A New Intersection Model and Improved Algorithms for Tolerance Graphs

George B. Mertzios[1], Ignasi Sau[2], and Shmuel Zaks[3]

[1] Department of Computer Science, RWTH Aachen University, Germany.
mertzios@cs.rwth-aachen.de
[2] Mascotte joint Project of INRIA/CNRS/UNSA, Sophia-Antipolis, France; and Graph Theory and Combinatorics Group, Applied Maths. IV Dept. of UPC, Barcelona, Spain. ignasi.sau@sophia.inria.fr
[3] Department of Computer Science, Technion, Haifa, Israel.
zaks@cs.technion.ac.il

Abstract. Tolerance graphs model interval relations in such a way that intervals can tolerate a certain degree of overlap without being in conflict. This class of graphs, which generalizes in a natural way both interval and permutation graphs, has attracted many research efforts since their introduction in [9], as it finds many important applications in constraint-based temporal reasoning, resource allocation, and scheduling problems, among others. In this article we propose the first non-trivial intersection model for general tolerance graphs, given by three-dimensional parallelepipeds, which extends the widely known intersection model of parallelograms in the plane that characterizes the class of bounded tolerance graphs. Apart from being important on its own, this new representation also enables us to improve the time complexity of three problems on tolerance graphs. Namely, we present optimal $\mathcal{O}(n \log n)$ algorithms for computing a minimum coloring and a maximum clique, and an $\mathcal{O}(n^2)$ algorithm for computing a maximum weight independent set in a tolerance graph with n vertices, thus improving the best known running times $\mathcal{O}(n^2)$ and $\mathcal{O}(n^3)$ for these problems, respectively.

Keywords: Tolerance graphs, parallelogram graphs, intersection model, minimum coloring, maximum clique, maximum weight independent set.

1 Introduction

A graph $G = (V, E)$ on n vertices is a *tolerance graph* if there is a set $I = \{I_i \mid i = 1, \ldots, n\}$ of closed intervals on the real line and a set $T = \{t_i \mid i = 1, \ldots, n\}$ of positive real numbers, called *tolerances*, such that for any two vertices $v_i, v_j \in V$, $v_i v_j \in E$ if and only if $|I_i \cap I_j| \geq \min\{t_i, t_j\}$, where $|I|$ denotes the length of the interval I. These sets of intervals and tolerances form a *tolerance representation* of G. If G has a tolerance representation such that $t_i \leq |I_i|$ for $i = 1, \ldots, n$, then G is called a *bounded tolerance graph* and its representation is a *bounded tolerance representation.*

Tolerance graphs were introduced in [9], mainly motivated by the need to solve scheduling problems in which resources that would be normally used exclusively, like rooms or vehicles, can tolerate some sharing among users. Since then, tolerance graphs have been widely studied in the literature [1, 2, 5, 10, 11, 14, 16, 20], as they naturally generalize both interval graphs (when all tolerances are equal) and permutation graphs (when $|I_i| = t_i$ for $i = 1, \ldots, n$) [9]. For more details, see [12].

Notation. All the graphs considered in this paper are finite, simple, and undirected. Given a graph $G = (V, E)$, we denote by n the cardinality of V. An edge between vertices u and v is denoted by uv, and in this case vertices u and v are said to be *adjacent.* $\overline{G}$ denotes the *complement* of G, i.e. $\overline{G} = (V, \overline{E})$, where $uv \in \overline{E}$ if and only if $uv \notin E$. Given a subset of vertices $S \subseteq V$, the graph $G[S]$ denotes the graph *induced* by the vertices in S, i.e. $G[S] = (S, F)$, where for any two vertices $u, v \in S$, $uv \in F$ if and only if $uv \in E$. A subset $S \subseteq V$ is an *independent set* in G if the graph $G[S]$ has no edges. For a subset $K \subseteq V$, the induced subgraph $G[K]$ is a *complete subgraph* of G, or a *clique*, if each two of its vertices are adjacent (equivalently, K is an independent set in $\overline{G}$). The maximum cardinality of a clique in G is denoted by $\omega(G)$ and is termed the *clique number* of G. A *proper coloring* of G is an assignment of different colors to adjacent vertices, which results in a partition of V into independent sets. The minimum number of colors for which there exists a proper coloring is denoted by $\chi(G)$ and is termed the *chromatic number* of G. A partition of V into $\chi(G)$ independent sets is a *minimum coloring* of G.

