The Recognition of Triangle Graphs

George B. Mertzios[1]

1 Department of Computer Science, Technion, Haifa, Israel*
mertzios@cs.technion.ac.il

Abstract

Trapezoid graphs are the intersection graphs of trapezoids, where every trapezoid has a pair of opposite sides lying on two parallel lines L_1 and L_2 of the plane. Strictly between permutation and trapezoid graphs lie the *simple-triangle* graphs – also known as *PI* graphs (for Point-Interval) – where the objects are triangles with one point of the triangle on L_1 and the other two points (i.e. interval) of the triangle on L_2, and the *triangle* graphs – also known as *PI** graphs – where again the objects are triangles, but now there is no restriction on which line contains one point of the triangle and which line contains the other two. The complexity status of both triangle and simple-triangle recognition problems (namely, the problems of deciding whether a given graph is a triangle or a simple-triangle graph, respectively) have been the most fundamental open problems on these classes of graphs since their introduction two decades ago. Moreover, since triangle and simple-triangle graphs lie naturally between permutation and trapezoid graphs, and since they share a very similar structure with them, it was expected that the recognition of triangle and simple-triangle graphs is polynomial, as it is also the case for permutation and trapezoid graphs. In this article we surprisingly prove that the recognition of triangle graphs is NP-complete, even in the case where the input graph is known to be a trapezoid graph.

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1 Introduction

A graph $G = (V, E)$ with n vertices is the *intersection graph* of a family $F = \{S_1, \ldots, S_n\}$ of subsets of a set S if there exists a bijection $\mu : V \to F$ such that for any two distinct vertices $u, v \in V$, $uv \in E$ if and only if $\mu(u) \cap \mu(v) \neq \emptyset$. Then, F is called an *intersection model* of G. Note that every graph has a trivial intersection model based on adjacency relations [18]. However, some intersection models provide a natural and intuitive understanding of the structure of a class of graphs, and turn out to be very helpful to obtain structural results, as well as to find efficient algorithms to solve optimization problems [18]. Many important graph classes can be described as intersection graphs of set families that are derived from some kind of geometric configuration.

Consider two parallel horizontal lines on the plane, L_1 (the upper line) and L_2 (the lower line). Various intersection graphs can be defined on objects formed with respect to these two lines. In particular, for *permutation* graphs, the objects are line segments that have one

* Current Address: Caesarea Rothschild Institute for Computer Science, University of Haifa, Haifa, Israel.

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endpoint on L_1 and the other one on L_2. Generalizing to objects that are trapezoids with one interval on L_1 and the opposite interval on L_2, *trapezoid* graphs have been introduced independently in [5] and [6]. Given a trapezoid graph G, an intersection model of G with trapezoids between L_1 and L_2 is called a *trapezoid representation* of G. Trapezoid graphs are perfect graphs [3, 9] and generalize in a natural way both interval graphs (when the trapezoids are rectangles) and permutation graphs (when the trapezoids are trivial, i.e. lines). In particular, the main motivation for the introduction of trapezoid graphs was to generalize some well known applications of interval and permutation graphs on channel routing in integrated circuits [6].

Moreover, two interesting subclasses of trapezoid graphs have been introduced in [5]. A trapezoid graph G is a *simple-triangle* graph if it admits a trapezoid representation, in which every trapezoid is a triangle with one point on L_1 and the other two points (i.e. interval) on L_2. Similarly, G is a *triangle* graph if it admits a trapezoid representation, in which every trapezoid is a triangle, but now there is no restriction on which line between L_1 and L_2 contains one point of the triangle and which one contains the other two points (i.e. the interval) of the triangle. Such an intersection model of a simple-triangle (resp. triangle) graph G with triangles between L_1 and L_2 is called a *simple-triangle* (resp. *triangle* representation of G). Simple-triangle and triangle graphs are also known as *PI* and *PI** graphs, respectively [3–5,15], where PI stands for "Point-Interval"; note that, using this notation, permutation graphs are *PP* (for "Point-Point") graphs, while trapezoid graphs are *II* (for "Interval-Interval") graphs [5]. In particular, both interval and permutation graphs are strictly contained in simple-triangle graphs, which are strictly contained in triangle graphs, which are strictly contained in trapezoid graphs [3, 5].

Due to both their interesting structure and their practical applications, trapezoid graphs have attracted many research efforts. In particular, efficient algorithms for several optimization problems that are NP-hard in general graphs have been designed for trapezoid graphs [2, 7, 10, 12, 13, 16, 25], which also apply to triangle and simple-triangle graphs. Furthermore, several efficient algorithms appeared for the recognition problems of both permutation [9, 17] and trapezoid graphs [14, 16, 21]; see [26] for an overview.

In spite of this, the complexity status of both triangle and simple-triangle recognition problems have been the most fundamental open problems on these classes of graphs since their introduction two decades ago [3]. Since, on the one hand, very few subclasses of perfect graphs are known to be NP-hard to recognize (for instance, *perfectly orderable* graphs [23], *EPT* graphs [11], and recently *tolerance* and *bounded tolerance* graphs [22]) and, on the other hand, triangle and simple-triangle graphs lie naturally between permutation and trapezoid graphs, while they share a very similar structure with them, it was expected that the recognition of triangle and simple-triangle graphs was polynomial.

