Solution of Parameter-Varying Linear Matrix Inequalities in Toeplitz Form

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Abstract

In this paper the necessary and sufficient conditions are given for the solution of a system of parameter varying linear inequalities of the form $\mathbf{A}(t)\,\mathbf{x} \geq \mathbf{b}(t)$ for all $t \in T$, where T is an arbitrary set, $\mathbf{x}$ is the unknown vector, $\mathbf{A}(t)$ is a known triangular Toeplitz matrix and $\mathbf{b}(t)$ is a known vector. For every $t \in T$ the corresponding inequality defines a polyhedron, in which the solution should exist. The solution of the linear system is the intersection of the corresponding polyhedrons for every $t \in T$. A general modular decomposition method has been developed, which is based on the successive reduction of the initial system of inequalities by reducing iteratively the number of variables and by considering an equivalent system of inequalities.

Keywords: Linear matrix inequalities, parameter varying systems, constrained optimization, polyhedron, robust control theory, Toeplitz matrices.

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1 Introduction

A wide variety of problems arising in system and control theory can be reduced to constrained optimization problems, having as design constraints a simple reformulation in terms of linear matrix inequalities [1],[5]. Parameter varying Linear Matrix Inequalities (LMIs) have been proved to be a powerful tool, having important applications in a vast variety of systems and control theory problems including robustness analysis, robust control synthesis, stochastic control and identification [3],[2], synthesis of dynamic output feedback controllers [7], analysis and synthesis of control systems [4], error and sensitivity analysis, problems encountered in filtering, estimation, etc. Specifically, LMIs appear in the solution of continuous and discrete-time H_∞ control problems, in finding solvability conditions for regular and singular problems, in parameterization of H_∞ and H_2 suboptimal controllers, including reduced-order controllers [6], in finding explicit controller formulas of the H_∞ synthesis [1],[5], as well as in multiobjective synthesis and in linear parameter-varying synthesis.

LMI techniques offer the advantage of operational simplicity in contrast with the classical approaches, which necessitate the cumbersome material of Riccati equations [1]. Using LMIs, a small number of concepts and principles are sufficient to develop tools, which can then be used in practice. Also, the LMI techniques are effective numerical tools exploiting a branch of convex programming. Many LMI control methods make use of Lyapunov variables and possibly additional variables, often called scalings or multipliers, which in some sense translate how ideal behaviors are altered by uncertainties or perturbations.

Another application of LMIs is the domain of graphical manipulation in dynamic environments, where the types of interactive controls are restricted by reducing the problem of graphical manipulation to a constrained optimization problem, dictating how a user configures a set of graphical objects to achieve the desired goals. Thus, the possible configurations of the objects are represented by the object's state vector having a set of real-valued parameters and the graphical interaction problem is reduced to a problem of resolving the corresponding system of LMIs [8].

In this paper we provide necessary and sufficient conditions for the existence of the solution of the system of inequalities $\mathbf{A}(t)\mathbf{x} \geq \mathbf{b}(t), \forall t \in T$ and restrictions of this solution, if such exists, in the general case, where T may be an infinite, or even a super countable set. Specifically, t is a variable within an arbitrary set T, which may represent the domain of external disturbances or parameter variations of a system in the most general form, $\mathbf{x} \in \mathrm{R}^N$ is the unknown vector, $\mathbf{A}(t) \in \mathrm{R}^{N\times N}$ is a given triangular Toeplitz matrix dependent on t and $\mathbf{b}(t) \in \mathrm{R}^N$ is a given vector of parameters dependent on t. A Toeplitz matrix $\mathbf{A} = [a_{ij}] \in \mathrm{R}^{N\times N}$is a highly structured matrix, where $a_{i+1,j+1} = a_{i,j}$, for each appropriate $i, j \in \{1, 2, ..., N\}$, containing at most $2N - 1$ different element values. The use of a triangular Toeplitz matrix finds many applications in control theory and signal processing, since every element of the vector $\mathbf{A}(t)\mathbf{x}$ is a discrete-time convolution between the sequence of the functions in $\mathbf{A}(t)$ and the sequence in $\mathbf{x}$ and so the inequality $\mathbf{A}(t)\mathbf{x} \geq \mathbf{b}(t)$ represents a convolution that is greater than or equal to a given function, at every moment. Also a Toeplitz Matrix is the covariance matrix of a weak stationary stochastic process.

The case, where $T = \{t_0\}$ is an one-element set, can be solved with various methods, like the ellipsoid-algorithm [10]. Then, the case of a finite set T is a generalization of the latter case, in the sense that one can consider $|T|$ times the special problem on an one-element set. On the other hand, the most general cases, where the set T is infinite and in particular where T is super countable (for example when $T = \mathrm{R}^k, k \in \mathrm{N}$), are of major importance and are considered here. Although the system of equations $\mathbf{A}(t)\mathbf{x} = \mathbf{b}(t), \forall t \in T$ has numerous methods of solutions, there is no available algorithm allowing computing the solutions of a system of inequalities $\mathbf{A}(t)\mathbf{x} \geq \mathbf{b}(t), \forall t \in T$ in the general case of infinite T [11],[9].

The underlying idea in the present paper for the solution of the LMIs $\mathbf{A}(t)\mathbf{x} \geq \mathbf{b}(t), \forall t \in T$ is the General Modular (GM) decomposition of the involved inequalities into simpler inequalities, considering the cases where each element $a_i(t)$ of $\mathbf{A}(t) \in \mathrm{R}^{N \times N}$ takes zero, positive or negative values. This is possible, since a given inequality is reduced to different simpler inequalities for different ranges of $t \in T$. Following this reasoning, in Section 2 an arbitrary inequality with $k = 1$ variable is decomposed into three inequalities, the first of them including only known coefficients, including no variable and the other two expressing explicitly the upper and the lower range respectively of this one variable. Also an arbitrary inequality, including $k \geq 2$ variables is decomposed into four inequalities, each one including $k-1$ variables, using the GM decomposition. In both decompositions we derive a set of inequalities, which have a solution, if and only if the initial inequality has a solution. In Section 3 the decompositions described in Section 2 are applied successively $k-1$ times to an arbitrary inequality with $k \geq 2$ variables, thus arriving at a set of inequalities including exactly one of the k variables. Each of these inequalities of one variable is further decomposed into three inequalities. The main results of the present contribution are (a) the necessary and sufficient conditions of the existence of a solution $\mathbf{x}$ of the system and (b) the restrictions of the solution, which are expressed in the form of a hypercube, i.e. the upper and lower bound for each unknown variable x_r, $1 \leq r \leq N$, in the case where such a solution exists, which are derived in Section 4 in analytic form.

2 Decomposition of Inequalities

In this section, the decomposition of a given inequality for $t \in T$ into simpler inequalities that hold for t belonging in subsets of T, so that the polynomials $a_i(t)$, $i = 1, 2, ..., N$ take zero, positive and negative values, are described. These sets constitute a partition of T. Here T is arbitrary and plays the role of an external parameter-set, which may represent an one- or multidimensional variable (vector) that is dependent on time and other parameters. This partition of T is given in Definition 1.

Definition 1. Let $a_i(t), \forall i \in \{1, 2, ..., k\}$, be a certain sequence of functions dependent on $t \in T$, for an arbitrary set T. Then, we define for each $i \in \{1, 2, ..., k\}$ the partition sets of T:

$$S_i^1 = \{t \in T : a_i(t) = 0\}, \quad S_i^2 = \{t \in T : a_i(t) > 0\}, \quad S_i^3 = \{t \in T : a_i(t) < 0\}.$$

The underlying idea is that the partition of the set T into three subsets S_i^1, S_i^2, S_i^3 leads to inequalities having the restriction that the functions $a_i(t), \forall i \in \{1, 2, ..., N\}$ are zero-, positive- or negative-valued respectively, where $a_i(t)$ are the elements of the Toeplitz matrices $\mathbf{A}(t) \in \mathrm{R}^{N \times N}$ appearing in the LMIs $\mathbf{A}(t)\mathbf{x} \geq \mathbf{b}(t), \forall t \in T$.