Motivation and previous work. Besides generalizing interval and permutation graphs in a natural way, the class of tolerance graphs has other important subclasses and superclasses. Let us briefly survey some of them.

A graph is *perfect* if the chromatic number of every induced subgraph equals the clique number of that subgraph. Perfect graphs include many important families of graphs, and serve to unify results relating colorings and cliques in those families. For instance, in all perfect graphs, the graph coloring problem, maximum clique problem, and maximum independent set problem can all be solved in polynomial time using the Ellipsoid method [13]. Since tolerance graphs were shown to be perfect [10], there exist polynomial time algorithms for these problems. However, these algorithms are not very efficient and therefore, as it happens for most known subclasses of perfect graphs, it makes sense to devise specific fast algorithms for these problems on tolerance graphs.

A *comparability* graph is a graph which can be transitively oriented. A *co-comparability* graph is a graph whose complement is a comparability graph. Bounded tolerance graphs are co-comparability graphs [9], and therefore all known polynomial time algorithms for co-comparability graphs apply to bounded tolerance graphs. This is one of the main reasons why for many problems the existing algorithms have better running time in bounded tolerance graphs than in general tolerance graphs.

A graph $G = (V, E)$ is the *intersection graph* of a family $F = \{S_1, \ldots, S_n\}$ of distinct nonempty subsets of a set S if there exists a bijection $\mu : V \to F$ such that for any two distinct vertices $u, v \in V$, $uv \in E$ if and only if $\mu(u) \cap \mu(v) \neq \emptyset$. In that case, we say that F is an *intersection model* of G. It is easy to see that each graph has a trivial intersection model based on adjacency relations [18]. Some intersection models provide a natural and intuitive understanding of the structure of a class of graphs, and turn out to be very helpful to find efficient algorithms to solve optimization problems [18]. Therefore, it is of great importance to establish non-trivial intersection models for families of graphs. A graph G on n vertices is a *parallelogram graph* if we can fix two parallel lines L_1 and L_2, and for each vertex $v_i \in V(G)$ we can assign a parallelogram $\overline{P}_i$ with parallel sides along L_1 and L_2 so that G is the intersection graph of $\{\overline{P}_i \mid i = 1, \ldots, n\}$. It was proved in [1, 17] that a graph is a bounded tolerance graph if and only if it is a parallelogram graph. This characterization provides a useful way to think about bounded tolerance graphs. However, this intersection model cannot cope with general tolerance graphs, in which the tolerance of an interval can be greater than its length.

Our contribution. In this article we present the first non-trivial intersection model for general tolerance graphs, which generalizes the widely known parallelogram representation of bounded tolerance graphs. The main idea is to exploit the third dimension to capture the information given by unbounded tolerances, and as a result parallelograms are replaced with parallelepipeds. The proposed intersection model is very intuitive and can be efficiently constructed from a tolerance representation (actually, we show that it can be constructed in linear time).

Apart from being important on its own, this new representation proves to be a powerful tool for designing efficient algorithms for general tolerance graphs. Indeed, using our intersection model we improve the best existing running times of three problems on tolerance graphs. We present algorithms to find a minimum coloring and a maximum clique in $\mathcal{O}(n \log n)$ time, which turns out to be optimal. The best existing algorithm was $\mathcal{O}(n^2)$ [11, 12]. We also present an algorithm to find a maximum weight independent set in $\mathcal{O}(n^2)$ time, whereas the best known algorithm was $\mathcal{O}(n^3)$ [12]. We note that [20] proposes an $\mathcal{O}(n^2 \log n)$ algorithm to find a maximum *cardinality* independent set on a general tolerance graph, and that [12] refers to an algorithm transmitted by personal communication with running time $\mathcal{O}(n^2 \log n)$ to find a maximum weight independent set on a general tolerance graph; to the best of our knowledge, this algorithm has not been published.

It is important to note that the complexity of recognizing bounded and general tolerance graphs is a challenging open problem [3,12,20], and this is the reason why we assume throughout this paper that along with the input tolerance graph we are also given a tolerance representation of it. The only "positive" result in the literature concerning recognition of tolerance graphs is a linear time algorithm for the recognition of bipartite tolerance graphs [3].