Our contribution

In this article we establish the complexity of recognizing triangle graphs. Namely, we prove that this problem is surprisingly NP-hard, by providing a reduction from the 3SAT problem. Specifically, given a boolean formula formula ϕ in conjunctive normal form with three literals in every clause (3-CNF), we construct a trapezoid graph G_ϕ, which is a triangle graph if and only if ϕ is satisfiable. Therefore, as the recognition problems for both triangle and simple-triangle graphs are in the complexity class NP, it follows in particular that the triangle graph recognition problem is NP-complete. This complements the recent surprising result that the recognition of *parallelogram* graphs (i.e. the intersection graphs of parallelograms between two parallel lines L_1 and L_2), which coincides with *bounded tolerance* graphs, is NP-complete [22].

Organization of the paper.

Background definitions and properties of trapezoid graphs and their representations are presented in Section 2. In Section 3 we introduce the notion of a *standard trapezoid representation*, the existence of which is a sufficient condition for a trapezoid graph to be a triangle graph. In Sections 4 and 5, we investigate the structure of some specific trapezoid and triangle graphs, respectively, and prove special properties of them. We use these graphs as parts of the gadgets in our reduction of 3SAT to the recognition problem of triangle graphs, which we present in Section 6. Finally, we discuss the presented results and further research in Section 7. Due to space limitations, some proofs are omitted; a full version can be found in [19].

2 Triangle and simple-triangle graphs

In this section we provide some notation and properties of trapezoid graphs and their representations, which will be mainly applied in the sequel to triangle and simple-triangle graphs.

Notation. We consider in this article simple undirected and directed graphs with no loops or multiple edges. In an undirected graph G, the edge between vertices u and v is denoted by uv, and in this case u and v are said to be *adjacent* in G. Given a graph $G = (V, E)$ and a subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of G on the vertices in S. Furthermore, we denote for simplicity by $G-S$ the induced subgraph $G[V \setminus S]$ of G. Moreover, given a graph G, we denote its vertex set by $V(G)$. A connected graph $G = (V, E)$ is called *k-connected*, where $k \geq 1$, if k is the smallest number of vertices that have to be removed from G such that the resulting graph is disconnected. Furthermore, a vertex v of a 1-connected graph G is called a *cut vertex* of G, if $G - \{v\}$ is disconnected. By possibly performing a small shift of the endpoints, we assume throughout the article without loss of generality that all endpoints of the trapezoids (resp. triangles) in a trapezoid (resp. triangle or simple-triangle) representation are distinct [8, 10, 12]. Given a trapezoid (resp. triangle or simple-triangle) graph G along with a trapezoid (resp. triangle or simple-triangle) representation R, we may not distinguish in the following between a vertex of G and the corresponding trapezoid (resp. triangle) in R, whenever it is clear from the context. Moreover, given an induced subgraph H of G, we denote by $R[H]$ the restriction of the representation R on the trapezoids (resp. triangles) of H.

Consider a trapezoid graph $G = (V, E)$ and a trapezoid representation R of G, where for any vertex $u \in V$ the trapezoid corresponding to u in R is denoted by T_u. Since trapezoid graphs are also cocomparability graphs (there is a transitive orientation of the complement) [9], we can define the partial order $(V, \ll_R)$, such that $u \ll_R v$, or equivalently $T_u \ll_R T_v$, if and only if T_u lies completely to the left of T_v in R (and thus also $uv \notin E$). Otherwise, if neither $T_u \ll_R T_v$ nor $T_v \ll_R T_u$, we will say that T_u *intersects* T_v in R (and thus also $uv \in E$). Furthermore, we define the total order $<_R$ on the lines L_1 and L_2 in R as follows. For two points a and b on L_1 (resp. on L_2), if a lies to the left of b on L_1 (resp. on L_2), then we will write $a <_R b$.

There are several trapezoid representations of a particular trapezoid graph G. For instance, given one such representation R, we can obtain another one R' by *vertical axis flipping* of R, i.e. R' is the mirror image of R along an imaginary line perpendicular to L_1 and L_2. Moreover, we can obtain another representation R'' of G by *horizontal axis flipping* of R, i.e. R'' is the mirror image of R along an imaginary line parallel to L_1 and L_2. We will

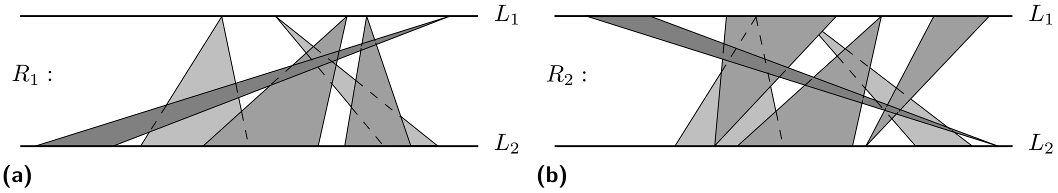


Figure 1 (a) A simple-triangle representation R_1 and (b) a triangle representation R_2.

use extensively these two basic operations throughout the article. For every trapezoid T_u in R, where $u \in V$, we define by $l(u)$ and $r(u)$ (resp. $L(u)$ and $R(u)$) the lower (resp. upper) left and right endpoint of T_u, respectively (cf. the trapezoid T_v in Figure 2). Since every triangle and simple-triangle representation is a special type of a trapezoid representation, all the above notions can be also applied to triangle and simple-triangle graphs. Note here that, if R is a simple-triangle representation of $G = (V, E)$, then $L(u) = R(u)$ for every $u \in V$; similarly, if R is a triangle representation of G, then $L(u) = R(u)$ or $l(u) = r(u)$ for every $u \in V$. An example of a simple-triangle and a triangle representation is shown in Figure 1.

