_2, \dots, Q_m$ ,  $m \leq n$  and suppose that the corresponding pick to  $Q_i$

is the matrix element  $H_G(a_i, b_i)$ , where  $i \in \{1, \dots, m\}$ . Since the maximal cliques of  $G$ , i.e. the stairs of  $H_G$ , are linearly ordered, it holds that  $1 \leq a_1 \leq \dots \leq a_m \leq n$  and  $1 \leq b_1 \leq \dots \leq b_m \leq n$ . Denote for simplicity  $a_0 = b_0 = 0$  and  $a_{m+1} = b_{m+1} = n + 1$ . Then, Algorithm 1 computes the values  $s(w)$  and  $e(w)$  for all vertices  $w \in \{1, \dots, n\}$ , as it is illustrated in Figure 1. Since  $m \leq n$ , the runtime of Algorithm 1 is  $O(n)$ .



**Fig. 1.** The computation of  $s(w)$  and  $e(w)$ .

---

**Algorithm 1** Compute  $s(w)$  and  $e(w)$  for all vertices  $w$

---

```

1: for  $i = 0$  to  $m$  do
2:   for  $w = a_i + 1$  to  $a_{i+1}$  do
3:      $s(w) \leftarrow b_{i+1}$ 
4:   for  $w = b_i$  to  $b_{i+1} - 1$  do
5:      $e(w) \leftarrow a_i$ 

```

---

The vertices  $\{i, \dots, j\}$  of  $G$ , where  $i \leq j$ , constitute a submatrix  $H_{i,j}$  of  $H_G$  that is equivalent to the induced subgraph  $G_{i,j}$  of these vertices. Since the proper interval graphs are hereditary, this subgraph remains a proper interval graph as well. In particular,  $H_{1,n} = H_G$  is equivalent to  $G_{1,n} = G$ .

**Definition 3.** A vertex  $w$  of  $G_{i,j}$  is called singular point of  $G_{i,j}$ , if there exist two consecutive cliques  $Q, Q'$  of  $G_{i,j}$ , such that

$$|Q \cap Q'| = \{w\} \tag{2}$$

Otherwise,  $w$  is called regular point of  $G_{i,j}$ . The set of all singular points of  $G_{i,j}$  is denoted by  $S(G_{i,j})$ .

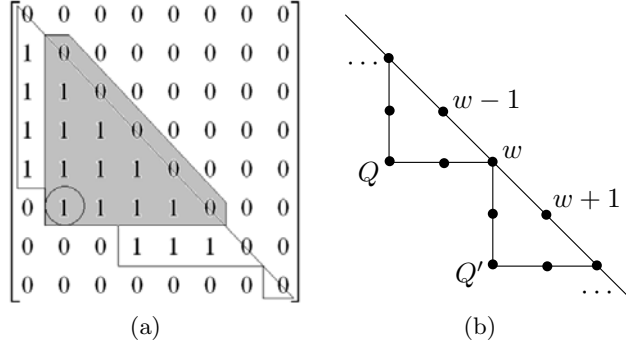


Fig. 2. (a) The SNIR matrix  $H_G$ , (b) a singular point  $w$  of  $G_{i,j}$ .

**Proposition 1.** For every singular point  $w$  of  $G_{i,j}$ , it holds  $i + 1 \leq w \leq j - 1$ .

*Proof.* Since  $w$  is a singular point of  $G_{i,j}$ , there exist two consecutive maximal cliques  $Q, Q'$  of  $G_{i,j}$  with  $Q \cap Q' = \{w\}$ . Then, as it is illustrated in Figure 2(b), both  $Q$  and  $Q'$  contain at least another vertex than  $w$ , since otherwise one of them would be included in the other, which is a contradiction. It follows that  $i + 1 \leq w \leq j - 1$ .

**Definition 4.** Consider a connected proper interval graph  $G$  and two indices  $i \leq j \in \{1, \dots, n\}$ . The submatrix  $H_{i,j}$  of  $H_G$  is called two-way matrix, if all vertices of  $G_{i,j}$  are regular points of it. Otherwise,  $H_{i,j}$  is called one-way matrix.