Organization of the paper. We provide the new intersection model of general tolerance graphs in Section 2. In Section 3 we present a canonical representation of tolerance graphs, and then show how it can be used in order to obtain optimal $\mathcal{O}(n \log n)$ algorithms for finding a minimum coloring and a maximum clique in a tolerance graph. In Section 4 we present an $\mathcal{O}(n^2)$ algorithm for finding a maximum weight independent set. Finally, Section 5 is devoted to conclusions and open problems. Some proofs have been omitted due to space limitations; a full version can be found in [19].

2 A New Intersection Model for Tolerance Graphs

One of the most natural representations of bounded tolerance graphs is given by parallelograms between two parallel lines in the Euclidean plane [1, 12, 17]. In this section we extend this representation to a three-dimensional representation of general tolerance graphs.

Given a tolerance graph $G = (V, E)$ along with a tolerance representation of it, recall that vertex $v_i \in V$ corresponds to an interval $I_i = [a_i, b_i]$ on the real line with a tolerance $t_i \geq 0$. W.l.o.g. we may assume that $t_i > 0$ for every vertex v_i [12].

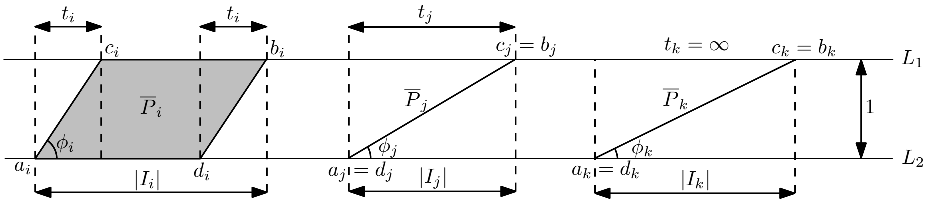

Fig. 1. Parallelograms $\overline{P}_i$ and $\overline{P}_j$ correspond to bounded vertices v_i and v_j, respectively, whereas $\overline{P}_k$ corresponds to an unbounded vertex v_k.

Definition 1. *Given a tolerance representation of a tolerance graph $G = (V, E)$, vertex v_i is* bounded *if $t_i \leq |I_i|$. Otherwise, v_i is* unbounded. *V_B and V_U are the sets of bounded and unbounded vertices in V, respectively. Clearly $V = V_B \cup V_U$.*

We can also assume w.l.o.g. that $t_i = \infty$ for any unbounded vertex v_i, since if v_i is unbounded, then the intersection of any other interval with I_i is strictly smaller than t_i. Let L_1 and L_2 be two parallel lines at distance 1 in the Euclidean plane.

Definition 2. *Given an interval $I_i = [a_i, b_i]$ with tolerance t_i, $\overline{P}_i$ is the parallelogram defined by the points c_i, b_i in L_1 and a_i, d_i in L_2, where $c_i = \min\{b_i, a_i + t_i\}$ and $d_i = \max\{a_i, b_i - t_i\}$. The* slope *$\phi_i$ of $\overline{P}_i$ is $\phi_i = \arctan\left(\frac{1}{c_i - a_i}\right)$.*

An example is depicted in Figure 1, where $\overline{P}_i$ and $\overline{P}_j$ correspond to bounded vertices v_i and v_j, and $\overline{P}_k$ corresponds to an unbounded vertex v_k. Observe that when vertex v_i is bounded, the values c_i and d_i coincide with the *tolerance points* defined in [7, 12, 15], and $\phi_i = \arctan\left(\frac{1}{t_i}\right)$. On the other hand, when vertex v_i is unbounded, the values c_i and d_i coincide with the endpoints b_i and a_i of I_i, respectively, and $\phi_i = \arctan\left(\frac{1}{|I_i|}\right)$. Observe also that in both cases $t_i = b_i - a_i$ and $t_i = \infty$, parallelogram $\overline{P}_i$ is reduced to a line segment (c.f. $\overline{P}_j$ and $\overline{P}_k$ in Figure 1). Since $t_i > 0$ for every vertex v_i, it follows that $0 < \phi_i < \frac{\pi}{2}$. Furthermore, we can assume w.l.o.g. that all points a_i, b_i, c_i, d_i and all slopes ϕ_i are distinct [7, 12, 15].

