Fast Convergence of Routing Games with Splittable Flows

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Abstract

In this paper we investigate the splittable routing game in a series-parallel network with two selfish players. Every player wishes to route optimally, i.e. at minimum cost, an individual flow demand from the source to the destination, giving rise to a non-cooperative game. We allow a player to split his flow along any number of paths. One of the fundamental questions in this model is the convergence of the best response dynamics to a Nash equilibrium, as well as the time of convergence. We prove that this game converges indeed to a Nash equilibrium in a logarithmic number of steps. Our results hold for increasing and convex player-specific latency functions. Finally, we prove that our analysis on the convergence time is tight for affine latency functions.

1. Introduction

We investigate in this paper the splittable routing game in a network with two selfish players. The aim of each player is to route a flow demand from a source to a destination at minimum cost. This gives rise to a non-cooperative game. The players have possibly different demands (*weighted flow*). At each step of the game, one of the players reallocates his flow, such that his individual cost is minimized, assuming that the flow allocation of the other player remains unchanged (*best response dynamics*). The players are allowed to split their flow arbitrarily along any number of paths from the source to the destination (*splittable flow*).

This model was introduced in [11], where it is noted that the existence of Nash equilibrium points (NEP) follows from a classical result about convex games [13]. In [11] the uniqueness of NEP is investigated mainly. In particular, it has been proved that in a network with two nodes and multiple parallel links the NEP is unique under reasonable convexity assumptions on the latency functions. Recently, it has been proved that only the class of nearly parallel networks ensures uniqueness of NEP for any set of players and latency functions [12]. This model has been considered also from the perspective of the efficiency of equilibria, i.e. the social optimality [2], [3], [15], [16]. For the special case of symmetric players in a network the unique NEP is characterized as the minimum of a convex optimization problem [2]. This implies that the game with symmetric players is an exact potential game and thus the best response dynamics converge to a NEP [9]. For a survey on finite (weighted) congestion games, we refer to [10].

The stability of NEP is a fundamental question in game theory and has been stated as a major issue for further research in the model under consideration [11]. The only result until now about this issue is that the two player game on a restricted network with only two nodes and two parallel links converges to the unique NEP [11]. To the best of our knowledge, nothing is known about the time of convergence in this model. The convergence issue in the closely related model of unsplittable flows has been investigated in the literature as well. Two main results in this context are the existence of exponentially long best response paths to NEP [5] and fast convergence to constant factor solutions on random best response paths [8]. Furthermore, the number of steps required to reach a NEP has been investigated for a variety of load balancing models using a potential-based argument in [4].

Another related model is that of a cost sharing mechanism. In contrast to our model, this mechanism is non-increasing in the number of players using an edge. This crucial property has been used for the multicast game with selfish players, in order to obtain a potential function for both splittable and unsplittable versions [1]. The splittable version of this model is similar to ours. Though, the existence of this potential function, which extends that of Rosenthal [14], seems not to be extensible to non-decreasing cost functions, as it is in our case.

In this article we prove that the game with two selfish players on a series-parallel network G converges in a logarithmic number of steps to a NEP, starting at an arbitrary initial configuration. Here, we use the notion of convergence to a NEP in the sense of [11], i.e. that the strategy configuration, which in the present case is a point in a Eucledian space, is close to a NEP. Throughout this article we make the assumption that the latency functions are increasing and convex, which complies with the convexity assumptions of [11].

The proof of convergence relies on a potential-based argument. In particular, we define for every step $t \in \mathbb{N}$ a non-negative function (potential) $\Phi_G(t)$, which equals the amount of flow that is reallocated in the network G during step t. We prove that for every $\varepsilon > 0$ the value of this function is at most ε after a number of steps that is logarithmic in ε^{-1}. In a series-parallel network with m edges, the asymptotic convergence time is given in the main Theorem 1:

Theorem 1: The game converges to a Nash equilibrium in a logarithmic number of steps. In particular, after

$$t(\varepsilon) = O(\log(m/\varepsilon))$$

steps, the potential $\Phi_G(t(\varepsilon))$ is at most ε, for every $\varepsilon > 0$.