Based on the above approach, in the rest of this Section the following results are presented:

- Lemma 1 describes the Special Decomposition of an arbitrary inequality in $k = 1$ variable, into three equivalent inequalities, the first of them having only known quantities with no variables and the other two expressing explicitly the upper and lower bound for this one variable, in order to satisfy the initial inequality.

- Theorem 1 provides the necessary and sufficient conditions for the existence of a solution of an arbitrary inequality in $k \geq 2$ variables, in the form of two inequalities each one of them including $k-1$ variables and

- Theorem 2 describes the General Modular (GM) Decomposition of an arbitrary inequality in $k \geq 2$ variables into four equivalent inequalities, each one of them including $k-1$ variables.

Lemma 1 (Special Decomposition of inequalities in one variable)**.** *There exists $x_1 \in \mathrm{R}$, so that:*

$$a_1(t)\, x_1 \geq b(t)\,,\ \forall t \in T \tag{1}$$

if and only if exists $x_1 \in \mathrm{R}$, such that the following inequalities hold:

$$0 \geq b(t)\,,\ \forall t \in S_1^1, \tag{2}$$

$$\frac{b(t)}{a_1(t)} \leq x_1,\ \forall t \in S_1^2, \tag{3}$$

$$x_1 \leq \frac{b(t)}{a_1(t)},\ \forall t \in S_1^3, \tag{4}$$

The above three inequalities (2), (3) and (4) constitute the Decomposition of (1).

Proof. (a). Necessary condition. Suppose that exists $x_1 \in \mathrm{R}$, so that (1) holds. Then (2) holds since $a_1(t) = 0, \forall t \in S_1^1$. Also for $t \in S_1^2$ and $t \in S_1^3$, the relations (3) and (4) hold respectively. Therefore x_1 satisfies (2)-(4) depending on the range of t in the sets S_1^1, S_1^2, S_1^3 and the necessity part has been proved.

(b). Sufficient condition. Conversely, suppose that $\exists x_1 \in \mathrm{R}$, so that (2)-(4) hold. Then

- for $a_1(t) = 0,\ \forall t \in S_1^1$and from (2) it results that $a_1(t)\, x_1 = 0 \geq b(t)\,,\ \forall t \in S_1^1$

- for $a_1(t) > 0,\ \forall t \in S_1^2$and from (3) it results that $a_1(t)\, x_1 \geq b(t)\,,\ \forall t \in S_1^2$

- for $a_1(t) < 0,\ \forall t \in S_1^3$and multiplying (4) with the negative quantity $a_1(t) = -|a_1(t)|$, it results that $a_1(t)\, x_1 \geq b(t)\,,\ \forall t \in S_1^3$.

Thus, it holds $a_1(t)\, x_1 \geq b(t)\,,\ \forall t \in T = S_1^1 \cup S_1^2 \cup S_1^3$, from which the sufficient part of Lemma 1 is concluded. □

Theorem 1. *Suppose we have the inequality:*

$$\sum_{i=1}^{k} a_{k-i+1}(t)\, x_i \geq b(t)\,, \forall t \in T, \quad k \geq 2 \tag{5}$$

where $a_i(t)\,,\ i \in \{1, 2, ..., k\}$ and $b(t)$are varying coefficients dependent on t and $x_i,\ i \in \{1, 2, ..., k\}$ are unknown real variables independent of t. There exists a vector $\mathbf{x} = [x_1, x_2, ..., x_k]^T \in \mathrm{R}^k$ satisfying (5), if *and only if there exists a vector* $\mathbf{x}' = [x'_1, x'_2, ..., x'_k]^T \in \mathrm{R}^k$ *satisfying the conditions:*

$$\sum_{i=1}^{k-1} a_{k-i+1}(t)\, x'_i \geq b(t)\,, \forall t \in S_1^1, \tag{6}$$

$$\sum_{i=1}^{k-1} \left[\frac{a_{k-i+1}(t_2)}{|a_1(t_2)|} + \frac{a_{k-i+1}(t_3)}{|a_1(t_3)|}\right] x'_i \geq \left[\frac{b(t_2)}{|a_1(t_2)|} + \frac{b(t_3)}{|a_1(t_3)|}\right], \forall (t_2, t_3) \in S_1^2 \times S_1^3. \tag{7}$$

Proof. (a). Necessary condition. Suppose that there exist some $\mathbf{x} = [x_1, x_2, ..., x_k]^T \in \mathrm{R}^k$, such that (5) holds. For $a_1(t) = 0,\ \forall t \in S_1^1$ it is seen from (5) that (6) holds, while for $a_1(t) > 0,\ \forall t \in S_1^2$ and $a_1(t) < 0,\ \forall t \in S_1^3$, we have respectively:

$$x_k \geq \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t)}{a_1(t)},\ \forall t \in S_1^2$$

and

$$x_k \leq \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t)}{a_1(t)},\ \forall t \in S_1^3,$$

which are satisfied only when:

$$\frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t_2)}{a_1(t_2)} \leq \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t_3)}{a_1(t_3)},\ \forall (t_2, t_3) \in S_1^2 \times S_1^3. \tag{8}$$

The use of the Cartesian product in (8) dictates the use of the auxiliary independent variables $t_2 \in S_1^2$, $t_3 \in S_1^3$. Inequality (8) is equivalent to (7) for $\mathbf{x}' = \mathbf{x}$, since $a_1(t_2) = |a_1(t_2)|\,,\ \forall t_2 \in S_1^2$ and $a_1(t_3) = -|a_1(t_3)|\,,\ \forall t_3 \in S_1^3$. Therefore both conditions (6) and (7) are satisfied for $\mathbf{x}' = \mathbf{x}$ and the necessity part has been proved.

(b). Sufficient condition. Conversely, suppose that there exists some $\mathbf{x} = [x_1, x_2, ..., x_k]^T \in \mathrm{R}^k$, so that (6) and (7) hold. It will be shown that exists a vector $\mathbf{x}' = [x'_1, x'_2, ..., x'_k]^T \in \mathrm{R}^k$, in general different from $\mathbf{x}$, for which (5) also holds.

Inequality (7) is equivalent to (8) (when substituting $\mathbf{x}'$ by $\mathbf{x}$), which may be written as:

$$\exists c \in \mathrm{R}: \ \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t_2)}{a_1(t_2)} \leq c \leq \frac{[b - a_k x_1 - ... - a_2 x_{k-1}](t_3)}{a_1(t_3)},\ \forall (t_2, t_3) \in S_1^2 \times S_1^3. \tag{9}$$

Multiplying the left and right part of the inequalities in (9) with $a_1(t_2) > 0$ and $a_1(t_3) < 0$ respectively and summarizing the results, it results that (9) is equivalent to the inequality

$$a_k(t)\, x_1 + ... + a_2(t)\, x_{k-1} + a_1(t)\, c \geq b(t)\,,\ \forall t \in S_1^2 \cup S_1^3. \tag{10}$$

Since $a_1(t) = 0,\ \forall t \in S_1^1$, we obtain from (6) (substituting also $\mathbf{x}'$ by $\mathbf{x}$):

$$\sum_{i=1}^{k-1} a_{k-i+1}(t)\, x_i = a_k(t)\, x_1 + a_{k-1}(t)\, x_2 + ... + a_2(t)\, x_{k-1} + a_1(t)\, c \geq b(t)\,, \forall t \in S_1^1. \tag{11}$$

Now, from (10) and (11) it follows that:

$$a_k(t)x_1 + ... + a_2(t)x_{k-1} + a_1(t)c \geq b(t), \ \forall t \in T = S_1^1 \cup S_1^2 \cup S_1^3, \tag{12}$$

from which we can see that there exists a solution $\mathbf{x}' = [x_1, x_2, ..., x_{k-1}, c]^T \in \mathrm{R}^k$ for (5). This proves the sufficient part of Theorem 1. □

Theorem 1 gives the necessary and sufficient conditions (6) and (7) for the existence of solutions of (5). Using this equivalence, where only one variable is eliminated, we loose information about the conditions that this variable should satisfy. Indeed, in (6) and (7) the variable x_k has been removed and the information about the range of the values that x_k may take in an eventual solution of (5) is lost.