It can be easily seen that every triangle (resp. single-triangle) graph G has a triangle (resp. single-triangle) representation of G, in which the endpoints of the triangles in both lines L_1 and L_2 are integers. That is, every triangle (resp. single-triangle) graph G with n vertices has a representation with size polynomial on n, and thus the recognition problems of both both triangle and simple-triangle graphs are in NP, as the next observation states.

▶ Observation 1. The triangle and simple-triangle graph recognition problems are in the complexity class NP.

3 Standard trapezoid representations

In this section we investigate several properties of trapezoid and triangle graphs and their representations. In particular, we introduce the notion of a *standard trapezoid representation*. We prove that a sufficient condition for a trapezoid graph G to be a triangle graph is that G admits such a standard representation. These properties of trapezoid and triangle graphs, as well as the notion of a standard trapezoid representation will then be used in our reduction for the triangle graph recognition problem. In order to define the notion of a standard trapezoid representation (cf. Definition 3), we first provide the following two definitions regarding an arbitrary trapezoid T_v in a trapezoid representation.

▶ **Definition 1.** Let R be a trapezoid representation of a trapezoid graph $G = (V, E)$ and T_v be a trapezoid in R, where $v \in V$. Let R' and R'' be the representations obtained by vertical axis flipping and by horizontal axis flipping of R, respectively. Then,

- T_v is *upper-right-closed in* R if there exist two vertices $u, w \in N(v)$, such that $T_u \ll_R T_w$, $L(w) <_R R(v)$, and $r(v) <_R l(w)$; otherwise T_v is *upper-right-open in* R,
- T_v is *upper-left-closed in* R if T_v is upper-right-closed in R'; otherwise T_v is *upper-left-open in* R,
- T_v is *lower-right-closed in* R if T_v is upper-right-closed in R''; otherwise T_v is *lower-right-open in* R,
- T_v is *lower-left-closed in* R if T_v is lower-right-closed in R'; otherwise T_v is *lower-left-open in* R.

▶ **Definition 2.** Let R be a trapezoid representation of a trapezoid graph $G = (V, E)$ and T_v be a trapezoid in R, where $v \in V$. Then,

- T_v is *right-closed in* R if T_v is both upper-right-closed and lower-right-closed in R; otherwise T_v is *right-open in* R,
- T_v is *left-closed in* R if T_v is both upper-left-closed and lower-left-closed in R; otherwise T_v is *left-open in* R,
- T_v is *closed* in R if T_v is both right-closed and left-closed in R; otherwise T_v is *open* in R.

As an example for Definitions 1 and 2, consider the trapezoid representation R in Figure 2. In this figure, the trapezoid T_v is upper-left-closed and lower-left-closed, as well as upper-right-closed and lower-right-open. Therefore, T_v is left-closed and right-open in R, i.e. T_v is open in R. For better visibility, we place in Figure 2 three bold bullets on the upper right, upper left, and lower left endpoints of the trapezoid T_v, in order to indicate that T_v is upper-right-closed, upper-left-closed, and lower-left-closed, respectively.

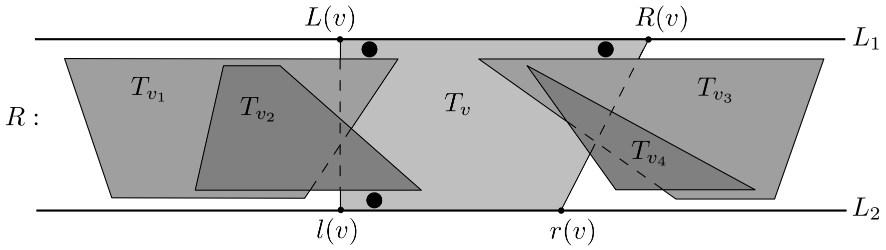


Figure 2 A standard trapezoid representation R, in which the trapezoid T_v is left-closed, upper-right-closed, and lower-right-open.

We are now ready to define the notion of a standard trapezoid representation.

▶ **Definition 3.** Let $G = (V, E)$ be a trapezoid graph and R be a trapezoid representation of G. If, for every $v \in V$, the trapezoid T_v is open in R or T_v is a triangle in R, then R is a *standard trapezoid representation.*

For example, the trapezoid representation R in Figure 2 is a standard. Indeed, none of the trapezoids $T_{v_1}, T_{v_2}, T_{v_3}$ is right-closed or left-closed, while T_v is lower-right-open (and therefore also right-open by Definition 2). Thus, each of the trapezoids T_v, T_{v_1}, T_{v_2}, and T_{v_3} is open in R. Moreover, T_{v_4} is a triangle in R.