The intuition resulting from Definition 4 is the following. If  $H_{i,j}$  is an one-way matrix, then  $G_{i,j}$  has at least one singular point  $w$ . In this case, no vertex among  $\{i, \dots, w - 1\}$  is connected to any vertex among  $\{w + 1, \dots, j\}$ , as it is illustrated in Figure 2(b). Thus, every Hamiltonian path of  $G_{i,j}$  passes only once from the vertices  $\{i, \dots, w - 1\}$  to the vertices  $\{w + 1, \dots, j\}$ , through vertex  $w$ . Otherwise, if  $H_{i,j}$  is a two-way matrix, a Hamiltonian path may pass more than once from  $\{i, \dots, w - 1\}$  to  $\{w + 1, \dots, j\}$  and backwards, where  $w$  is an arbitrary vertex of  $G_{i,j}$ . The next corollary follows directly from Proposition 1.

**Corollary 1.** An arbitrary vertex  $w$  of  $G$  is a regular point of the subgraphs  $G_{i,w}$  and  $G_{w,j}$ , for every  $i \leq w$  and  $j \geq w$ .

### 3 The 2HP problem on proper interval graphs

#### 3.1 Necessary and sufficient conditions

In this section we solve the 2HP problem on proper interval graphs. In particular, given two fixed vertices  $u, v$  of a proper interval graph  $G$ , we provide necessary and sufficient conditions for the existence of a Hamiltonian path in  $G$  with endpoints  $u$  and  $v$ . An algorithm with runtime  $O(n)$  follows directly from these conditions, where  $n$  is the number of vertices of  $G$ .

Denote by  $2HP(G, u, v)$  this particular instance of 2HP on  $G$ . Since  $G$  is equivalent to the SNIR matrix  $H_G$  and since this matrix specifies a particular ordering of its vertices, we identify w.l.o.g. the vertices of  $G$  with their indices in this ordering. Observe at first that if  $G$  is not connected, then there is no Hamiltonian path at all in  $G$ . On the other hand, if  $G$  has only two vertices  $u, v$  and  $G$  is connected, there exists trivially a Hamiltonian path with  $u$  and  $v$  as endpoints. Thus, we assume in the following that  $G$  is connected and  $n \geq 3$ . The next Theorems 1 and 2 provide necessary and sufficient conditions for the existence of a Hamiltonian path with endpoints  $u, v$  in a connected proper interval graph  $G$ .

**Theorem 1.** *Let  $G$  be a connected proper interval graph and  $u, v$  be two vertices of  $G$ , with  $v \geq u + 2$ . There is a Hamiltonian path in  $G$  with  $u, v$  as endpoints if and only if the submatrices  $H_{1,u+1}$  and  $H_{v-1,n}$  of  $H_G$  are two-way matrices.*

*Proof.* Suppose that  $H_{1,u+1}$  is an one-way matrix. Then, due to Definition 4,  $G_{1,u+1}$  has at least one singular point  $w$ . Since  $G_{1,u+1}$  is connected as an induced subgraph of  $G$ , Proposition 1 implies that  $2 \leq w \leq u$ .

In order to obtain a contradiction, let  $P$  be a Hamiltonian path in  $G$  with  $u$  and  $v$  as its endpoints. Suppose first that for the singular point  $w$  it holds  $w < u$ . Then, due to the stair-shape of  $H_G$ , the path  $P$  has to visit  $w$  in order to reach the vertices  $\{1, \dots, w-1\}$ . On the other hand,  $P$  has to visit  $w$  again in order to reach  $v$ , since  $w < v$ . This is a contradiction, since  $P$  visits  $w$  exactly once. Suppose now that  $w = u$ . The stair-shape of  $H_G$  implies that  $u$  has to be connected in  $P$  with at least one vertex of  $\{1, \dots, u-1\}$  and with at least one vertex of  $\{u+1, \dots, n\}$ . This is also a contradiction, since  $u$  is an endpoint of  $P$ . Therefore, there exists no such path  $P$  in  $G$ , if  $H_{i,u+1}$  is an one-way matrix. Similarly, we obtain that