Observation 1 *Let $v_i \in V_U, v_j \in V_B$. Then $|I_i| < t_j$ if and only if $\phi_i > \phi_j$.*

We are ready to give the main definition of this article.

Definition 3. *Let $G = (V, E)$ be a tolerance graph with a tolerance representation $\{I_i = [a_i, b_i], t_i \mid i = 1, \ldots, n\}$. For every $i = 1 \ldots, n$, P_i is the parallelepiped in $\mathbb{R}^3$ defined as follows:*

(a) If $t_i \leq b_i - a_i$ (that is, v_i is bounded), then $P_i = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \overline{P}_i,\ 0 \leq z \leq \phi_i\}$.
(b) If $t_i > b_i - a_i$ (v_i is unbounded), then $P_i = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \overline{P}_i,\ z = \phi_i\}$.

The set of parallelepipeds $\{P_i \mid i = 1, \ldots, n\}$ is a parallelepiped representation *of G.*

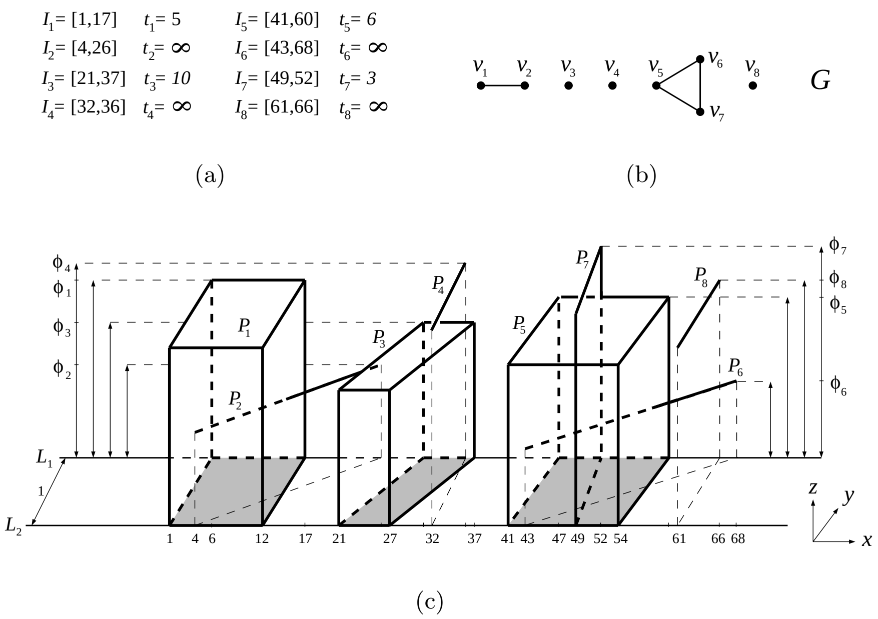

Fig. 2. The intersection model for tolerance graphs: (a) a set of intervals $I_i = [a_i, b_i]$ and tolerances t_i, $i = 1, \ldots, 8$, (b) the corresponding tolerance graph G and (c) a parallelepiped representation of G.

Observe that for each interval I_i, the parallelogram $\overline{P}_i$ of Definition 2 (see also Figure 1) coincides with the projection of the parallelepiped P_i on the plane $z = 0$. An example of the construction of these parallelepipeds is given in Figure 2, where a set of eight intervals with their associated tolerances is given in Figure 2(a). The corresponding tolerance graph

G is depicted in Figure 2(b), while the parallelepiped representation is illustrated in Figure 2(c). In the case $t_i < b_i - a_i$, the parallelepiped P_i is three-dimensional, c.f. P_1, P_3, and P_5, while in the border case $t_i = b_i - a_i$ it degenerates to a two-dimensional rectangle, c.f. P_7. In these two cases, each P_i corresponds to a bounded vertex v_i. In the remaining case $t_i = \infty$ (that is, v_i is unbounded), the parallelepiped P_i degenerates to a one-dimensional line segment above plane $z = 0$, c.f. P_2, P_4, P_6, and P_8.

We prove now that these parallelepipeds form a three-dimensional intersection model for the class of tolerance graphs (namely, that every tolerance graph G can be viewed as the intersection graph of the corresponding parallelepipeds P_i).