Note that every triangle representation is a standard trapezoid representation by Definition 3. We now provide the main theorem of this section, which states a sufficient condition for a trapezoid graph to be triangle.

▶ **Theorem 4.** *Let $G = (V, E)$ be a trapezoid graph. If there exists a standard trapezoid representation of G, then G is a triangle graph.*

4 Basic constructions of trapezoid graphs

In this section we investigate some small trapezoid graphs and prove special properties of them. These graphs will then be used as parts of the gadgets in our reduction of 3SAT to the recognition problem of triangle graphs in Section 6. For simplicity of the presentation, we do not distinguish in the sequel of the article between a vertex v of a trapezoid graph G and the trapezoid T_v of v in a trapezoid representation of G.

▶ **Lemma 5.** *Let $G = (V, E)$ be the trapezoid graph induced by the trapezoid representation of Figure 3a. Then, in* any *trapezoid representation R of G, such that $v \ll_R v'$,*

- *v is upper-right-closed in R and v' is lower-left-closed in R, or*
- *v is lower-right-closed in R and v' is upper-left-closed in R.*

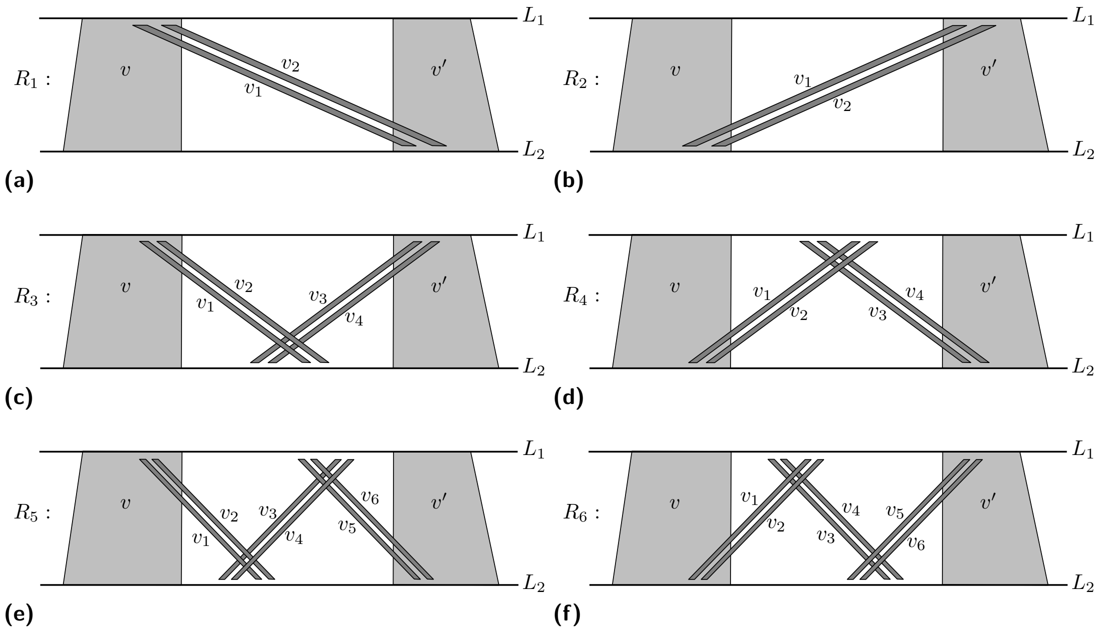


Figure 3 Six basic trapezoid representations.

The next two lemmas concern similar properties of the graphs induced by the trapezoid representations of Figures 3c and 3e, respectively.

▶ **Lemma 6.** *Let $G = (V, E)$ be the trapezoid graph induced by the trapezoid representation of Figure 3c. Then, in* any *trapezoid representation R of G, such that $v \ll_R v'$,*

- *v is upper-right-closed in R and v' is upper-left-closed in R, or*
- *v is lower-right-closed in R and v' is lower-left-closed in R.*

▶ **Lemma 7.** *Let $G = (V, E)$ be the trapezoid graph induced by the trapezoid representation of Figure 3e. Then, in* any *trapezoid representation R of G, such that $v \ll_R v'$,*

- *v is upper-right-closed in R and v' is lower-left-closed in R, or*
- *v is lower-right-closed in R and v' is upper-left-closed in R.*

5 Basic constructions of triangle graphs

In this section we investigate the structure of some specific triangle graphs and devise special properties of them. As triangle graphs are also trapezoid graphs, in order to prove these properties, we use some of the results provided in Section 4. Similarly to the trapezoid graphs investigated in Section 4, also the investigated graphs of the present section will then be used as gadgets in our reduction for the triangle graph recognition problem in Section 6. Before investigating any specific triangle graph, we first provide in the next theorem a generic result that concerns the triangle representations of the 1-connected triangle graphs.

▶ **Theorem 8.** *Let $G = (V, E)$ be a 1-connected triangle graph and $v \in V$ be a cut vertex of G. Then, in* any *triangle representation R of G, the trapezoid of v is open in R.*

We now use the generic Theorem 8, as well as the results of Section 4, in order to prove some properties of the trapezoid representations of Figure 4. Note that, although the representations of Figure 4 are not triangle representations, they are standard trapezoid representations, and thus the graphs induced by these representations are triangle graphs by Theorem 4.