Note that this potential is not a function defined over the strategy configurations, but a measure of the distance between two consecutive steps in the best-response path. Thus, the existence of this function does not imply that the game under consideration is a potential game.

Furthermore, a lower bound on the convergence time is presented. Our analysis is tight in the case of affine latency functions. We remind that the routing problem in a network with a single convex objective can be solved in a standard way using convex programming techniques [6], [7]. Thus, we assume that exact minimization is achieved at each step. Hence, we are interested only in the number of steps, in order to study the convergence time to a NEP.

The article is organized as follows. In Section 2 we investigate a network with two nodes and multiple parallel links as a special case of a series-parallel network. For this class of networks we provide a potential function that decreases strictly at every step after the second one. In Section 3 we generalize our analysis to arbitrary series-parallel networks. In particular, we provide a potential function that generalizes that of Section 2. In Section 3.3 this potential is used to prove Theorem 1. In Section 3.4 we obtain a lower bound on the convergence time, which is tight in the case of affine latency functions. Finally, some conclusions and open problems are discussed in Section 4.

2. A network of parallel links

2.1. Notation and terminology

We consider a network G with source u, destination v and a set of m parallel links $E = \{1, 2, ..., m\}$, where the selfish players $j \in \{1, 2\}$ wish to route an individual flow demand d_j from u to v at minimum cost each. W.l.o.g. we assume that the demands d_j are scaled in the interval $(0, 1]$. Let f_e and $x_{e,j}$ denote the total flow and the flow of player j, respectively on link e. Player j has on link e an increasing and convex player-specific latency function $\ell_{e,j}(f_e)$, which denotes the cost per unit of flow of player j on this edge. This latency implies that the cost function of player j on edge e is $c_{e,j}(x_{e,j}, f_e) = x_{e,j}\ell_{e,j}(f_e)$. The marginal cost function $g_{e,j}(x_{e,j}, f_e)$ equals the first derivative of the cost function $c_{e,j}$ with respect to $x_{e,j}$, i.e. $g_{e,j}(x_{e,j}, f_e) = \ell_{e,j}(f_e) + x_{e,j}\ell'_{e,j}(f_e)$.

Suppose that the players play alternating and let a step of the game denote the best response of the corresponding player. Denote by $j(t)$ the player moving at step t, as well as by $f_e^{(t)}$ and $x_{e,j}^{(t)}$ the quantities f_e and $x_{e,j}$, respectively after the execution of this step. Let furthermore $f_e^{(0)}$ be the flow on link e in the initial allocation of the network. For every step $t \in \mathbb{N}$ let $\Delta_e^{(t)} = f_e^{(t)} - f_e^{(t-1)}$ and define the sets $E^{(t)+} := \{e \in E : \Delta_e^{(t)} > 0\}$ and $E^{(t)-} := \{e \in E : \Delta_e^{(t)} < 0\}$.

Denote by $g_{e,j(t)}^{(s)} = g_{e,j(t)}(x_{e,j(t)}^{(s)}, f_e^{(s)})$ the marginal cost of player $j(t)$ on link e after the execution of step s. Denote furthermore by $S_j^{(t)} \subseteq E$ the support of player j after the execution of step t, i.e. the set of the network links on which player j allocates a positive amount of flow. Recall that, since the cost of player $j(t)$ is minimized at step t, his marginal cost $g_{e,j(t)}^{(t)}$ is equal to a quantity $g^{(t)}$ on every link $e \in S_{j(t)}^{(t)}$, while the marginal cost on every of the remaining links is at least $g^{(t)}$.