Theorem 1. *Let $G = (V, E)$ be a tolerance graph with a tolerance representation $\{I_i = [a_i, b_i], t_i \mid i = 1, \ldots, n\}$. Then for every $i \neq j$, $v_i v_j \in E$ if and only if $P_i \cap P_j \neq \emptyset$.*

Proof. We distinguish three cases according to whether vertices v_i and v_j are bounded or unbounded:

(a) Both vertices are bounded, that is $t_i \leq b_i - a_i$ and $t_j \leq b_j - a_j$. It follows that $v_i v_j \in E(G)$ if and only if $\overline{P}_i \cap \overline{P}_j \neq \emptyset$ [12]. However, due to the definition of the parallelepipeds P_i and P_j, in this case $P_i \cap P_j \neq \emptyset$ if and only if $\overline{P}_i \cap \overline{P}_j \neq \emptyset$ (c.f. P_1 and P_3, or P_5 and P_7, in Figure 2).

(b) Both vertices are unbounded, that is $t_i = t_j = \infty$. Since no two unbounded vertices are adjacent, $v_i v_j \notin E(G)$. On the other hand, the line segments P_i and P_j lie on the disjoint planes $z = \phi_i$ and $z = \phi_j$ of $\mathbb{R}^3$, respectively, since we assumed that the slopes ϕ_i and ϕ_j are distinct. Thus, $P_i \cap P_j = \emptyset$ (c.f. P_2 and P_4).

(c) One vertex is unbounded (that is, $t_i = \infty$) and the other is bounded (that is, $t_j \leq b_j - a_j$). If $\overline{P}_i \cap \overline{P}_j = \emptyset$, then $v_i v_j \notin E$ and $P_i \cap P_j = \emptyset$ (c.f. P_1 and P_6). Suppose that $\overline{P}_i \cap \overline{P}_j \neq \emptyset$. We distinguish two cases:

  (i) $\phi_i < \phi_j$. It is easy to check that $|I_i \cap I_j| \geq t_j$ and thus $v_i v_j \in E$. Since $\overline{P}_i \cap \overline{P}_j \neq \emptyset$ and $\phi_i < \phi_j$, then necessarily the line segment P_i intersects with the parallelepiped P_j on the plane $z = \phi_i$, and thus $P_i \cap P_j \neq \emptyset$ (c.f. P_1 and P_2).

  (ii) $\phi_i > \phi_j$. Clearly $|I_i \cap I_j| < t_i = \infty$. Furthermore, since $\phi_i > \phi_j$, Observation 1 implies that $|I_i \cap I_j| \leq |I_i| < t_j$. It follows that $|I_i \cap I_j| < \min\{t_i, t_j\}$, and thus $v_i v_j \notin E$. On the other hand, $z = \phi_i$ for all points $(x, y, z) \in P_i$, while $z \leq \phi_j < \phi_i$ for all points $(x, y, z) \in P_j$, and therefore $P_i \cap P_j = \emptyset$ (c.f. P_3 and P_4). ■

Clearly, for each $v_i \in V$ the parallelepiped P_i can be constructed in constant time. Therefore, given a tolerance representation of a tolerance graph G with n vertices, a parallelepiped representation of G can be constructed in $\mathcal{O}(n)$ time.

3 Coloring and Clique Algorithms in $\mathcal{O}(n \log n)$

In this section we present optimal $\mathcal{O}(n \log n)$ algorithms for constructing a minimum coloring and a maximum clique in a tolerance graph $G = (V, E)$ with n vertices, given a parallelepiped representation of G. These algorithms improve the best known running time $\mathcal{O}(n^2)$ of these problems on tolerance graphs [11, 12]. First, we introduce a canonical representation of tolerance graphs in Section 3.1, and then we use it to obtain the algorithms for the minimum coloring and the maximum clique problems in Section 3.2.

3.1 A canonical representation of tolerance graphs

We associate with every vertex v_i of G the point $p_i = (x_i, y_i)$ in the Euclidean plane, where $x_i = b_i$ and $y_i = \frac{\pi}{2} - \phi_i$. Since all endpoints of the parallelograms $\overline{P}_i$ and all slopes ϕ_i are distinct, all coordinates of the points p_i are distinct as well. Similarly to [11, 12], we state the following two definitions.