The idea which is used in order to recover the information about x_k is the additional elimination of another variable, say of x_{k-1}, so that a second pair of inequalities similar to (6) and (7) are derived, which have a solution if and only if (5) has a solution. Thus, by the elimination of two variables x_k and x_{k-1}, we arrive at the following Theorem 2, which describes the General Modular (GM) Decomposition of the initial inequality (5) into a set of four equivalent inequalities, each one of them including $k-1$ variables, without losing information about the range of the variables in the solution.

Theorem 2 (General Modular (GM) Decomposition of (5))**.** *The inequality (5) can be decomposed equivalently into the following four inequalities:*

$$\sum_{i=1}^{k-1} a_{k-i+1}(t)\,x_i \geq b(t), \ \forall t \in S_1^1, \tag{13}$$

$$\sum_{i=1}^{k-1} a_{k-i+1,1}(\bar{t})\,x_i \geq b_1(\bar{t}), \ \forall \bar{t} = (t_2, t_3) \in S_1^2 \times S_1^3 \tag{14}$$

$$\sum_{\substack{i=1 \\ i \neq k-1}}^{k} a_{k-i+1}(t)\,x_i \geq b(t), \ \forall t \in S_2^1, \tag{15}$$

$$\sum_{\substack{i=1 \\ i \neq k-1}}^{k} a_{k-i+1,2}(\bar{t})\,x_i \geq b_2(\bar{t}), \ \forall \bar{t} = (t_2, t_3) \in S_2^2 \times S_2^3, \tag{16}$$

where:

$$a_{k-i+1,1}(\bar{t}) = \frac{a_{k-i+1}(t_2)}{|a_1(t_2)|} + \frac{a_{k-i+1}(t_3)}{|a_1(t_3)|}, \ b_1(\bar{t}) = \frac{b(t_2)}{|a_1(t_2)|} + \frac{b(t_3)}{|a_1(t_3)|}, \ \forall \bar{t} = (t_2, t_3) \in S_1^2 \times S_1^3, \tag{17}$$

$$a_{k-i+1,2}(\bar{t}) = \frac{a_{k-i+1}(t_2)}{|a_2(t_2)|} + \frac{a_{k-i+1}(t_3)}{|a_2(t_3)|}, \ b_2(\bar{t}) = \frac{b(t_2)}{|a_2(t_2)|} + \frac{b(t_3)}{|a_2(t_3)|}, \ \forall \bar{t} = (t_2, t_3) \in S_2^2 \times S_2^3. \tag{18}$$

This set of inequalities (13)-(16) has a solution if and only if the inequality (5) has a solution.

Proof. It follows directly from Theorem 1 that each set of inequalities (13), (14) and (15), (16) constitute a set of equivalent conditions for the solution of (5). Moreover, the use of both pairs of inequalities guarantees that no information about the range of the variables is lost. □

In order to simplify the solution of the problem, we further decompose iteratively the initial inequality (5) into inequalities that contain a smaller number of variables, according to the GM decomposition. The technical advantage of the GM decomposition is, that only the sets S_1^1, S_1^2, S_1^3 and S_2^1, S_2^2, S_2^3, in which the coefficients of the two last variables x_{k-1} and x_k are respectively null, positive or negative, are used. This decomposition constitutes the substructure for the determination of the complete set of the conditions that the solutions of (5) should satisfy. These conditions determine the hypercube, where the solutions lie.

3 Reduction of an arbitrary inequality

In this section the initial inequality of the form (5) is reduced to a number of equivalent simpler inequalities that will be called "implicit" inequalities. This reduction is presented in Theorem 3 and is achieved in two steps:

- *Step 1.* Application of the GM decomposition successively $(k-1)$ times to an arbitrary inequality on $k \geq 2$ variables, leading at the end to a set of inequalities, each one of them containing implicitly one variable.
- *Step 2.* Application of the Special Decomposition described in Lemma 1 to each one of the inequalities resulted from Step 2, leading to a set of inequalities equivalent to the initial inequality, each one of them containing either only known quantities with no variables or explicitly only one variable.

At the 0^{th} decomposition-level consider that there is the inequality (5), while at the 1^{st} decomposition-level the inequalities (13)-(16) appear. Continuing in this way and applying iteratively the GM decomposition, we arrive at the $j^{th},\ j \in \{1, 2, ..., k-1\}$ decomposition-level.

It is seen from (14) and (16) that the coefficients of the variables after the application of the GM decomposition are functions of the coefficients $a_i(t),\ i \in \{1, 2, ..., k\}$ of the given inequality (5), while in (13) and (15) the coefficients remain the same. It results from this fact that one can define in a general form the dependence of the coefficients appearing after the application of the GM decomposition at any arbitrary decomposition-level on some coefficients appearing in (5). Specifically, any arbitrary coefficient appearing in a decomposition-level may be defined as a function, having as index a sequence of natural numbers that correspond to the specific coefficients in (5), on which this coefficient depends. It follows from the structure of the Theorem 2 that the indices of all coefficients that appear at a particular inequality have the same length.

The index of every coefficient that appears at any decomposition-level has at least length 1, so it may be written as $m\bar{l}$, where $m \in \mathrm{N}$ and $\bar{l}$ is a sequence of length at least zero. Whenever this index has length at least two, it may be written as $m\bar{l}n$, where $m, n \in \mathrm{N}$. At any arbitrary decomposition-level, a coefficient, which has as index a sequence of $j \geq 2$ natural numbers, is denoted as $a_{m\bar{l}n}(t)$, where $m, n \in \mathrm{N}$ and $\bar{l}$ is a sequence of $j-2$ natural numbers. Similarly, $b_{\bar{l}n}(t)$ denotes the corresponding constant term in the same inequality, where a coefficient $a_{m\bar{l}n}(t)$ appears.

Below, in Definition 2, the coefficients $a_{m\bar{l}n}(t)$ and $b_{\bar{l}n}(t)$ are expressed recursively having as initial conditions $a_i(t)$ and $b(t)$. In Definition 2 the general case is presented, where the indices of the coefficients have at least length 2 and the index of the corresponding constant term has at least length 1, since the trivial case has already been presented in (5).

According to the GM decomposition, the sets $P_{m\bar{l}n}$ on which the corresponding inequality is defined, may be calculated recursively. Moreover, $R^1_{m\bar{l}n}, R^2_{m\bar{l}n}, R^3_{m\bar{l}n}$ denote the sets, on which a specific coefficient in the corresponding inequality is zero-, positive- or negative-valued. Finally, the auxiliary sets $S^1_{m\bar{l}n}, S^2_{m\bar{l}n}, S^3_{m\bar{l}n}$ may be defined as a generalization of S_i^1, S_i^2, S_i^3, denoting the sets, on which the corresponding coefficient $a_{m\bar{l}n}(t)$ is zero-, positive- and negative-valued. All these definitions are represented in the following Definition 2 and are of critical importance for the analysis concerning the reduction of a given arbitrary inequality of the form (5) to the implicit inequalities.