2.2. The potential function

The potential at step $t \in \mathbb{N}$ is defined by

$$\Phi(t) := \sum_{e \in E} |\Delta_e^{(t)}| \geq 0 \tag{1}$$

For every step $t \in \mathbb{N}$, the potential $\Phi(t)$ equals the sum of the amounts of flow that are reallocated on all links during step t. Since the demands of the players remain constant, it holds that

$$\Phi(t) = 2 \sum_{e \in E^{(t)+}} |\Delta_e^{(t)}| = 2 \sum_{e \in E^{(t)-}} |\Delta_e^{(t)}| \qquad (2)$$

Define now

$$\lambda := \min_{\substack{j \in \{1,2\} \\ e \in E}} \left\{ \frac{\inf\limits_{0 \leq x \leq d_1 + d_2} \{\ell'_{e,j}(x)\}}{\sup\limits_{\substack{0 \leq x \leq d_1 \\ 0 \leq y \leq d_2}} \{[x \ell_{e,j}(x+y)]''\}} \right\} \qquad (3)$$

which is a constant that depends only on the latency functions of the network. This constant λ will be used in the sequel to indicate the rate, by which the potential $\Phi(t)$ decreases at every step. Due to the monotonicity and convexity of the latency functions, it holds that

$$\begin{aligned}
&\sup_{\substack{0 \leq x \leq d_1 \\ 0 \leq y \leq d_2}} \{[x \ell_{e,j}(x+y)]''\} \geq \\
&\qquad \geq [x_{e,j} \ell_{e,j}(f_e)]'' = 2\ell'_{e,j}(f_e) + x_{e,j} \ell''_{e,j}(f_e) \\
&\qquad \geq 2\ell'_{e,j}(f_e) \geq 2 \inf_{0 \leq x \leq d_1 + d_2} \{\ell'_{e,j}(x)\} > 0
\end{aligned}$$

for all values of $x_{e,j}$ and f_e. It follows that

$$0 < \lambda \leq \frac{1}{2} \qquad (4)$$

The following lemma proves that $\Phi(t)$ decreases strictly at every step $t \geq 3$ of the game.

Lemma 2: For every $t \geq 3$, it holds that $\Phi(t) \leq (1-\lambda)\Phi(t-1)$.

Proof: We denote for the purposes of the proof the quantities $f_e^{(t-2)}$, $x_{e,j(t)}^{(t-2)}$ and $j(t)$ by f_e, $x_{e,j}$ and j, respectively. For the marginal cost of player j on an arbitrary link $e \in E$ after the execution of step t, it holds that

$$\begin{aligned}
g_{e,j}^{(t)} = \ell_{e,j}&\left(f_e + \Delta_e^{(t-1)} + \Delta_e^{(t)}\right) \\
&+ \left(x_{e,j} + \Delta_e^{(t)}\right) \ell'_{e,j}\left(f_e + \Delta_e^{(t-1)} + \Delta_e^{(t)}\right)
\end{aligned} \qquad (5)$$

We distinguish the following cases.

Case 1. Suppose that $g^{(t)} \geq g^{(t-2)}$. Consider an arbitrary $e \in E^{(t)-}$. Due to the monotonicity and convexity of $\ell_{e,j}(f_e)$, the marginal cost function $g_{e,j}(x_{e,j}, f_e)$ is non-decreasing in f_e. Since $\Delta_e^{(t)} < 0$, player j allocated a positive amount of flow on e after step $t-2$, i.e. $e \in S_j^{(t-2)}$. It follows that his marginal cost on e after step $t-2$ was equal to $g^{(t-2)}$. On the other side, since $\Delta_e^{(t)} < 0$, his marginal cost on e after the execution of step $t-1$ was greater than $g^{(t)}$ and therefore greater that $g^{(t-2)}$, due to the assumption that $g^{(t)} \geq g^{(t-2)}$. It follows that the total flow f_e has been increased during step $t-1$ by the other player, i.e. $e \in E^{(t-1)+}$. Thus, $E^{(t)-} \subseteq E^{(t-1)+} \cap S_j^{(t-2)}$.