Definition 4. *An unbounded vertex $v_i \in V_U$ of a tolerance graph G is called* inevitable *(for a certain parallelepiped representation), if replacing P_i with $\{(x, y, z) \mid (x, y) \in P_i, 0 \leq z \leq \phi_i\}$ creates a new edge in G. Otherwise, v_i is called* evitable.

Definition 5. *Let $v_i \in V_U$ be an inevitable unbounded vertex of a tolerance graph G (for a certain parallelepiped representation). A vertex v_j is called a* hovering vertex *of v_i if $a_j < a_i$, $b_i < b_j$, and $\phi_i > \phi_j$.*

It is now easy to see that, by Definition 5, if v_j is a hovering vertex of v_i, then $v_i v_j \notin E$. Note that, in contrast to [11], in Definition 4, an isolated vertex v_i might be also inevitable unbounded, while in Definition 5, a hovering vertex might be also unbounded. Definitions 4 and 5 imply the following lemma:

Lemma 1. *Let $v_i \in V_U$ be an inevitable unbounded vertex of the tolerance graph G (for a certain parallelepiped representation). Then, there exists a hovering vertex v_j of v_i.*

Definition 6. *A parallelepiped representation of a tolerance graph G is called* canonical *if every unbounded vertex is inevitable.*

For example, in the tolerance graph depicted in Figure 2, v_4 and v_8 are inevitable unbounded vertices, v_3 and v_6 are hovering vertices of v_4 and v_8, respectively, while v_2 and v_6 are evitable unbounded vertices. Therefore, this representation is not canonical for the graph G. However, if we replace P_i with $\{(x, y, z) \mid (x, y) \in P_i, 0 \leq z \leq \phi_i\}$ for $i = 2, 6$, we get a canonical representation for G.

In the following, we present an algorithm that constructs a canonical representation of a given tolerance graph G.

Definition 7. *Let $\alpha = (x_\alpha, y_\alpha)$ and $\beta = (x_\beta, y_\beta)$ be two points in the plane. Then α* dominates *β if $x_\alpha > x_\beta$ and $y_\alpha > y_\beta$. Given a set A of points, the point $\gamma \in A$ is called an* extreme point *of A if there is no point $\delta \in A$ that dominates γ. $Ex(A)$ is the set of the extreme points of A.*

Algorithm 1 Construction of a canonical representation of a tolerance graph G

Input: A parallelepiped representation R of a given tolerance graph G with n vertices
Output: A canonical representation R' of G