Definition 2. The sets $S^i_{m\bar{l}n}$, $i = 1, 2, 3$ are stepwise defined in terms of $a_{m\bar{l}n}(t)$ and $S^i_{n\bar{l}}$, $i = 2, 3$. Also, the coefficients $a_{m\bar{l}n}(t)$ and $b_{\bar{l}n}(t)$ are recursively defined in terms of $a_{m\bar{l}}(t)$, $a_{n\bar{l}}(t)$, $b_{\bar{l}}(t)$ and $S^i_{n\bar{l}}$, $i = 2, 3$, as follows:

Initial Conditions

$$S_j^i, \quad i = 1, 2, 3, \quad a_j(t), \quad b(t), \quad j = 1, 2, ..., k.$$

Recursions

$$a_{m\bar{l}n}(\bar{t}) := \frac{a_{m\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|} + \frac{a_{m\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}, \quad b_{\bar{l}n}(\bar{t}) := \frac{b_{\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|} + \frac{b_{\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}, \quad \forall\, \bar{t} = (t_2, t_3) \in S^2_{n\bar{l}} \times S^3_{n\bar{l}},$$

$$S^1_{m\bar{l}n} := \left\{t \in S^2_{n\bar{l}} \times S^3_{n\bar{l}} : a_{m\bar{l}n}(t) = 0\right\}, S^2_{m\bar{l}n} := \left\{t \in S^2_{n\bar{l}} \times S^3_{n\bar{l}} : a_{m\bar{l}n}(t) > 0\right\},$$
$$S^3_{m\bar{l}n} := \left\{t \in S^2_{n\bar{l}} \times S^3_{n\bar{l}} : a_{m\bar{l}n}(t) < 0\right\},$$

where $m, n \in \mathrm{N}$, $\bar{l} = l_1 l_2 ... l_j \in \mathrm{N}^j$ for every $j \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}$, with $l_1, l_2, ..., l_j \in \mathrm{N}$ pair wise distinct and $\bar{l} = \emptyset$ for $j = 0$.

Definition 3. The sets $P_{m\bar{l}n}$ and $R^1_{m\bar{l}n}, R^2_{m\bar{l}n}, R^3_{m\bar{l}n}$ are recursively defined as follows:

Initial Conditions

$$P_{1,2} = T, \text{ for } m = 1,\ n = m + 1 = 2 \text{ and } \bar{l} = \emptyset,$$

$$R^i_{1,2} = S^i_1,\ i = 1, 2, 3.$$

Recursions

$$P_{m\,\bar{l}\,n} = \left\{ \begin{array}{ll} R^2_{(m-i)\left(\bar{l}\setminus(m-i)\right)m} \times R^3_{(m-i)\left(\bar{l}\setminus(m-i)\right)m} & , \text{ if } \bar{l} = \overline{l'}\,(m-i) \\ \bigcup_{i=1}^{m-1} R^1_{(m-i)\,\bar{l}\,m} & , \text{ if } (m-i) \notin \bar{l} \end{array} \right\}, \text{ if } n = m+1, \text{ with } m > 1,$$

$$P_{m\,\bar{l}\,n} = \left\{ \begin{array}{ll} \left[P_{m\left(\bar{l}\setminus(n-1)\right)(n-1)} \cap S^2_{(n-1)\left(\bar{l}\setminus(n-1)\right)}\right] \times & \\ \times \left[P_{m\left(\bar{l}\setminus(n-1)\right)(n-1)} \cap S^3_{(n-1)\left(\bar{l}\setminus(n-1)\right)}\right] & , \text{ if } \bar{l} = \overline{l'}\,(n-1) \\ P_{m\,\bar{l}\,(n-1)} \cap S^1_{(n-1)\,\bar{l}} & , \text{ if } (n-1) \notin \bar{l} \end{array} \right\}, \text{ if } n > m+1,$$

$$R^1_{m\,\bar{l}\,n} = \left\{t \in P_{m\,\bar{l}\,n} : a_{m\,\bar{l}}(t) = 0\right\}, R^2_{m\,\bar{l}\,n} = \left\{t \in P_{m\,\bar{l}\,n} : a_{m\,\bar{l}}(t) > 0\right\},$$
$$R^3_{m\,\bar{l}\,n} = \left\{t \in P_{m\,\bar{l}\,n} : a_{m\,\bar{l}}(t) < 0\right\}.$$

In the sequel, we denote as "first index part" of a function the first integer that appears in its index, which is a sequence of natural numbers and as "second index part" the rest sequence of the index. Thus, the first index part of $a_{m\,\bar{l}}(t)$ is the integer m and the second index part is $\bar{l}$.

Lemma 2. *1. the coefficients and the corresponding constant terms have the form of $a_{m\,\bar{l}}(t)$ and $b_{\bar{l}}(t)$ respectively, as defined in Definition 2,*

2. *the indices of all coefficients coincide, except for their first part,*

3. *the indices of all coefficients have the same length,*

4. *the common second index part of them is exactly the index of the corresponding constant term,*

5. *whenever the variable x_r, $r \in \{1, 2, ..., k\}$ appears, the first index part m of the coefficient of x_r, remains constant and equal to $k + 1 - r$, i.e. in the inequality appears the product $a_{(k+1-r)\,\bar{l}\,n}(t)\, x_r$ and*

6. *all indices r of x_r, $r \in \{1, 2, ..., k\}$ that appear are either:*

   - *successive natural numbers, or*
   - *successive natural numbers except for the most right one, which can be arbitrary bigger than the others.*

Proof. The proof is presented in Appendix 1. □

Corollary 1. *An inequality at an arbitrary decomposition-level may be uniquely specified only in terms of the indices of the two most right coefficients that appear in the particular inequality.*

Proof. Suppose that the indices of the two most right coefficients that appear in a particular inequality are known, i.e. $a_{n\,\bar{l}}(t)$ and $a_{m\,\bar{l}}(t)$, with $n \geq m+1$. Then, due to Lemma 2, the index of the constant term and the second index part of all the coefficients is equal to $\bar{l}$. The first index part of every of the rest coefficients, i.e. the coefficient of x_r, is equal to $k+1-r$. Also, due to Lemma 2, all indices r of x_r, $r \in \{1,2,...,k\}$ that appear at the left of the first two known coefficients are successive natural numbers. Thus, the only inequality, that has $a_{n\,\bar{l}}(t)$ and $a_{m\,\bar{l}}(t)$ as the two most right coefficients is:

$$a_{k\,\bar{l}}(t)\,x_1 + a_{k-1,\,\bar{l}}(t)\,x_2 + \cdots + a_{n+1,\,\bar{l}}(t)\,x_{k-n} + a_{n\,\bar{l}}(t)\,x_{k+1-n} + a_{m\,\bar{l}}(t)\,x_{k+1-m} \geq b_{\bar{l}}(t)\,. \qquad (19)$$

□

Lemma 3. *The set $P_{m\,\bar{l}\,n}$ is the set on which inequality (19) is defined.*

Proof. The proof is presented in Appendix 2. □

In the sequel, Theorem 3 is presented. The implicit inequalities in Theorem 3 provide analytically the ranges, where the variables x_r, $r = 1,2,...,k$ lie, provided that the initial inequality (5) has at least one solution.