Since $g_{e,j}^{(t)} \geq g^{(t)}$ holds for any link $e \in E$ and due to the assumption that $g^{(t)} \geq g^{(t-2)}$, it follows that $g_{e,j}^{(t)} \geq g^{(t-2)}$ for every $e \in E$. Suppose now that $|\Delta_e^{(t)}| \geq |\Delta_e^{(t-1)}|$ for some link $e \in E^{(t)-}$. Then, since $\Delta_e^{(t)} < 0$ and $\Delta_e^{(t-1)} > 0$, it holds that $x_{e,j} + \Delta_e^{(t)} < x_{e,j} + \Delta_e^{(t-1)} + \Delta_e^{(t)} \leq x_{e,j}$. Therefore, (5) implies, due to the monotonicity and convexity of $\ell_{e,j}$, that $g_{e,j}^{(t)} < \ell_{e,j}(f_e) + x_{e,j}\ell'_{e,j}(f_e)$. Since $e \in S_j^{(t-2)}$, the right hand side of the latter quantity equals $g^{(t-2)}$ and therefore, $g_{e,j}^{(t)} < g^{(t-2)}$. This is a contradiction, since $g_{e,j}^{(t)} \geq g^{(t-2)}$ for every $e \in E$.

It follows that for every $e \in E^{(t)-}$ it holds $|\Delta_e^{(t)}| < |\Delta_e^{(t-1)}|$, i.e. $|\Delta_e^{(t)}| = (1-\lambda_e)\,|\Delta_e^{(t-1)}|$, where $\lambda_e \in (0,1]$. By substituting this in the inequality $g_{e,j}^{(t)} \geq g^{(t-2)} = \ell_{e,j}(f_e) + x_{e,j}\ell'_{e,j}(f_e)$, we obtain from (5), since $\Delta_e^{(t)} = -|\Delta_e^{(t)}|$ and $\Delta_e^{(t-1)} = |\Delta_e^{(t-1)}|$, that

$$\begin{aligned}
&\ell_{e,j}(f_e + \lambda_e|\Delta_e^{(t-1)}|) \\
&\quad + (x_{e,j} + (\lambda_e - 1)|\Delta_e^{(t-1)}|)\ell'_{e,j}(f_e + \lambda_e|\Delta_e^{(t-1)}|) \\
&\quad \geq \ell_{e,j}(f_e) + x_{e,j}\ell'_{e,j}(f_e)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&[(x_{e,j} + \lambda_e|\Delta_e^{(t-1)}|)\ell_{e,j}(f_e + \lambda_e|\Delta_e^{(t-1)}|)]' \\
&\qquad -[x_{e,j}\ell_{e,j}(f_e)]' \geq |\Delta_e^{(t-1)}|\ell'_{e,j}(f_e + \lambda_e|\Delta_e^{(t-1)}|) \\
&\qquad \geq |\Delta_e^{(t-1)}| \inf_{0 \leq x \leq d_1 + d_2} \{\ell'_{e,j}(x)\}
\end{aligned}$$

The left hand side of the latter inequality is at most as

$$\lambda_e|\Delta_e^{(t-1)}| \sup_{\substack{0 \leq x \leq d_1 \\ 0 \leq y \leq d_2}} \{[x\ell_{e,j}(x+y)]''\}$$

from which it follows due to (3) that $\lambda_e \geq \lambda$. Thus, since $|\Delta_e^{(t)}| = (1-\lambda_e)|\Delta_e^{(t-1)}|$ holds for every $e \in E^{(t)-}$ and since $E^{(t)-} \subseteq E^{(t-1)+}$, it holds that

$$\sum_{e \in E^{(t)-}} |\Delta_e^{(t)}| \leq (1-\lambda) \sum_{E^{(t-1)+}} |\Delta_e^{(t-1)}| \qquad (6)$$

Now, the lemma follows from (2) and (6).