▶ **Lemma 9.** *Let $G = (V, E)$ be the triangle graph induced by the trapezoid representation of Figure 4a. Then, in* any *triangle representation R of G, such that $a_7 \ll_R u$, u is left-open in R if and only if w is right-open in R.*

Proof. Let R be a triangle representation of G, such that $a_7 \ll_R u$. Note that $G - \{u, w\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6]$, and thus one of these two induced subgraphs of G lies completely to the left of the other in R. If $v \ll_R a_7 \ll_R u$, then a_7 would intersect with a triangle of G_2, which is a contradiction, since $a_7 \in V(G_1)$. Furthermore, if $a_7 \ll_R v \ll_R u$, then v would intersect with a triangle of G_1, which is a contradiction, since $v \in V(G_2)$. Therefore $a_7 \ll_R u \ll_R v$; similarly, $a_7 \ll_R w \ll_R v$. Therefore, every triangle of G_1 must lie completely to the left of every triangle of G_2 in R.

($\Rightarrow$) Suppose that u is left-open in R, i.e. u is upper-left-open or lower-left-open in R. By possibly performing a horizontal axis flipping of R, we may assume without loss of generality that u is lower-left-open in R. Consider the induced subgraphs $H_1 = G[\{a_7, a_1, a_2, u\}]$ and $H_2 = G[\{a_7, a_1, a_2, w\}]$ of G. Note that both H_1 and H_2 are isomorphic to the graph investigated in Lemma 5. Since u is assumed to be lower-left-open in R (and thus also in the restriction $R[H_1]$ of the triangle representation R), Lemma 5 implies that u is upper-left-closed and a_7 is lower-right-closed in $R[H_1]$. Therefore, a_7 is lower-right-closed also in the restriction $R[H_1 - \{u\}] = R[H_2 - \{w\}]$ of R. Thus, Lemma 5 implies that a_7 is lower-right-closed and w is upper-left-closed in the restriction $R[H_2]$ of R, and thus w is upper-left-closed in R.

Consider now the induced subgraphs $H_3 = G[\{a_7, a_3, a_4, u\}]$ and $H_4 = G[\{a_7, a_3, a_4, a_5, a_6, w\}]$ of G. Note that H_3 is isomorphic to the graph investigated in Lemma 5, while H_4 is isomorphic to the graph investigated in Lemma 6. Since u is assumed to be lower-left-open in R (and thus also in $R[H_3]$), Lemma 5 implies that u is upper-left-closed and a_7 is lower-right-closed in $R[H_3]$. Therefore, a_7 is lower-right-closed also in the restriction $R[H_3 - \{u\}] = R[H_4 - \{a_5, a_6, w\}]$ of the triangle representation R. Thus, Lemma 6 implies that a_7 is lower-right-closed and w is lower-left-closed in the restriction $R[H_4]$ of R, and thus w is lower-left-closed in R. Therefore, since w is also upper-left-closed in R by the previous paragraph, it follows that w is left-closed in R.

Recall that R is a triangle representation by assumption, and thus the restriction $R[G - \{u\}]$ is also a triangle representation. Moreover, since w is left-closed in R, it follows that w is also left-closed in $R[G - \{u\}]$. Note now that the connected graph $G - \{u\}$ satisfies the conditions of Theorem 8. Indeed, w is a cut vertex of $G - \{u\}$ and $(G - \{u\}) - \{w\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6]$. Therefore, since w is left-closed in $R[G - \{u\}]$, Theorem 8 implies that w is right-open in $R[G - \{u\}]$, and thus also w is right-open in R.

($\Leftarrow$) Consider the triangle representation R' of G that is obtained by performing a vertical axis flipping of R. Note that $v \ll_{R'} w$, since $w \ll_R v$. Furthermore, note that there is a trivial automorphism of G, which maps vertex u to w, vertex a_7 to v, and the vertices $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ to the vertices $\{b_1, b_2, b_3, b_4, b_5, b_6\}$. That is, the relation $a_7 \ll_R u$

in the representation R is mapped by this automorphism to the relation $v \ll_{R'} w$ in the representation R'. It follows now directly by the necessity part ($\Rightarrow$) that, if w is left-open in R', then u is right-open in R'. That is, if w is right-open in R, then u is left-open in R. ◀