Theorem 3. *Applying successively $(k-1)$ times the GM decomposition and then one time the decomposition of Lemma 1 to the given inequality (5), we obtain the following set of inequalities, for every $r \in \{1,2,...,k\}$:*

$$\max_{\bar{l}}\left\{\sup_{t\in R^1_{(k-r+1)\,\bar{l}\,(k+1)}}\left\{b_{\bar{l}}(t)\right\}\right\} \leq 0, \qquad (20)$$

$$\max_{\bar{l}}\left\{\sup_{t\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}}\left\{\frac{b_{\bar{l}}(t)}{a_{(k-r+1)\,\bar{l}}(t)}\right\}\right\} \leq x_r \leq \min_{\bar{l}}\left\{\inf_{t\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}}\left\{\frac{b_{\bar{l}}(t)}{a_{(k-r+1)\,\bar{l}}(t)}\right\}\right\}, \qquad (21)$$

where maxima and minima are taken over every possible $\bar{l} = \overline{l^a}\,\overline{l^b} \in \mathrm{N}^{j_1+j_2}$, with $\overline{l^a} = l^a_1 l^a_2 ... l^a_{j_1} \in \mathrm{N}^{j_1}$, $\overline{l^b} = l^b_1 l^b_2 ... l^b_{j_2} \in \mathrm{N}^{j_2}$, $j_1 \in \{0,1,...,k-r\}$, $j_2 \in \{0,1,...,r-1\}$, such that:

$$\begin{aligned} &l^a_1 \in \{1,...,k-r\}\,,\ l^b_1 \in \{k-r+2,...,k\}\,,\\ &l^a_j \in \left(\bigcap_{i=1}^{j-1}\{l^a_i-1,...,k-r\}\right) \setminus \{l^a_1,...,l^a_{j-1}\}\,, for\ 2 \leq j \leq j_1,\\ &l^b_j \in \{l^b_{j-1}+1,...,k\}\,, for\ 2 \leq j \leq j_2. \end{aligned}$$

Proof. The given inequality (5) is decomposed initially into the four inequalities (13)-(16). Then the application of the GM decomposition to (13)-(16), produces a quadruplet of equivalent inequalities for each one of them and in total 4^2 inequalities. Proceeding in the same way and decomposing the 4^2

inequalities, we arrive at 4^3 inequalities and so on. In general, at the j^{th} decomposition-level 4^j inequalities are produced.

After $(k-1)$ successively applications of the GM decomposition to the initial inequality (5), as described above, we obtain an inequality with only one variable x_r; $r \in \{1, 2, ..., k\}$ and corresponding coefficient $a_{(k-r+1)\,\bar{l}}(t)$, for some appropriate $\bar{l}$, while a second one does not exist at all, since all the others have been eliminated during the successive applications of the GM decomposition. Considering in this inequality a zero-valued coefficient of an imaginary variable x_0 as the second one from the right, having $(k+1)$ as first index part, it is seen that the definition domain of this inequality is $P_{(k-r+1)\,\bar{l}\,(k+1)}$, for some appropriate $\bar{l}$. Indeed, $P_{m\,\bar{l}\,n}$ depends only on the coefficients having $i \in \{1, 2, \cdots, n-1\}$ and not n as first index part, as can be seen in Definition 3. Thus, after $(k-1)$ successively applications of the GM decomposition to the initial inequality (5) the following inequalities may be obtained:

$$a_{(k-r+1)\,\bar{l}}(t)\, x_r \geq b_{\bar{l}}(t)\,, \quad \forall t \in P_{(k-r+1)\,\bar{l}\,(k+1)}, \tag{22}$$

for every appropriate $\bar{l} = l_1 l_2 ... l_j \in \mathrm{N}^j, j \geq 0$.

In Appendix 3 it is proved that the possible integer-sequences $\bar{l}$ that may appear in $P_{(k-r+1)\,\bar{l}\,(k+1)}$ are exactly those of the form $\bar{l} = \overline{l^a}\overline{l^b} \in \mathrm{N}^{j_1+j_2}$, with $\overline{l^a}$ and $\overline{l^b}$ as given in the statement of Theorem 3.

Now, applying the special decomposition of Lemma 1 to (22), it results:

$$0 \geq b_{\bar{l}}(t)\,, \quad \forall t \in R^1_{(k-r+1)\,\bar{l}\,(k+1)}$$

$$\frac{b_{\bar{l}}(t_2)}{a_{(k-r+1)\,\bar{l}}(t_2)} \leq x_r \leq \frac{b_{\bar{l}}(t_3)}{a_{(k-r+1)\,\bar{l}}(t_3)}, \forall\,(t_2, t_3) \in R^2_{(k-r+1)\,\bar{l}\,(k+1)} \times R^3_{(k-r+1)\,\bar{l}\,(k+1)}$$

or equivalently:

$$\sup_{t \in R^1_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ b_{\bar{l}}(t) \right\} \leq 0$$

$$\sup_{t \in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{\bar{l}}(t)}{a_{(k-r+1)\,\bar{l}}(t)} \right\} \leq x_r \leq \inf_{t \in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{\bar{l}}(t)}{a_{(k-r+1)\,\bar{l}}(t)} \right\}$$

for every appropriate $\bar{l}$, as described above, or equivalently we obtain (20) and (21). Thus, Theorem 3 is proved. □

4 Main Results

In the following, in Theorem 4, the results obtained in Theorem 3 are used for deriving the necessary and sufficient conditions for the existence of a solution of a system of LMIs in Toeplitz form, along with some bounds of the solution, if such exists.

Theorem 4. *The necessary and sufficient conditions for the existence of a solution* $\mathbf{x} = [x_1, x_2, ..., x_N]^T \in \mathrm{R}^N$*, satisfying the inequality:*

$$\mathbf{A}(t)\,\mathbf{x} = \begin{bmatrix} a_1(t) & 0 & \cdots & 0 \\ a_2(t) & a_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_N(t) & a_{N-1}(t) & \cdots & a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \geq \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix} = \mathbf{b}(t), \forall t \in T, \tag{23}$$

are the following:

$$\max_{\substack{k\in\{r,r+1,...,N\} \\ \bar{l}}} \left\{ \sup_{\bar{t}\in R^1_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ b_{k\,\bar{l}}\left(\bar{t}\right) \right\} \right\} \leq 0, \forall r \in \{1, 2, ..., N\}, \tag{24}$$

$$\max_{\substack{k\in\{r,r+1,...,N\} \\ \bar{l}}} \left\{ \sup_{\bar{t}\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\left(\bar{t}\right)}{a_{(k-r+1)\,\bar{l}}\left(\bar{t}\right)} \right\} \right\} \leq$$

$$\leq \min_{\substack{k\in\{r,r+1,...,N\} \\ \bar{l}}} \left\{ \inf_{\bar{t}\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\left(\bar{t}\right)}{a_{(k-r+1)\,\bar{l}}\left(\bar{t}\right)} \right\} \right\}, \forall r \in \{1, 2, ..., N\} \tag{25}$$

and the solution $\mathbf{x} = [x_1, x_2, ..., x_r, ..., x_N]^T \in \mathrm{R}^N$ *is bounded by:*

$$x_r \in \left[\max_{\substack{k\in\{r,r+1,...,N\} \\ \bar{l}}} \left\{ \sup_{\bar{t}\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\left(\bar{t}\right)}{a_{(k-r+1)\,\bar{l}}\left(\bar{t}\right)} \right\} \right\}, \right.$$

$$\left. \min_{\substack{k\in\{r,r+1,...,N\} \\ \bar{l}}} \left\{ \inf_{\bar{t}\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\left(\bar{t}\right)}{a_{(k-r+1)\,\bar{l}}\left(\bar{t}\right)} \right\} \right\} \right] \tag{26}$$

where maxima and minima are taken over k *and over every possible* $\bar{l} = \overline{l^a}\,\overline{l^b} \in \mathrm{N}^{j_1+j_2}$*, with* $\overline{l^a} = l^a_1 l^a_2 ... l^a_{j_1} \in \mathrm{N}^{j_1}$*,* $\overline{l^b} = l^b_1 l^b_2 ... l^b_{j_2} \in \mathrm{N}^{j_2}$*,* $j_1 \in \{0, 1, ..., k-r\}$*,* $j_2 \in \{0, 1, ..., r-1\}$*, such that:*

$$\begin{aligned} & l^a_1 \in \{1, ..., k-r\}, \quad l^b_1 \in \{k-r+2, ..., k\}, \\ & l^a_j \in \left(\bigcap_{i=1}^{j-1} \{l^a_i - 1, ..., k-r\} \right) \setminus \left\{ l^a_1, ..., l^a_{j-1} \right\}, for\ 2 \leq j \leq j_1, \\ & l^b_j \in \left\{ l^b_{j-1} + 1, ..., k \right\}, for\ 2 \leq j \leq j_2. \end{aligned}$$