Case 2. Suppose that $g^{(t)} < g^{(t-2)}$. Consider an arbitrary $e \in E^{(t)+}$. Similarly, since $\Delta_e^{(t)} > 0$, player j allocates on e a positive amount of flow after the execution of step t, i.e. $e \in S_j^{(t)}$. It follows that his marginal cost on e after step t is equal to $g^{(t)}$. On the other side, since $\Delta_e^{(t)} > 0$, his marginal cost on e after the execution of step $t-1$ was less than $g^{(t)}$ and therefore less that $g^{(t-2)}$, due to the assumption that $g^{(t)} < g^{(t-2)}$. It follows that the total flow f_e has been decreased during step $t-1$ by the other player, i.e. $e \in E^{(t-1)-}$. It follows that $E^{(t)+} \subseteq E^{(t-1)-} \cap S_j^{(t)}$.

Since $e \in S_j^{(t)}$, it holds that $g_{e,j}^{(t)} = g^{(t)}$. Furthermore, since $g^{(t-2)} \le g_{e,j}^{(t-2)}$ holds for every $e \in E$ and due to the assumption that $g^{(t)} < g^{(t-2)}$, we obtain that $g_{e,j}^{(t)} < g_{e,j}^{(t-2)}$. Suppose now that $|\Delta_e^{(t)}| \ge |\Delta_e^{(t-1)}|$ for some link $e \in E^{(t)+}$. Then, since $\Delta_e^{(t)} > 0$ and $\Delta_e^{(t-1)} < 0$, it holds that $x_{e,j} + \Delta_e^{(t)} > x_{e,j} + \Delta_e^{(t-1)} + \Delta_e^{(t)} \ge x_{e,j}$. Therefore, (5) implies, due to the monotonicity and convexity of $\ell_{e,j}$, that $g_{e,j}^{(t)} > \ell_{e,j}(f_e) + x_{e,j}\ell'_{e,j}(f_e) = g_{e,j}^{(t-2)}$. This is a contradiction, since $g_{e,j}^{(t)} < g_{e,j}^{(t-2)}$ for every $e \in E^{(t)+}$.

It follows that for every $e \in E^{(t)+}$ it holds $|\Delta_e^{(t)}| < |\Delta_e^{(t-1)}|$, i.e. $|\Delta_e^{(t)}| = (1-\lambda_e)\,|\Delta_e^{(t-1)}|$, where $\lambda_e \in (0,1]$. By substituting this in the inequality $g_{e,j}^{(t)} < g_{e,j}^{(t-2)} = \ell_{e,j}(f_e) + x_{e,j}\ell'_{e,j}(f_e)$, we obtain from (5), since $\Delta_e^{(t)} = |\Delta_e^{(t)}|$ and $\Delta_e^{(t-1)} = -|\Delta_e^{(t-1)}|$, that

$$\begin{aligned}
&\ell_{e,j}(f_e - \lambda_e|\Delta_e^{(t-1)}|) \\
&\quad + (x_{e,j} + (1-\lambda_e)|\Delta_e^{(t-1)}|)\ell'_{e,j}(f_e - \lambda_e|\Delta_e^{(t-1)}|) \\
&\quad < \ell_{e,j}(f_e) + x_{e,j}\ell'_{e,j}(f_e)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&[x_{e,j}\ell_{e,j}(f_e)]' \\
&\quad -[(x_{e,j} - \lambda_e|\Delta_e^{(t-1)}|)\ell'_{e,j}(f_e - \lambda_e|\Delta_e^{(t-1)}|)]' \\
&\quad > |\Delta_e^{(t-1)}|\ell'_{e,j}(f_e - \lambda_e|\Delta_e^{(t-1)}|) \\
&\quad \ge |\Delta_e^{(t-1)}| \inf_{0 \le x \le d_1 + d_2}\{\ell'_{e,j}(x)\}
\end{aligned}$$

The left hand side of the latter inequality is at most as

$$\lambda_e|\Delta_e^{(t-1)}| \sup_{\substack{0 \le x \le d_1 \\ 0 \le y \le d_2}} \{[x\ell_{e,j}(x+y)]''\}$$

from which it follows due to (3) that $\lambda_e > \lambda$. Thus, since $|\Delta_e^{(t)}| = (1-\lambda_e)|\Delta_e^{(t-1)}|$ holds for every $e \in E^{(t)+}$ and since $E^{(t)+} \subseteq E^{(t-1)-}$, it holds that

$$\sum_{e \in E^{(t)+}} |\Delta_e^{(t)}| < (1-\lambda) \sum_{E^{(t-1)-}} |\Delta_e^{(t-1)}| \qquad (7)$$

Now, the lemma follows from (2) and (7). □

3. Series-parallel networks

3.1. Notation and terminology

In this section we extent our model to a series-parallel network G with source u, destination v and m edges, which is a generalization of the network presented in Section 2. We remind here the definition of such a network.