---

**Algorithm 2** Construct a Hamiltonian path  $P$  in  $G$  with  $u, v$  as endpoints

---

```

1:  $t \leftarrow u; P \leftarrow \{u\}$ 
2: while  $t > 1$  do
3:    $p \leftarrow s(t)$  {the adjacent vertex of  $t$  with the smallest index}
4:    $P \leftarrow P \circ p; t \leftarrow p$ 
5: while  $t < v - 1$  do
6:   if  $(t + 1) \notin P$  then
7:      $P \leftarrow P \circ (t + 1); t \leftarrow t + 1$ 
8:   else
9:     if  $(t + 2) \notin P \cup \{v\}$  then
10:       $P \leftarrow P \circ (t + 2); t \leftarrow t + 2$ 
11: while  $t < n$  do
12:    $p \leftarrow e(t)$  {the adjacent vertex of  $t$  with the greatest index}
13:    $P \leftarrow P \circ p; t \leftarrow p$ 
14: while  $t > v$  do
15:   if  $(t - 1) \notin P$  then
16:      $P \leftarrow P \circ (t - 1); t \leftarrow t - 1$ 
17:   else
18:      $P \leftarrow P \circ (t - 2); t \leftarrow t - 2$ 
19: return  $P$ 

```

---

there exists again no such path  $P$  in  $G$ , if  $H_{v-1,n}$  is an one-way matrix. This completes the necessity part of the proof.

For the sufficiency part, suppose that both  $H_{1,u+1}$  and  $H_{v-1,n}$  are two-way matrices. Then, Algorithm 2 constructs a Hamiltonian path  $P$  in  $G$  having  $u$  and  $v$  as endpoints, as follows. In the while-loop of the lines 2-4 of Algorithm 2,  $P$  starts from vertex  $u$  and reaches vertex 1 using sequentially the uppermost diagonal elements, i.e. vertices, of the visited stairs of  $H_G$ . Since  $H_{1,u+1}$  is a two-way matrix,  $P$  does not visit any two neighbored diagonal elements until it reaches vertex 1. In the while-loop of the lines 5-10,  $P$  continues visiting all unvisited vertices until vertex  $v - 1$ . Let  $t$  be the actual visited vertex of  $P$  during these lines. Since  $P$  did not visit any two neighbored diagonal elements in lines 2-4, at least one of the vertices  $t + 1, t + 2$  have been not visited yet. Thus, always one of the lines 7 and 10 is executed.

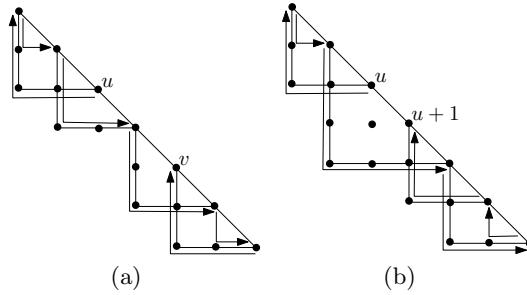
Next, in the while-loop of the lines 11-13,  $P$  starts from vertex  $v - 1$  and reaches vertex  $n$  using sequentially the lowermost diagonal elements of the visited stairs of  $H_G$ . During the execution of lines 11-13, since  $H_{v-1,n}$  is a two-way matrix,  $P$  does not visit any two neighbored diagonal elements until it reaches vertex  $n$ . Finally, in the while-loop of the lines 14-18,  $P$  continues visiting all unvisited vertices until  $v$ . Similarly to the lines 5-10, let  $t$  be the actual visited vertex of  $P$ . Since  $P$  did not visit



any two neighbored diagonal elements in lines 11-13, at least one of the vertices  $t-1, t-2$  have been not visited yet. Thus, always one of the lines 16 and 18 is executed. Figure 3(a) illustrates the construction of such a Hamiltonian path by Algorithm 2 in a small example.