Proof. The LMIs in (23) are written for $k = 1, 2, ..., N$ and $\forall t \in T$ in the form:

$$\sum_{i=1}^{k} a_{k-i+1}(t)\, x_i \geq b_k(t);\ \ k = 1, 2, ..., N \tag{27}$$

It is seen from (27), that for any $r \in \{1,2,...,N\}$, the restrictions on x_r are imposed only from the inequalities of the rows $r, r+1, ..., N$. For the k^{th}, $k = r, r+1, ..., N$ inequality, the restrictions on x_r are described in (20) and (21). Summarizing the restrictions on x_r from the inequalities of the rows $r, r+1, ..., N$ of (27) and considering all $r \in \{1,2,...,N\}$, it results that the necessary and sufficient conditions, such that a solution $\mathbf{x} = [x_1, x_2, ..., x_r, ..., x_N]^T \in \mathrm{R}^N$ exists, satisfying (27) are the following:

$$\max_{\substack{k\in\{r,r+1,...,N\}\\ \bar{l}}} \left\{ \sup_{\bar{t}\in R^1_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ b_{k\,\bar{l}}\,(\bar{t}) \right\} \right\} \leq 0,\ \forall r \in \{1,2,...,N\}, \tag{28}$$

$$\max_{\substack{k\in\{r,r+1,...,N\}\\ \bar{l}}} \left\{ \sup_{t_2\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\,(t_2)}{a_{(k-r+1)\,\bar{l}}\,(t_2)} \right\} \right\} \leq x_r \leq$$
$$\leq \min_{\substack{k\in\{r,r+1,...,N\}\\ \bar{l}}} \left\{ \inf_{t_3\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\,(t_3)}{a_{(k-r+1)\,\bar{l}}\,(t_3)} \right\} \right\}, \forall r \in \{1,2,...,N\}, \tag{29}$$

where maxima and minima are taken over k and over every appropriate $\bar{l}$, as described above. In (29) a $x_r \in \mathrm{R}$ exists if and only if the upper bound of x_r is greater than or equal to the corresponding lower bound. Therefore, the necessary and sufficient conditions, such that some $\mathbf{x} \in \mathrm{R}^N$ exists, satisfying (27), are the inequalities (24) and (25).

Now, suppose that conditions (24) and (25) are satisfied, so that a solution $\mathbf{x} = [x_1, x_2, ..., x_r, ..., x_N]^T \in \mathrm{R}^N$ of the system in (27) exists. We will find out where all these solutions lie. The conditions (28) and (29) have been derived by using only the decompositions of Theorem 2 and Lemma 1. In Theorem 2 (Lemma 1) it is proved that the solution of an inequality is also a solution of the four (two) produced inequalities. Continuing in this way, it results that $\mathbf{x} \in \mathrm{R}^N$ satisfies also the produced set of inequalities in (29). Therefore, it results from (29) that the arbitrary component x_r of the solution $\mathbf{x} \in \mathrm{R}^N$, if such exists, lies in the following set:

$$x_r \in \left[\max_{\substack{k\in\{r,r+1,...,N\}\\ \bar{l}}} \left\{ \sup_{t_2\in R^2_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\,(t_2)}{a_{(k-r+1)\,\bar{l}}\,(t_2)} \right\} \right\}, \right.$$
$$\left. \min_{\substack{k\in\{r,r+1,...,N\}\\ \bar{l}}} \left\{ \inf_{t_3\in R^3_{(k-r+1)\,\bar{l}\,(k+1)}} \left\{ \frac{b_{k\,\bar{l}}\,(t_3)}{a_{(k-r+1)\,\bar{l}}\,(t_3)} \right\} \right\} \right]$$

where maxima and minima are taken over k and over every possible $\bar{l} = \overline{l^a}\,\overline{l^b} \in \mathrm{N}^{j_1+j_2}$, with $\overline{l^a}$ and $\overline{l^b}$ as given in the statement of Theorem 4. Thus Theorem 4 is proved. □

Throughout the paper the general case has been considered, where all the sets $S^1_{m\,\bar{l}\,n}$, $S^2_{m\,\bar{l}\,n}$, $S^3_{m\,\bar{l}\,n}$, $R^1_{m\,\bar{l}\,n}$, $R^2_{m\,\bar{l}\,n}$, $R^3_{m\,\bar{l}\,n}$ (and thus also $P_{m\,\bar{l}\,n}$) that occur are not empty. If some of these are empty, then (24) and (25) degenerate, thus reducing the restrictions of the desired solutions. If one side of (25)

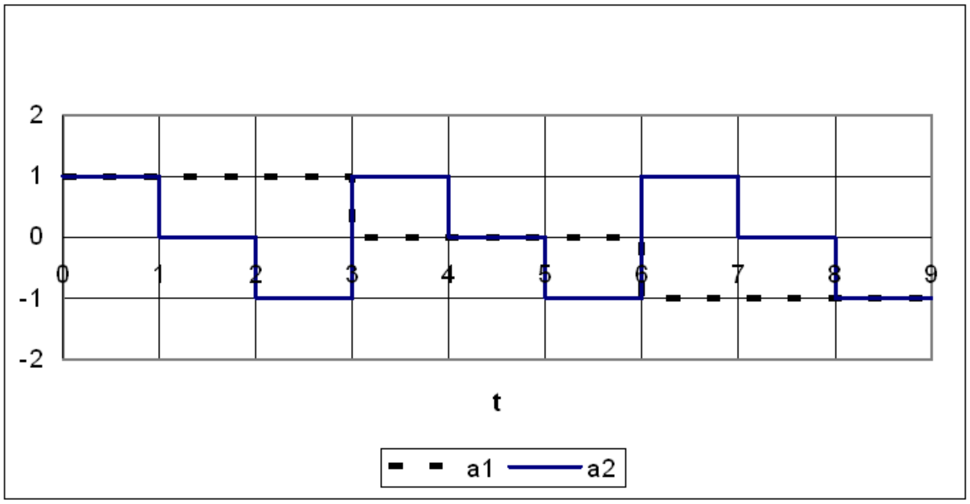

Figure 1: The functions $a_1(t)$ and $a_2(t)$.

vanishes for some $r \in \{1, 2, ..., N\}$, then it should be required that the other side is finite, since in the opposite case no finite real value for x_r exists. Assuming that all elements of $\mathbf{A}$ and $\mathbf{b}$ are bounded in T, then both sides in (25) are finite for every $r \in \{1, 2, ..., N\}$; thus an inequality can be ignored, whenever one side of it vanishes. In this case the necessary and sufficient conditions for the existence of the solution of the system, as well as the restrictions of the final solution, are also derived without any other modification.