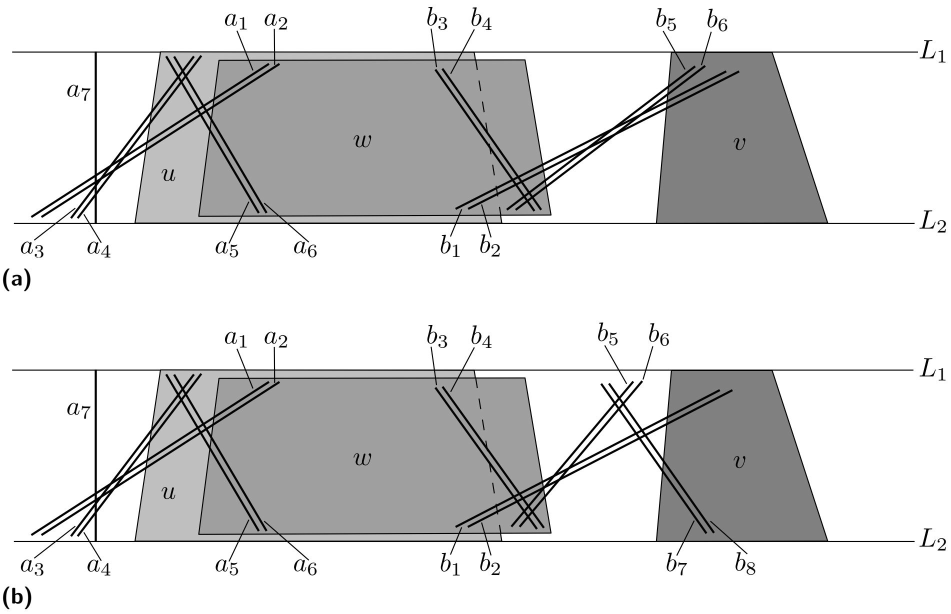


Figure 4 Two basic trapezoid representations.

Now, using Lemma 9, we can prove the next two lemmas.

▶ **Lemma 10.** *Let $G = (V, E)$ be the triangle graph induced by the trapezoid representation of Figure 4a. Then, in* any *triangle representation R of G, such that $a_7 \ll_R u$, u is left-open in R if and only if v is left-open in R.*

▶ **Lemma 11.** *Let $G = (V, E)$ be the triangle graph induced by the trapezoid representation of Figure 4b. Then, in* any *triangle representation R of G, such that $a_7 \ll_R u$, u is left-open in R if and only if v is left-closed in R.*

6 The recognition of triangle graphs

In this section we provide a reduction from the *three-satisfiability (3SAT)* problem to the problem of recognizing whether a given graph is a triangle graph. Given a boolean formula ϕ in conjunctive normal form with three literals in each clause (3-CNF), ϕ is *satisfiable* if there is a truth assignment of ϕ, such that every clause contains at least one true literal. The problem of deciding whether a given 3-CNF formula ϕ is satisfiable is one of the most known NP-complete problems. We can assume without loss of generality that each clause has literals that correspond to three distinct variables. Given the formula ϕ, we construct in polynomial time a trapezoid graph G_ϕ, such that G_ϕ is a triangle graph if and only if ϕ is satisfiable. Before constructing the whole trapezoid graph G_ϕ, we construct first some smaller trapezoid graphs for each clause and each variable that appears in the given formula ϕ.

6.1 The construction for each clause

Consider a 3-CNF formula $\phi = \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k$ with k clauses $\alpha_1, \alpha_2, \ldots, \alpha_k$ and n boolean variables $x_1, x_2, \ldots, x_n$, such that $\alpha_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ for $i = 1, 2, \ldots, k$. For the literals $\ell_{i,1}, \ell_{i,2}, \ell_{i,3}$ of the clause α_i, let $\ell_{i,1} \in \{x_{r_{i,1}}, \overline{x_{r_{i,1}}}\}$, $\ell_{i,2} \in \{x_{r_{i,2}}, \overline{x_{r_{i,2}}}\}$, and $\ell_{i,3} \in \{x_{r_{i,3}}, \overline{x_{r_{i,3}}}\}$, where $1 \leq r_{i,1} < r_{i,2} < r_{i,3} \leq n$. Let L_1 and L_2 be two parallel lines in the plane. For every clause α_i, where $i = 1, 2, \ldots, k$, we correspond the trapezoid representation R_{α_i} with 7 trapezoids that is illustrated in Figure 5. Note that the trapezoid of the vertex z_i in R_{α_i} is trivial, i.e. line. In this construction, the trapezoids of the vertices $v_{i,1}$, $v_{i,2}$, and $v_{i,3}$ correspond to the literals $\ell_{i,1}$, $\ell_{i,2}$, and $\ell_{i,3}$, respectively. Furthermore, by the construction of R_{α_i}, the left line of $v_{i,1}$ lies completely to the left of the left line of $v_{i,2}$ in R_{α_i}, while the left line of $v_{i,2}$ lies completely to the left of the left line of $v_{i,3}$ in R_{α_i}.

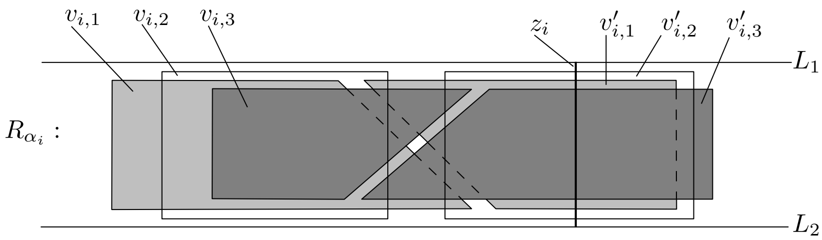


Figure 5 The construction R_{α_i} that corresponds to the clause α_i of the formula ϕ, where $i = 1, 2, \ldots, k$.

We prove now two basic properties of the construction R_{α_i} in Figure 5 for the clause α_i that will be then used in the proof of correctness of our reduction.