**Theorem 2.** *Let  $G$  be a connected proper interval graph with  $n \geq 3$  vertices and  $u$  be a vertex of  $G$ . There is a Hamiltonian path in  $G$  with  $u, u+1$  as endpoints if and only if  $H_G$  is a two-way matrix and either  $u \in \{1, n\}$  or the vertices  $u-1$  and  $u+2$  are adjacent.*

If the conditions of Theorems 1 and 2 are satisfied, Algorithm 2 constructs a Hamiltonian path with endpoints  $u, v$ . Algorithm 2 operates on every vertex of  $G$  at most twice. Thus, since all values  $s(t)$  and  $e(t)$  can be computed in time  $O(n)$ , its runtime is  $O(n)$  as well. Figure 3 illustrates the construction of such a Hamiltonian path by Algorithm 2 in a small example, for both cases  $v \geq u+2$  and  $v = u+1$ .



**Fig. 3.** The construction of the HP with endpoints  $u, v$ , where (a)  $v \geq u+2$ , (b)  $v = u+1$ .

### 3.2 The decision of 2HP in time $O(n)$

We can use now the results of Section 3.1 in order to decide in time  $O(n)$  whether a given proper interval graph  $G$  has a Hamiltonian path  $P$  with two specific endpoints  $u, v$  and to construct  $P$ , if it exists. The values  $s(w)$  and  $e(w)$  for all vertices  $w \in \{1, \dots, n\}$  can be computed in time  $O(n)$ . Due to the stair-shape of  $H_G$ , the graph  $G$  is not connected if and only if there is a vertex  $w \in \{1, \dots, n-1\}$ , for which it holds  $e(w) = w$  and thus, we can check the connectivity of  $G$  in time  $O(n)$ . If  $G$  is not connected, then it has no Hamiltonian path at all. Finally, a vertex  $w$  is

singular if and only if  $e(w - 1) = s(w + 1) = w$  and thus, the singular points of  $G$  can be computed in  $O(n)$ .

Since the proper interval graphs are hereditary, the subgraphs  $G_{1,u+1}$  and  $G_{v-1,n}$  of  $G$  remain proper interval graphs as well. Thus, if  $G$  is connected, we can check in time  $O(n)$  whether these graphs have singular points, or equivalently, whether  $H_{1,u+1}$  and  $H_{v-1,n}$  are two-way matrices. On the other hand, we can check in constant time whether the vertices  $u - 1$  and  $u + 2$  are adjacent. Thus, we can decide in time  $O(n)$  whether  $G$  has a Hamiltonian path with endpoints  $u, v$ , due to Theorems 1 and 2. In the case of existence, Algorithm 2 constructs in time  $O(n)$  the desired Hamiltonian path.

## 4 The $k$ PC problem on proper interval graphs

### 4.1 The algorithm

In this section we propose Algorithm 3, which solves in  $O(n)$  the  $k$ -fixed-endpoint path cover ( $k$ PC) problem on a proper interval graph  $G$  with  $n$  vertices, for any  $k \leq n$ . This algorithm uses the characterization of the 2HP problem of the previous section. We assume that for the given set  $T = \{t_1, t_2, \dots, t_k\}$  it holds  $t_1 < t_2 < \dots < t_k$ . Denote also for simplicity  $t_{k+1} = n + 1$ .

---

**Algorithm 3** Compute  $C(G, T)$  for a proper interval graph  $G$

---

```

1: if  $G = \emptyset$  then
2:   return  $\emptyset$ 
3: Compute the values  $s(w)$  and  $e(w)$  for every vertex  $w$  by Algorithm 1
4:  $w \leftarrow 1$ 
5: while  $w < n$  do
6:   if  $e(w) = w$  then { $G$  is not connected}
7:      $T_1 \leftarrow T \cap \{1, 2, \dots, w\}$ ;  $T_2 \leftarrow T \setminus T_1$ 
8:     return  $C(G_{1,w}, T_1) \cup C(G_{w+1,n}, T_2)$ 
9:    $w \leftarrow w + 1$ 
10: if  $k \leq 1$  then
11:   Call Algorithm 4
12: if  $t_1 \in S(G)$  then
13:    $P_1 \leftarrow 1 \circ \dots \circ t_1$ 
14:   return  $\{P_1\} \cup C(G_{t_1+1,n}, T \setminus \{t_1\})$ 
15: Call Algorithm 5