Example

Consider the linear system for $N = 2$:

$$\begin{bmatrix} a_1(t) & 0 \\ a_2(t) & a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1(t)\, x_1 \\ a_2(t)\, x_1 + a_1(t)\, x_2 \end{bmatrix} \geq \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}, \forall t \in T = [0, 9),$$

where:

$$a_1(t) = \begin{cases} 1, & t \in [0,3) \\ 0, & t \in [3,6) \\ -1, & t \in [6,9) \end{cases}, \; a_2(t) = \begin{cases} 1, & t \in [0,1) \cup [3,4) \cup [6,7) \\ 0, & t \in [1,2) \cup [4,5) \cup [7,8) \\ -1, & t \in [2,3) \cup [5,6) \cup [8,9) \end{cases}, \; b_1(t) = t^2 - 82, \; b_2(t) = t - 10.$$

The functions $a_1(t)$ and $a_2(t)$, for all $t \in T$, are graphically shown in Figure 1. Thus, according to Definition 1, it holds:

$$S_1^1 = [3, 6), \quad S_1^2 = [0, 3), \quad S_1^3 = [6, 9),$$

$$S_2^1 = [1, 2) \cup [4, 5) \cup [7, 8), \quad S_2^2 = [0, 1) \cup [3, 4) \cup [6, 7), \quad S_2^3 = [2, 3) \cup [5, 6) \cup [8, 9).$$

It is seen from (24) and (25) that for every $r \in \{1,2\}$ only $k \in \{r, r+1, \cdots, N\}$ is considered. For $r=1$ we obtain $k \in \{1,2\}$. For $k=1$ the only possible $\bar{l}$ is $\emptyset$. For $k=2$ the possible $\bar{l}$'s are $\emptyset$ and 1 (as described in Theorem 4). For $r=2$ we obtain only $k=2$. Now, the possible $\bar{l}$'s are $\emptyset$ and 2. The conditions (24) and (25) for $r=1$ and $r=2$ take the form of (30),(31) and (32),(33) respectively, as follows:

$$\max\left\{\sup_{\bar{t}\in R^1_{1,2}}\left\{b_1\left(\bar{t}\right)\right\},\ \sup_{\bar{t}\in R^1_{2,3}}\left\{b_2\left(\bar{t}\right)\right\},\ \sup_{\bar{t}\in R^1_{2,1,3}}\left\{b_{2,1}\left(\bar{t}\right)\right\}\right\} \leq 0, \tag{30}$$

$$\max\left\{\sup_{\bar{t}\in R^1_{1,3}}\left\{b_2\left(\bar{t}\right)\right\},\ \sup_{\bar{t}\in R^1_{1,2,3}}\left\{b_{2,2}\left(\bar{t}\right)\right\}\right\} \leq 0, \tag{31}$$

$$\begin{aligned}\max &\left\{\sup_{\bar{t}\in R^2_{1,2}}\left\{\frac{b_1\left(\bar{t}\right)}{a_1\left(\bar{t}\right)}\right\},\ \sup_{\bar{t}\in R^2_{2,3}}\left\{\frac{b_2\left(\bar{t}\right)}{a_2\left(\bar{t}\right)}\right\},\ \sup_{\bar{t}\in R^2_{2,1,3}}\left\{\frac{b_{2,1}\left(\bar{t}\right)}{a_{2,1}\left(\bar{t}\right)}\right\}\right\} \leq \\ &\leq \min\left\{\inf_{\bar{t}\in R^3_{1,2}}\left\{\frac{b_1\left(\bar{t}\right)}{a_1\left(\bar{t}\right)}\right\},\ \inf_{\bar{t}\in R^3_{2,3}}\left\{\frac{b_2\left(\bar{t}\right)}{a_2\left(\bar{t}\right)}\right\},\ \inf_{\bar{t}\in R^3_{2,1,3}}\left\{\frac{b_{2,1}\left(\bar{t}\right)}{a_{2,1}\left(\bar{t}\right)}\right\}\right\},\end{aligned} \tag{32}$$

$$\max\left\{\sup_{\bar{t}\in R^2_{1,3}}\left\{\frac{b_2\left(\bar{t}\right)}{a_1\left(\bar{t}\right)}\right\},\ \sup_{\bar{t}\in R^2_{1,2,3}}\left\{\frac{b_{2,2}\left(\bar{t}\right)}{a_{1,2}\left(\bar{t}\right)}\right\}\right\} \leq \min\left\{\inf_{\bar{t}\in R^3_{1,3}}\left\{\frac{b_2\left(\bar{t}\right)}{a_1\left(\bar{t}\right)}\right\},\ \inf_{\bar{t}\in R^3_{1,2,3}}\left\{\frac{b_{2,2}\left(\bar{t}\right)}{a_{1,2}\left(\bar{t}\right)}\right\}\right\}. \tag{33}$$

In order to compute (30)-(33), it is required to compute first the following sets and functions:

$$P_{1,2} = T = [0,9)\,, \quad P_{2,3} = R^1_{1,2} = S^1_1 = [3,6)\,, \quad P_{2,1,3} = R^2_{1,2} \times R^3_{1,2} = S^2_1 \times S^3_1 = [0,3) \times [6,9)\,,$$

$$P_{1,3} = P_{1,2} \cap S^1_2 = S^1_2 = [1,2) \times [4,5) \times [7,8)\,,$$

$$P_{1,2,3} = \left[P_{1,2} \cap S^2_2\right] \times \left[P_{1,2} \cap S^3_2\right] = S^2_2 \times S^3_2 = \left([0,1) \cup [3,4) \cup [6,7)\right) \times \left([2,3) \cup [5,6) \cup [8,9)\right),$$

$$b_{2,1}\left(t_2,t_3\right) = \frac{b_2\left(t_2\right)}{\left|a_1\left(t_2\right)\right|} + \frac{b_2\left(t_3\right)}{\left|a_1\left(t_3\right)\right|} = \frac{t_2-10}{|1|} + \frac{t_3-10}{|-1|} = t_2+t_3-20, \forall\left(t_2,t_3\right) \in S^2_1 \times S^3_1,$$

$$b_{2,2}\left(t_2,t_3\right) = \frac{b_2\left(t_2\right)}{\left|a_2\left(t_2\right)\right|} + \frac{b_2\left(t_3\right)}{\left|a_2\left(t_3\right)\right|} = \frac{t_2-10}{|1|} + \frac{t_3-10}{|-1|} = t_2+t_3-20, \forall\left(t_2,t_3\right) \in S^2_2 \times S^3_2,$$

$$a_{2,1}\left(t_2,t_3\right) = \frac{a_2\left(t_2\right)}{\left|a_1\left(t_2\right)\right|} + \frac{a_2\left(t_3\right)}{\left|a_1\left(t_3\right)\right|} = \frac{a_2\left(t_2\right)}{|1|} + \frac{a_2\left(t_3\right)}{|-1|} = a_2\left(t_2\right) + a_2\left(t_3\right), \forall\left(t_2,t_3\right) \in S^2_1 \times S^3_1,$$

$$a_{1,2}\left(t_2,t_3\right) = \frac{a_1\left(t_2\right)}{\left|a_2\left(t_2\right)\right|} + \frac{a_1\left(t_3\right)}{\left|a_2\left(t_3\right)\right|} = \frac{a_1\left(t_2\right)}{|1|} + \frac{a_1\left(t_3\right)}{|-1|} = a_1\left(t_2\right) + a_1\left(t_3\right), \forall\left(t_2,t_3\right) \in S^2_2 \times S^3_2,$$

$$\begin{aligned}&R^1_{1,2} = \left\{t \in P_{1,2} : a_1(t) = 0\right\} = \left\{t \in T : a_1(t) = 0\right\} = S^1_1 = [3,6)\,, R^2_{1,2} = S^2_1 = [0,3)\,,\\ &R^3_{1,2} = S^3_1 = [6,9)\,,\end{aligned}$$