Definition 3 (Series-parallel network): A *series-parallel network* G is a directed network with a *source* u and a *destination* v that is defined recursively as follows:

1) The *primitive* series-parallel network consists of a source u, a destination v and a single directed edge from u to v.
2) The *parallel composition* $P = P(G_1, G_2)$ of the series-parallel networks G_1 and G_2 is the network created from the disjoint union of G_1 and G_2 by merging the sources and destinations of them to create the source and the destination of P, respectively.
3) The *series composition* $S = S(G_1, G_2)$ of the series-parallel networks G_1 and G_2 is the network created from the disjoint union of G_1 and G_2 by merging the destination of G_1 and the source of G_2. The source of S is then the source of G_1 and its destination is the destination of G_2.

Similarly to Section 2, we use here the following notation for a series-parallel network G with a set E of m edges. Denote by $\mathcal{P}_G$ the set of directed paths from u to v and by $S_j^{(t)} \subseteq \mathcal{P}_G$ the support of player j after the execution of step t, i.e. the set of paths on which player j allocates a positive amount of flow. The marginal cost of player $j(t)$ on a path $P \in \mathcal{P}_G$ after the execution of step s is denoted by $g_{P,j(t)}^{(s)} = \sum_{e \in P} g_{e,j(t)}^{(s)}$, where $g_{e,j(t)}^{(s)}$ denotes his marginal cost on edge $e \in E$ after step s, as in Section 2. Since the cost of player $j(t)$ is minimized at step t, it follows that his marginal cost $g_{P,j(t)}^{(t)}$ is equal to a quantity $g_G^{(t)}$ on every path $P \in S_{j(t)}^{(t)}$, while the marginal cost on every of the remaining paths is at least $g_G^{(t)}$.

3.2. The potential function

Denote now by $\Delta G^{(t)}$ the difference of the flows on G between steps t and $t-1$. This is a flow with value $\Delta_e^{(t)}$ on edge $e \in E$. Let furthermore $\mathcal{P}_G^{(t)} = \{P_i^{(t)}\}_{i \in I_G^{(t)}}$ be a path decomposition of the flow $\Delta G^{(t)}$ in directed paths from u to v, where $I_G^{(t)}$ is an appropriate index set. Denote the flow on $P_i^{(t)}$ by $f_i^{(t)}$. W.l.o.g. suppose that $\sum_{i \in I_G^{(t)}} |f_i^{(t)}|$ is minimum among all path decompositions of $\Delta G^{(t)}$. Then, the potential at step t is defined as

$$\Phi_G(t) := \sum_{i \in I_G^{(t)}} |f_i^{(t)}| \geq 0 \tag{8}$$

which equals the amount of flow that is reallocated in G during step t. Define now the subsets $I_G^{(t)+} := \{i \in I_G^{(t)} : f_i^{(t)} > 0\}$ and $I_G^{(t)-} := \{i \in I_G^{(t)} : f_i^{(t)} < 0\}$ of the index set $I_G^{(t)}$. In the case where G is a network of parallel links, the paths correspond to the links. Thus, the potential function of (8) degenerates to that of (1) and the sets $I_G^{(t)+}$ and $I_G^{(t)-}$ correspond to the sets $E^{(t)+}$ and $E^{(t)-}$ of Section 2, respectively. Since the demands of the players remain constant, it holds similarly to (2) that

$$\Phi_G(t) = 2 \sum_{i \in I_G^{(t)+}} |f_i^{(t)}| = 2 \sum_{i \in I_G^{(t)-}} |f_i^{(t)}| \tag{9}$$

for every $t \in \mathbb{N}$. The following lemma shows that also in this case the potential $\Phi_G(t)$ decreases strictly at every step $t \geq 3$.