```

---

Algorithm 3 computes an optimal path cover  $C(G, T)$  of  $G$ . In lines 4-9, it checks the connectivity of  $G$ . If it is not connected, the algorithm

computes in lines 7-8 recursively the optimal solutions of the first connected component and of the remaining graph. It reaches line 10 only if  $G$  is connected. In the case  $|T| = k \leq 1$ , Algorithm 3 calls Algorithm 4 as subroutine.

In lines 12-14, Algorithm 3 considers the case that  $G$  is connected,  $|T| \geq 2$  and  $t_1$  is a singular point of  $G$ . Then, Proposition 1 implies that  $2 \leq t_1 \leq n - 1$ . Since no vertex among  $\{1, \dots, t_1 - 1\}$  is connected to any vertex among  $\{t_1 + 1, \dots, n\}$  and since  $t_1 \in T$ , an optimal solution must contain at least two paths. Thus, it is always optimal to choose in line 13 a path that visits sequentially the first  $t_1$  vertices and then to compute recursively in line 14 an optimal solution in the remaining graph  $G_{t_1+1,n}$ . Algorithm 3 reaches line 15 if  $G$  is connected,  $|T| \geq 2$  and  $t_1$  is a regular point of  $G$ . In this case, it calls the following Algorithm 5 as subroutine.

Both Algorithms 4 and 5 compute by Algorithm 2 a Hamiltonian path of some subgraphs of  $G$  with two specific endpoints, if such a path exists. If the conditions of Theorems 1 and 2 are not satisfied, i.e. if there is no such Hamiltonian path in these subgraphs, Algorithm 2 returns “no” instead of a Hamiltonian path.

---

**Algorithm 4** Compute  $C(G, T)$ , if  $G$  is connected and  $|T| \leq 1$

---

```

1: if  $k = 0$  then
2:   return  $\{1 \circ 2 \circ \dots \circ n\}$ 
3: if  $k = 1$  then
4:   if  $t_1 \in \{1, n\}$  then
5:     return  $\{1 \circ 2 \circ \dots \circ n\}$ 
6:   else
7:      $P_1 \leftarrow 2HP(G, 1, t_1)$ 
8:      $P_2 \leftarrow 2HP(G, t_1, n)$ 
9:     if  $P_1 = \text{“no”}$  then
10:      if  $P_2 = \text{“no”}$  then
11:        return  $\{1 \circ \dots \circ t_1\} \cup \{(t_1 + 1) \circ \dots \circ n\}$ 
12:      else
13:        return  $\{P_2\}$ 
14:      else
15:        return  $\{P_1\}$ 

```

---

Algorithm 4 computes an optimal path cover  $C(G, T)$  of  $G$  in the case, where  $G$  is connected and  $|T| = k \leq 1$ . If  $k = 0$ , then the optimal solution includes clearly only one path that visits sequentially the vertices  $1, 2, \dots, n$ , since  $G$  is connected. Let now  $k = 1$ . If  $t_1 \in \{1, n\}$ , then the optimal solution is again the single path  $\{1, 2, \dots, n\}$ . Otherwise, suppose

that  $t_1 \in \{2, \dots, n-1\}$ . In this case, a trivial path cover is that with the paths  $\{1 \circ \dots \circ t_1\}$  and  $\{(t_1+1) \circ \dots \circ n\}$ . This path cover is not optimal if and only if  $G$  has a Hamiltonian path  $P$  with  $u = t_1$  as one endpoint. The other endpoint  $v$  of  $P$  lies either in  $\{1, \dots, t_1-1\}$  or in  $\{t_1+1, \dots, n\}$ . If  $v \in \{t_1+1, \dots, n\}$ , then  $H_{1,t_1+1}$  and  $H_{v-1,n}$  have to be two-way matrices, due to Theorems 1 and 2. However, due to Definition 1, if  $H_{v-1,n}$  is a two-way matrix, then  $H_{n-1,n}$  is also a two-way matrix, since  $H_{n-1,n}$  is a trivial submatrix of  $H_{v-1,n}$ .