▶ **Lemma 12.** *Let G_{α_i} be the trapezoid graph induced by the trapezoid representation R_{α_i} of Figure 5. Then, in* any *trapezoid representation R of G_{α_i}, such that $v_{i,1} \ll_R z_i$, one of $v_{i,1}, v_{i,2}, v_{i,3}$ is right-closed in R.*

▶ **Corollary 13.** *Consider the trapezoid representation R_{α_i} of Figure 5. For every $p \in \{1, 2, 3\}$, we can locally change appropriately in R_{α_i} the right lines of $v_{i,1}, v_{i,2}, v_{i,3}$ and the left lines of $v'_{i,1}, v'_{i,2}, v'_{i,3}$, such that $v_{i,p}$ is right-closed and $v_{i,p'}$ is right-open, for every $p' \in \{1, 2, 3\} \setminus \{p\}$.*

6.2 The construction for each variable

Let x_j be a variable of the formula ϕ, where $1 \leq j \leq n$. Let x_j appear in ϕ (either as x_j or negated as $\overline{x_j}$) in the m_j clauses $\alpha_{i_{j,1}}, \alpha_{i_{j,2}}, \ldots, \alpha_{i_{j,m_j}}$, where $1 \leq i_{j,1} < i_{j,2} < \ldots < i_{j,m_j} \leq k$. Then, we correspond to the variable x_j the trapezoid representation R_{x_j} with $2m_j + 7$ trapezoids that is illustrated in Figure 6. In this construction, the trapezoids of the vertices $u_{j,t}$ and $w_{j,t}$, where $1 \leq t \leq m_j$, correspond to the appearance of the variable x_j (either as x_j or negated as $\overline{x_j}$) in the clause $\alpha_{i_{j,t}}$ in ϕ. Note that the trapezoids of the vertices $a_j^1, a_j^2, \ldots, a_j^7$ are trivial, i.e. lines. By the construction of R_{x_j}, the right line of $u_{j,t}$ lies completely to the left of the right line of $w_{j,t}$ for all values of $j = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, m_j$. Furthermore, the right line of each of $\{u_{j,t}, w_{j,t}\}$ lies completely to the left of the right line of each of $\{u_{j,t'}, w_{j,t'}\}$ in R_{x_j}, whenever $t < t'$.

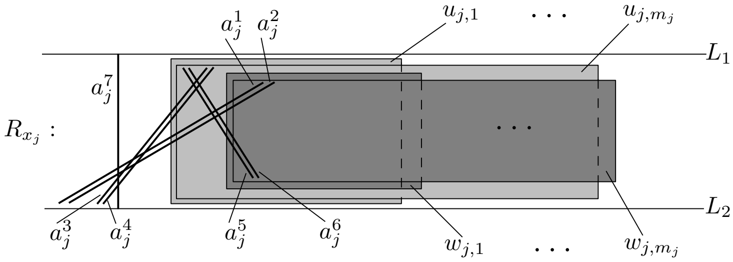


Figure 6 The construction R_{x_j} that corresponds to the variable x_j of the formula ϕ, where $j = 1, 2, \ldots, n$.

6.3 The construction the trapezoid graph G_ϕ

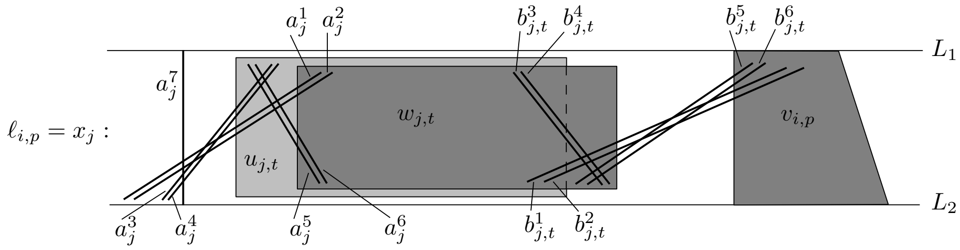
We construct now a trapezoid representation R_ϕ of the whole trapezoid graph G_ϕ, by composing the constructions R_{α_i} and R_{x_j} presented in Sections 6.1 and 6.1, as follows. First, we place in R_ϕ the k trapezoid representations R_{α_i}, where $i = 1, 2, \ldots, k$, between the lines L_1 and L_2 such that, whenever $i < i'$, every trapezoid of R_{α_i} lies completely to the left of every trapezoid of $R_{\alpha_{i'}}$. Then, we place in R_ϕ the n trapezoid representations R_{x_j}, where $j = 1, 2, \ldots, n$, between the lines L_1 and L_2 such that, whenever $j < j'$, the lines of $a_j^1, a_j^2, \ldots, a_j^7$ and the left lines of all $u_{j,t}, w_{j,t}$, lie completely to the left of the lines of $a_{j'}^1, a_{j'}^2, \ldots, a_{j'}^7$ and the left lines of all $u_{j',t'}, w_{j',t'}$. Moreover, for every $j, j' = 1, 2, \ldots, n$, the lines of $a_j^1, a_j^2, \ldots, a_j^7$ and the left lines of all $u_{j,t}, w_{j,t}$, lie in R_ϕ completely to the left of the right lines of all $u_{j',t'}, w_{j',t'}$. Thus, note in particular that every $u_{j,t}$ intersects every other $u_{j',t'}$ and every $w_{j',t'}$ in R_ϕ.