Lemma 4: It holds that $\Phi_G(t) \leq (1-\lambda)\ \Phi_G(t-1)$ for every $t \geq 3$.

Proof: The proof will be done by induction on the structure of G. If G is a network of parallel links, then the lemma follows from Lemma 2.

Suppose first that $G = S(G_1, G_2)$ for some series-parallel networks G_1, G_2. The paths of $\mathcal{P}_{G_1}^{(t)}$ and $\mathcal{P}_{G_2}^{(t)}$ cover the whole flow in $\Delta G_1^{(t)}$ and $\Delta G_2^{(t)}$, while $\Phi_{G_1}(t), \Phi_{G_2}(t)$ denote the sum of the absolute flows on the paths of $\mathcal{P}_{G_1}^{(t)}$ and $\mathcal{P}_{G_2}^{(t)}$, respectively. Due to the definition, the values $\Phi_{G_1}(t)$ and $\Phi_{G_2}(t)$ are minimum among all path decompositions of $\Delta G_1^{(t)}$ and $\Delta G_2^{(t)}$, respectively. W.l.o.g. it holds that $\Phi_{G_1}(t) \geq \Phi_{G_2}(t)$. Then, extend every path P of $\mathcal{P}_{G_2}^{(t)}$ by some paths of $\mathcal{P}_{G_1}^{(t)}$ of the same total flow with P. We cover this way the whole $\Delta G_2^{(t)}$ and a part of $\Delta G_1^{(t)}$ with paths, such that their absolute flows sum up to $\Phi_{G_2}(t)$. The sum of the absolute flows on the remaining paths of $\Delta G_1^{(t)}$ equals $\Phi_{G_1}(t) - \Phi_{G_2}(t)$. We extend all these paths of $\Delta G_1^{(t)}$ by a single path P_0 of G_2, covering thus the whole $\Delta G^{(t)}$ with paths of total absolute value $\Phi_{G_1}(t)$. It follows that

$$\Phi_G(t) = \max\{\Phi_{G_1}(t), \Phi_{G_2}(t)\} \tag{10}$$

for every $t \in \mathbb{N}$. Now, the induction hypothesis implies that

$$\Phi_{G_1}(t) \leq (1-\lambda)\Phi_{G_1}(t-1) \tag{11}$$

and

$$\Phi_{G_2}(t) \leq (1-\lambda)\Phi_{G_2}(t-1) \tag{12}$$

The lemma follows from (10), (11) and (12).

Suppose now that $G = P(G_1, G_2)$. The networks G_1 and G_2 do not share any common edges or paths. Thus, since the cost of player $j(t)$ is minimized at step t, it holds that $g_{G_1}^{(t)} = g_{G_2}^{(t)} = g_G^{(t)}$. We distinguish the following cases.

Case 1. Suppose that $g_G^{(t)} \geq g_G^{(t-2)}$. Due to Case 1 in the proof of Lemma 2, the induction hypothesis implies for both components G_1, G_2 of G that

$$\sum_{i \in I_{G_k}^{(t)-}} |f_i^{(t)}| \leq (1-\lambda) \sum_{i \in I_{G_k}^{(t-1)+}} |f_i^{(t-1)}|,\ k \in \{1,2\} \tag{13}$$

since $g_{G_1}^{(t)} \geq g_{G_1}^{(t-2)}$ and $g_{G_2}^{(t)} \geq g_{G_2}^{(t-2)}$. By adding the inequalities of (13) for both $k \in \{1,2\}$, we obtain

$$\sum_{i \in I_G^{(t)-}} |f_i^{(t)}| \leq (1-\lambda) \sum_{i \in I_G^{(t-1)+}} |f_i^{(t-1)}| \tag{14}$$

The lemma follows from (9) and (14).