Thus, if such a Hamiltonian path with endpoints  $t_1$  and  $v$  exists, then there exists also one with endpoints  $t_1$  and  $n$ . Similarly, if there exists a Hamiltonian path with endpoints  $v \in \{1, \dots, t_1-1\}$  and  $t_1$ , then there exists also one with endpoints 1 and  $t_1$ . Thus, Algorithm 4 calls  $P_1 = 2HP(G, 1, t_1)$  and  $P_2 = 2HP(G, t_1, n)$  in lines 7 and 8 respectively. If both outputs are “no”, then  $\{1 \circ \dots \circ t_1\}$  and  $\{(t_1+1) \circ \dots \circ n\}$  constitute an optimal solution. Otherwise, we return one of the obtained paths  $P_1$  or  $P_2$  in lines 15 or 13 respectively. Since the runtime of Algorithm 2 for the 2HP problem is  $O(n)$ , the runtime of Algorithm 4 is  $O(n)$  as well.

In lines 5-9 and 12-14, Algorithm 3 separates  $G$  in two subgraphs and computes their optimal solutions recursively. Thus, since the computation of all values  $s(w)$  and  $e(w)$  can be done in  $O(n)$  and since the runtime of Algorithms 4 and 5 is  $O(n)$ , Algorithm 3 runs in time  $O(n)$  as well.

## 4.2 Correctness of Algorithm 5

The correctness of Algorithm 5 follows from the technical Lemmas 2 and 3. In these Lemmas, we assume an optimal solution  $C$  of  $G$ . Denote by  $P_i$  the path in  $C$ , which has  $t_i$  as endpoint and let  $e_i$  be its second endpoint. Observe that, if  $e_1 = t_2$ , then  $P_1 = P_2$ .

**Lemma 2.** *If  $\{1, \dots, t_1-1\} \cap S(G) = \emptyset$ , then w.l.o.g.  $e_1 = t_2$ .*

**Lemma 3.** *If  $\{1, \dots, t_1-1\} \cap S(G) \neq \emptyset$ , then w.l.o.g.  $e_1 = 1$ .*

Algorithm 5 considers in lines 1-10 the case that there are no singular points of  $G$  among  $\{1, \dots, t_1-1\}$ . The proof of Lemma 2 implies for this case that  $e_1 = t_2$  and, in particular that  $P_1 = 2HP(G_{1,a}, t_1, t_2)$  for some  $a \in \{t_2, \dots, t_3-1\}$ . In order to maximize  $P_1$  as much as possible, we choose the greatest possible value of  $a$ , for which  $G_{1,a}$  has a Hamiltonian path with endpoints  $t_1, t_2$ . Namely, if  $G_{1,t_2+1}$  does not have such a Hamiltonian path, we set  $a = t_2$  in line 3. Suppose now that  $G_{1,t_2+1}$  has such a path. In the case that there is at least one singular point of  $G$  among  $\{t_2 +$