Let $j \in \{1, 2, \ldots, n\}$ and $t \in \{1, 2, \ldots, m_j\}$. Recall that, by the construction of R_{x_j} in Section 6.2, the pair of trapezoids $\{u_{j,t}, w_{j,t}\}$ corresponds to the appearance of the variable x_j in a clause α_i of ϕ, where $i = i_{j,t} \in \{1, 2, \ldots, k\}$. That is, either $\ell_{i,p} = x_j$ or $\ell_{i,p} = \overline{x_j}$ for some $p \in \{1, 2, 3\}$, where $\alpha_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$. Then, we place in R_ϕ the right lines of the trapezoids $u_{j,t}$ and $w_{j,t}$ directly before the left line of $v_{i,p}$ (i.e. no line of any other trapezoid intersects with or lies between the right lines of $u_{j,t}$ and $w_{j,t}$ and the left line of $v_{i,p}$).

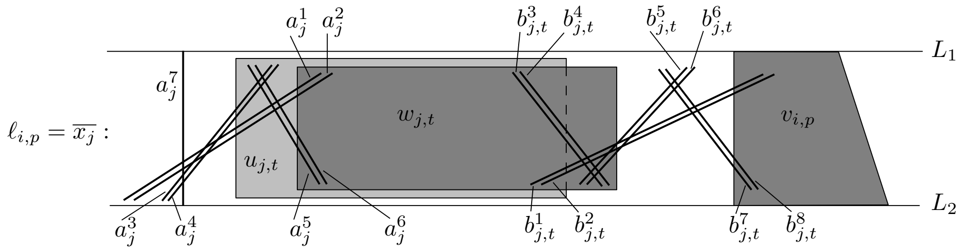
In order to finalize the construction of R_ϕ, we distinguish now the two cases regarding the literal $\ell_{i,p}$ of the clause α_i, in which the variable x_j appears. If $\ell_{i,p} = x_j$, then we add to R_ϕ six trivial trapezoids (i.e. lines) $\{b_{j,t}^1, b_{j,t}^2, \ldots b_{j,t}^6\}$, as it is shown in Figure 7a. On the other hand, if $\ell_{i,p} = \overline{x_j}$, then we add to R_ϕ eight trivial trapezoids (i.e. lines) $\{b_{j,t}^1, b_{j,t}^2, \ldots, b_{j,t}^8\}$, as it is shown in Figure 7b. In particular, we place these six (resp. eight) new lines in R_ϕ such that they intersect only the right lines of $u_{j,t}$ and $w_{j,t}$ and the left line of $v_{i,p}$ in R_ϕ. Note that the trapezoid graphs induced by the representations in Figures 7a and 7b are isomorphic to the graphs investigated in Lemmas 10 and 11, respectively. This completes the construction of the trapezoid representation R_ϕ, while G_ϕ is the trapezoid graph induced by R_ϕ.

It is now easy to verify that, by the construction of R_ϕ, all the trapezoids $u_{j,t}$ are upper-left-closed and right-closed in R_ϕ, while all the trapezoids $w_{j,t}$ are lower-right-closed and left-closed in R_ϕ. Furthermore, all the trapezoids $u_{j,t}$ are lower-left-open in R_ϕ and all the trapezoids $w_{j,t}$ are upper-right-open in R_ϕ. Consider now a trapezoid $v_{i,p}$ in R_ϕ. If $v_{i,p}$ corresponds to a positive literal $\ell_{i,p} = x_j$ (for some variable x_j), then $v_{i,p}$ is upper-left-closed and lower-left-open in R_ϕ (cf. Figure 7a). On the other hand, if $v_{i,p}$ corresponds to a negative literal $\ell_{i,p} = \overline{x_j}$, then $v_{i,p}$ is left-closed in R_ϕ (cf. Figure 7b).

We can prove that the formula ϕ is satisfiable if and only if G_ϕ is a triangle graph, cf. [19]. Therefore, since 3SAT is NP-complete, it follows that the recognition of triangle graphs



(a)



(b)

Figure 7 The composition of the trapezoids of R_{x_j} with the trapezoid $v_{i,p}$ of R_{α_i}, in the cases where (a) $\ell_{i,p} = x_j$ and (b) $\ell_{i,p} = \overline{x_j}$.

is NP-hard. Moreover, since the recognition of triangle graphs lies in NP by Observation 1, and since G_ϕ is a trapezoid graph, we can summarize our main result in the next theorem.

▶ **Theorem 14.** *Given a graph G, it is NP-complete to decide whether G is a triangle graph. The problem remains NP-complete even if the given graph G is known to be a trapezoid graph.*

7 Concluding Remarks

In this article we proved that the triangle graph (known also as PI* graph) recognition problem is NP-complete, by providing a reduction from the 3SAT problem, thus answering a longstanding open question. Our reduction implies that this problem remains NP-complete even in the case where the input graph is a trapezoid graph. The recognition of *simple-triangle* graphs [3], as well as the recognition of the related classes of *unit* and *proper tolerance* graphs [1,10] (these are subclasses of bounded tolerance, i.e. parallelogram, graphs [1]), *proper bitolerance* graphs [2,10] (they coincide with *unit bitolerance* graphs [2]), and *multitolerance* graphs [20] (they naturally generalize trapezoid graphs [20,24]) remain interesting open problems for further research.

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