Case 2. Suppose that $g_G^{(t)} < g_G^{(t-2)}$. Similarly, due to Case 2 in the proof of Lemma 2, the induction hypothesis implies for both components G_1, G_2 of G that

$$\sum_{i \in I_{G_k}^{(t)+}} |f_i^{(t)}| < (1-\lambda) \sum_{i \in I_{G_k}^{(t-1)-}} |f_i^{(t-1)}|,\ k \in \{1,2\} \tag{15}$$

since $g_{G_1}^{(t)} \leq g_{G_1}^{(t-2)}$ and $g_{G_2}^{(t)} \leq g_{G_2}^{(t-2)}$. By adding the inequalities of (15) for both $k \in \{1,2\}$, we obtain

$$\sum_{i \in I_G^{(t)+}} |f_i^{(t)}| < (1-\lambda) \sum_{i \in I_G^{(t-1)-}} |f_i^{(t-1)}| \tag{16}$$

Now, the lemma follows from (9) and (16). □

3.3. Proof of Theorem 1

Proof: Recall first that the demands d_1, d_2 are scaled in the interval $(0,1]$. Denote now $\lambda_0 = 1 - \lambda$. The flow $\Delta G^{(t)}$ can be decomposed in at most m paths in $\mathcal{P}_G^{(t)}$ with non-zero flow. Since player 2 moves at the second step of the game, it holds that $|f_i^{(2)}| \leq d_2 \leq 1$ for every path $P_i^{(2)}$ of $\mathcal{P}_G^{(2)}$. It follows that $\Phi_G(2) \leq m$. Due to Lemma 4, it holds that $\Phi_G(t) \leq \lambda_0^{t-2}\Phi_G(2) \leq \lambda_0^{t-2}m$ for every $t \geq 3$. Now, suppose that $\lambda_0^{t-2}m \leq \varepsilon$, for some $\varepsilon > 0$. It follows then that $\lambda_0^{-t+2} \geq m/\varepsilon$. Therefore, after

$$t(\varepsilon) := \lceil \log^{-1}(\lambda_0^{-1}) \cdot \log(m/\varepsilon) \rceil + 2 \tag{17}$$

steps the potential Φ_G is at most ε, for any given $\varepsilon > 0$. Thus, since $\lambda_0 < 1$ is a constant and since the potential equals the 1-norm of the difference between the configurgation vectors in two consecutive steps, the game converges in a logarithmic number of steps to a NEP and the theorem follows. □

3.4. Tight bounds

Due to (4) it holds that $\lambda_0 = 1 - \lambda \in [\frac{1}{2}, 1)$ in the proof of Theorem 1. Thus, $\log^{-1}(\lambda_0^{-1}) \geq 1$ and for the convergence time in (17) it holds that

$$t(\varepsilon) \geq \lceil \log(m/\varepsilon) \rceil + 2 \tag{18}$$

Consider now the special case that the player-specific latency functions are affine, i.e. for every $e \in E$ and $j \in \{1,2\}$ it holds that $\ell_{e,j}(x) = \alpha_{e,j} x + \beta_{e,j}$, with $\alpha_{e,j} > 0$ and $\beta_{e,j} \geq 0$. Then, directly substitution in (3) implies that

$$\lambda = \min_{j \in \{1,2\}, e \in E} \left\{ \frac{\alpha_{e,j}}{2\alpha_{e,j}} \right\} = \frac{1}{2} \tag{19}$$

Therefore, $\lambda_0 = \frac{1}{2}$ and $\log^{-1}(\lambda_0^{-1}) = 1$. It follows that in this case equality holds in (18), which shows that our analysis is tight.

4. Concluding remarks

This paper investigates the selfish routing of two players in a series-parallel network. Each player controls a demand of flow, which can be splitted arbitrarily on the available paths between the source and the destination. The main result is the convergence of the best response dynamics to a Nash equilibrium in a logarithmic number of steps, starting at an arbitrary initial configuration. The generalization of this result to the case of an arbitrary network, as well as to the atomic game with several players activated in a round robin fashion, remains an important open question for further research.

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