---

**Algorithm 5** Compute  $C(G, T)$ , where  $G$  is connected,  $|T| \geq 2$ ,  $t_1 \notin S(G)$ .

---

```

1: if  $\{1, \dots, t_1 - 1\} \cap S(G) = \emptyset$  then  $\{e_1 = t_2\}$ 
2:   if  $\mathcal{2HP}(G_{1, t_2+1}, t_1, t_2) = \text{"no"}$  then
3:      $a \leftarrow t_2$ 
4:   else
5:     if  $\{t_2 + 1, \dots, t_3 - 1\} \cap S(G) \neq \emptyset$  then
6:        $a \leftarrow \min\{\{t_2 + 1, \dots, t_3 - 1\} \cap S(G)\}$ 
7:     else
8:        $a \leftarrow t_3 - 1$ 
9:      $P_1 \leftarrow \mathcal{2HP}(G_{1, a}, t_1, t_2)$ 
10:     $C_2 \leftarrow C(G_{a+1, n}, T \setminus \{t_1, t_2\})$ 
11: else  $\{e_1 = 1\}$ 
12:   if  $\mathcal{2HP}(G_{1, t_1+1}, 1, t_1) = \text{"no"}$  then
13:      $a \leftarrow t_1$ 
14:   else
15:     if  $\{t_1 + 1, \dots, t_2 - 1\} \cap S(G) \neq \emptyset$  then
16:        $a \leftarrow \max\{\{t_1 + 1, \dots, t_2 - 1\} \cap S(G)\}$ 
17:     else
18:        $a \leftarrow t_2 - 1$ 
19:      $P_1 \leftarrow \mathcal{2HP}(G_{1, a}, 1, t_1)$ 
20:      $C_2 \leftarrow C(G_{a+1, n}, T \setminus \{t_1\})$ 
21: return  $\{P_1\} \cup C_2$ 

```

---

$1, \dots, t_3 - 1\}$ , we set  $a$  to be this one with the smallest index among them in line 6. Otherwise, we set  $a = t_3 - 1$  in line 8. Denote for simplicity  $G_{1, n+1} = G$ . Then, in the extreme cases  $t_3 = t_2 + 1$  or  $t_2 = n$ , the algorithm sets  $a = t_2 = t_3 - 1$ .

Next, in lines 11-20, Algorithm 5 considers the case that there are some singular points of  $G$  among  $\{1, \dots, t_1 - 1\}$ . Then, the proof of Lemma 3 implies that  $e_1 = 1$  and, in particular that  $P_1 = \mathcal{2HP}(G_{1, a}, 1, t_1)$ , for some  $a \in \{t_1, \dots, t_2 - 1\}$ . In order to maximize  $P_1$  as much as possible, we choose the greatest possible value of  $a$ , for which  $G_{1, a}$  has a Hamiltonian path with endpoints 1 and  $t_1$ . Namely, if  $G_{1, t_1+1}$  does not have such a Hamiltonian path, we set  $a = t_1$  in line 13. Suppose now that  $G_{1, t_1+1}$  has such a path. In the case that there is at least one singular point of  $G$  among  $\{t_1 + 1, \dots, t_2 - 1\}$ , we set  $a$  to be this one with the greatest index among them in line 16. Otherwise, we set  $a = t_2 - 1$  in line 18. Note that in the extreme case  $t_2 = t_1 + 1$ , the algorithm sets  $a = t_1 = t_2 - 1$ .

The algorithm computes  $P_1$  in time  $O(n)$  in lines 9 and 19 respectively. Then, it computes recursively the optimum path cover  $C_2$  of the remaining graph in lines 10 and 20 respectively and it outputs  $\{P_1\} \cup C_2$ . Since the

computation of a 2HP by Algorithm 2 can be done in time  $O(n)$ , the runtime of Algorithm 5 is  $O(n)$  as well.

## 5 Concluding remarks

In this article we proposed a simple algorithm for the  $k$ -fixed-endpoint path cover problem on proper interval graphs with runtime  $O(n)$ . Since any algorithm for this problem has to visit at least all  $n$  vertices of  $G$ , this runtime is optimal. The proposed algorithm is based on the characterization of proper interval graphs by the SNIR matrix. The complexity status of the  $k$ -fixed-endpoint path cover problem, as well as of 1HP and 2HP, on the general class of interval graphs remains an interesting open question for further research.