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Sort the vertices of G, such that a_i < a_j whenever i < j
ℓ_0 ← min{x_i : 1 ≤ i ≤ n}; r_0 ← max{x_i : 1 ≤ i ≤ n}
p_s ← (ℓ_0 − 1, π/2); p_t ← (r_0 + 1, 0)
P ← (p_s, p_t); R' ← R
for i = 1 to n do
  Find the point p_j having the smallest x_j with x_j > x_i
  if y_j < y_i then {no point of P dominates p_i}
    Find the point p_k having the greatest x_k with x_k < x_i
    Find the point p_ℓ having the greatest y_ℓ with y_ℓ < y_i
    if x_k ≥ x_ℓ then
      Replace points p_ℓ, p_{ℓ+1} ..., p_k with point p_i in the list P
    else
      Insert point p_i between points p_k and p_ℓ in the list P
    if v_i ∈ V_U then {v_i is an evitable unbounded vertex}
      Replace P_i with {(x, y, z) | (x, y) ∈ P_i, 0 ≤ z ≤ φ_i} in R'
  else {y_j > y_i; p_j dominates p_i}
    if v_i ∈ V_U then {v_i is an inevitable unbounded vertex}
      v_j is a hovering vertex of v_i
return R'
```

Given a tolerance graph $G = (V, E)$ with the set $V = \{v_1, v_2, \ldots, v_n\}$ of vertices (and its parallelepiped representation), we can assume w.l.o.g. that $a_i < a_j$ whenever $i < j$. Recall that with every vertex v_i we associated the point $p_i = (x_i, y_i)$, where $x_i = b_i$ and $y_i = \frac{\pi}{2} - \phi_i$, respectively. The following theorem shows that, given a parallelepiped representation of a tolerance graph G, we can construct in $\mathcal{O}(n \log n)$ a canonical representation of G. This result is crucial for the time complexity analysis of the algorithms of Section 3.2.

Theorem 2. *Every parallelepiped representation of a tolerance graph G with n vertices can be transformed by Algorithm 1 to a canonical representation of G in $\mathcal{O}(n \log n)$ time.*

3.2 Minimum coloring and maximum clique

In the next theorem we present an optimal $\mathcal{O}(n \log n)$ algorithm for computing a minimum coloring of a tolerance graph G with n vertices, given a parallelepiped representation of G. The informal description of the algorithm is identical to the one in [11], which has running time $\mathcal{O}(n^2)$; the difference is in the fact that we use our new representation, in order to improve the time complexity.

Theorem 3. *A minimum coloring of a tolerance graph G with n vertices can be computed in $\mathcal{O}(n \log n)$ time.*

In the next theorem we prove that a maximum clique of a tolerance graph G with n vertices can be computed in optimal $\mathcal{O}(n \log n)$ time, given a parallelepiped representation of G. This theorem follows from Theorem 2 and from the clique algorithm presented in [6], and it improves the best known $\mathcal{O}(n^2)$ running time mentioned in [11].

Theorem 4. *A maximum clique of a tolerance graph G with n vertices can be computed in $\mathcal{O}(n \log n)$ time.*

Based on a lower time bound of $\Omega(n \log n)$ for computing the length of a longest increasing subsequence in a permutation [6,8], it turns out that the time complexity $\mathcal{O}(n \log n)$ of the presented algorithms for the minimum coloring and the maximum clique problems presented in Theorems 3 and 4 are oprimal.

4 Weighted Independent Set Algorithm in $\mathcal{O}(n^2)$

In this section we present an algorithm for computing a maximum weight independent set in a tolerance graph $G = (V, E)$ with n vertices in $\mathcal{O}(n^2)$

time, given a parallelepiped representation of G, and a weight $w(v_i) > 0$ for every vertex v_i of G. The proposed algorithm improves the running time $\mathcal{O}(n^3)$ of the one presented in [12]. In the following, consider as above the partition of the vertex set V into the sets V_B and V_U of bounded and unbounded vertices of G, respectively.

Similarly to [12], we add two isolated bounded vertices v_s and v_t to G with weights $w(v_s) = w(v_t) = 0$, such that the corresponding parallelepipeds P_s and P_t lie completely to the left and to the right of all other parallelepipeds of G, respectively. Since both v_s and v_t are bounded vertices, we augment the set V_B by the vertices v_s and v_t. In particular, we define the set of vertices $V'_B = V_B \cup \{v_s, v_t\}$ and the tolerance graph $G' = (V', E)$, where $V' = V'_B \cup V_U$. Since $G'[V'_B]$ is a bounded tolerance graph, it is a co-comparability graph as well [10, 12]. A transitive orientation of the comparability graph $\overline{G'[V'_B]}$ can be obtained by directing each edge according to the upper left endpoints of the parallelograms $\overline{P}_i$. Formally, let $(V'_B, \prec)$ be the partial order defined on the bounded vertices V'_B, such that $v_i \prec v_j$ if and only if $v_i v_j \notin E$ and $c_i < c_j$. Recall that a *chain* of elements in a partial order is a set of mutually comparable elements in this order [4].

Observation 2 ([12]) *The independent sets of $G[V_B]$ are in one-to-one correspondence with the chains in the partial order $(V'_B, \prec)$ from v_s to v_t.*

Using a dynamic programming algorithm that exploits the properties of the new parallelepiped representation of tolerance graphs, we derive the next theorem. The details can be found in [19].

Theorem 5. *A maximum weight independent set of a tolerance graph G with n vertices can be computed in $\mathcal{O}(n^2)$ time.*

5 Conclusions and Further Research

In this article we proposed the first non-trivial intersection model for general tolerance graphs, given by parallelepipeds in the three-dimensional space. This representation generalizes the parallelogram representation of bounded tolerance graphs. Using this representation, we presented improved algorithms for computing a minimum coloring, a maximum clique, and a maximum weight independent set on a tolerance graph. The complexity of the recognition problem for tolerance and bounded tolerance graphs is the main open problem in this class of graphs. Even when the input graph is known to be a tolerance graph, it is not known how to obtain a tolerance representation for it [20].

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