## References

1. P. Hell, R. Shamir, and R. Sharan. A fully dynamic algorithm for recognizing and representing proper interval graphs. *SIAM J. Comput.*, 31(1):289–305, 2001.
2. W.L. Hsu. A simple test for interval graphs. In *WG '92: Proceedings of the 18th International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 11–16, London, 1993. Springer-Verlag.
3. D. Corneil, H. Kim, S. Natarajan, S. Olariu, and A.P. Sprague. Simple linear time recognition of unit interval graphs. *Inform. Process. Lett.*, 55:99–104, 1995.
4. D.G. Corneil, S. Olariu, and L. Stewart. The ultimate interval graph recognition algorithm? In *SODA '98: Proceedings of the ninth annual ACM-SIAM symposium on Discrete algorithms*, pages 175–180, 1998.
5. B.S. Panda and S.K. Das. A linear time recognition algorithm for proper interval graphs. *Information Processing Letters*, 87(3):153–161, 2003.
6. M.C. Golumbic and A.N. Trenk. *Tolerance graphs*. Cambridge University Press, Cambridge, 2004.
7. A.V. Carrano. Establishing the order to human chromosome-specific DNA fragments. In A. D. Woodhead and B. J. Barnhart, editors, *Biotechnology and the Human Genome*, pages 37–50. Plenum Press, New York, 1988.
8. M.R. Garey and D.S. Johnson. *Computers and intractability: a guide to the theory of NP-completeness*. W.H. Freeman, San Francisco, 1979.
9. U.I. Gupta, D.T. Lee, and J.Y.T. Leung. Efficient algorithms for interval graphs and circular-arc graphs. *Networks*, pages 459–467, 1982.
10. Ju Yuan Hsiao and Chuan Yi Tang. An efficient algorithm for finding a maximum weight 2-independent set on interval graphs. *Inf. Process. Lett.*, 43(5):229–235, 1992.
11. M.S. Chang, S.L. Peng, and J.L. Liaw. Deferred-query - an efficient approach for problems on interval and circular-arc graphs (extended abstract). In *WADS*, pages 222–233, 1993.
12. S.R. Arikati and C.P. Rangan. Linear algorithm for optimal path cover problem on interval graphs. *Information Processing Letters*, 35(3):149–153, 1990.

13. G.S. Adhar and S. Peng. Parallel algorithms for path covering, hamiltonian path and hamiltonian cycle in cographs. In *International Conference on Parallel Processing*, volume 3, pages 364–365, 1990.
14. R. Lin, S. Olariu, and G. Pruesse. An optimal path cover algorithm for cographs. *Comput. Math. Appl.*, 30:75–83, 1995.
15. R. Srikant, R. Sundaram, K.S. Singh, and C.P. Rangan. Optimal path cover problem on block graphs and bipartite permutation graphs. *Theoretical Computer Science*, 115:351–357, 1993.
16. M.R. Garey, D.S. Johnson, and R.E. Tarjan. The planar hamiltonian circuit problem is np-complete. *SIAM J. Comput.*, 5:704–714, 1976.
17. M.C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*, volume 57. Annals of Discrete Mathematics, Amsterdam, The Netherlands, 2004.
18. H. Müller. Hamiltonian circuits in chordal bipartite graphs. *Discrete Mathematics*, 156:291–298, 1996.
19. A.A. Bertossi and M.A. Bonucelli. Finding hamiltonian circuits in interval graph generalizations. *Information Processing Letters*, 23:195–200, 1986.
20. R.W. Hung and M.S. Chang. Solving the path cover problem on circular-arc graphs by using an approximation algorithm. *Discrete Applied Mathematics*, 154(1):76–105, 2006.
21. P. Damaschke. Paths in interval graphs and circular-arc graphs. *Discrete Mathematics*, 112:49–64, 1993.
22. K. Asdre and S.D. Nikolopoulos. A linear-time algorithm for the k-fixed-endpoint path cover problem on cographs. *Networks*, 50:231–240, 2007.
23. K. Asdre and S.D. Nikolopoulos. A polynomial solution to the k-fixed-endpoint path cover problem on proper interval graphs. In *18th International Conference on Combinatorial Algorithms (IWOCA'07)*, Newcastle, Australia, 2007.
24. G.B. Mertzios. A matrix characterization of interval and proper interval graphs. *Applied Mathematics Letters*, 21(4):332–337, 2008.
25. G.B. Mertzios. A polynomial algorithm for the k-cluster problem on interval graphs. *Discrete Mathematics*, 2008. doi:10.1016/j.disc.2007.12.030.
26. D.G. Corneil and Y. Perl. Clustering and domination in perfect graphs. *Discrete Applied Mathematics*, 9:27–39, 1984.