$$R^1_{2,3} = \{t \in P_{2,3} : a_2(t) = 0\} = \{t \in [3,6) : a_2(t) = 0\} = [4,5), \quad R^2_{2,3} = [3,4), \quad R^3_{2,3} = [5,6),$$

$$R^1_{2,1,3} = \{t \in P_{2,1,3} : a_{2,1}(t) = 0\} = \{(t_2,t_3) \in [0,3) \times [6,9) : a_2(t_2) + a_2(t_3) = 0\}$$
$$= ([0,1) \times [8,9)) \cup ([1,2) \times [7,8)) \cup ([2,3) \times [6,7)),$$

$$R^2_{2,1,3} = ([0,1) \times [6,7)) \cup ([0,1) \times [7,8)) \cup ([1,2) \times [6,7)),$$

$$R^3_{2,1,3} = ([1,2) \times [8,9)) \cup ([2,3) \times [7,8)) \cup ([2,3) \times [8,9)),$$

$$R^1_{1,3} = \{t \in P_{1,3} : a_1(t) = 0\} = \left\{t \in S^1_2 : a_1(t) = 0\right\} = [4,5),$$

$$R^2_{1,3} = [1,2), \quad R^3_{1,3} = [7,8),$$

$$R^1_{1,2,3} = \{t \in P_{1,2,3} : a_{1,2}(t) = 0\} =$$
$$= \{(t_2,t_3) \in ([0,1) \cup [3,4) \cup [6,7)) \times ([2,3) \cup [5,6) \cup [8,9)) : a_1(t_2) + a_1(t_3) = 0\} =$$
$$= ([0,1) \times [8,9)) \cup ([3,4) \times [5,6)) \cup ([6,7) \times [2,3)),$$

$$R^2_{1,2,3} = ([0,1) \times [2,3)) \cup ([0,1) \times [5,6)) \cup ([3,4) \times [2,3)),$$

$$R^3_{1,2,3} = ([3,4) \times [8,9)) \cup ([6,7) \times [5,6)) \cup ([6,7) \times [8,9)).$$

In the sequel, using the above quantities, we check whether inequalities (30)-(33) are satisfied. Indeed, (30)-(33) hold, since $\max\{-46,-5,-10\} = -5 \le 0$, $\max\{-5,-10\} = -5 \le 0$, $\max\{-73,-6,-11\} = -6 \le \min\{1,4,4\} = 1$ and $\max\{-8,-8\} = -8 \le \min\{2,2\} = 2$. Therefore a solution $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathrm{R}^2$ exists, which is bounded by $x_1 \in [-6,1]$, $x_2 \in [-8,2]$, as it follows from (26).

The exact set of solutions of the system and the bounds of these solutions, as given above, are graphically shown in Figure 2. The rectangle produced from these bounds is the smallest possible, since its erosion leads to loss of solutions.

Appendix 1: Proof of Lemma 2

The proof is done by induction. In (5) all coefficients are of the form $a_i(t), i \in \{1,2,...,k\}$ and the constant term is $b(t)$. Here $\overline{l} = \emptyset$ is the index of the constant term and also the second index part of every coefficient. All the indices of the coefficients coincide, except of their first index part i of them and thus they have all the same length. Also, the index of the coefficient of the variable $x_r,\ r \in \{1,2,...,k\}$ is equal to $k+1-r$ and the indices r of $x_r,\ r \in \{1,2,...,k\}$ are successive natural numbers. This shows the initialization of the induction procedure.

Suppose now that Lemma 2 holds until an arbitrary decomposition-level. Let we have at that level

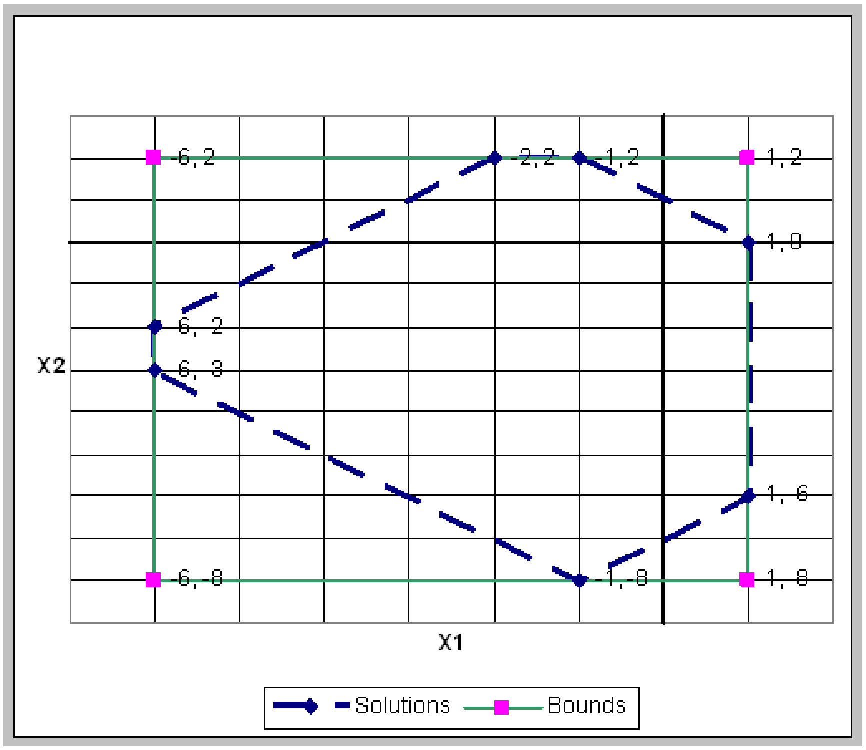

Figure 2: The set of the solutions of the system and their bounds.

the inequality:

$$a_{k\bar{l}}(t)\,x_1 + a_{k-1,\bar{l}}(t)\,x_2 + \cdots + a_{n+1,\bar{l}}(t)\,x_{k-n} + a_{n\bar{l}}(t)\,x_{k+1-n} + a_{m\bar{l}}(t)\,x_{k+1-m} \geq b_{\bar{l}}(t)\,, \forall t \in \Pi, \tag{A1.1}$$

for some set Π, where $n \geq m+1$. We apply now to (A1.1) the GM decomposition and we obtain from (13)-(16) respectively:

$$a_{k\bar{l}}(t)\,x_1 + \cdots + a_{n+1,\bar{l}}(t)\,x_{k-n} + a_{n\bar{l}}(t)\,x_{k+1-n} \geq b_{\bar{l}}(t)\,, \forall t \in \Pi_m^1 \tag{A1.2}$$

$$\left[\frac{a_{k\bar{l}}(t_2)}{\left|a_{m\bar{l}}(t_2)\right|} + \frac{a_{k\bar{l}}(t_3)}{\left|a_{m\bar{l}}(t_3)\right|}\right] x_1 + \cdots + \left[\frac{a_{n+1,\bar{l}}(t_2)}{\left|a_{m\bar{l}}(t_2)\right|} + \frac{a_{n+1,\bar{l}}(t_3)}{\left|a_{m\bar{l}}(t_3)\right|}\right] x_{k-n} +$$

$$+\left[\frac{a_{n\bar{l}}(t_2)}{\left|a_{m\bar{l}}(t_2)\right|} + \frac{a_{n\bar{l}}(t_3)}{\left|a_{m\bar{l}}(t_3)\right|}\right] x_{k+1-n} \geq \left[\frac{b_{\bar{l}}(t_2)}{\left|a_{m\bar{l}}(t_2)\right|} + \frac{b_{\bar{l}}(t_3)}{\left|a_{m\bar{l}}(t_3)\right|}\right], \forall\,(t_2,t_3) \in \Pi_m^2 \times \Pi_m^3 \tag{A1.3}$$

$$a_{k\bar{l}}(t)\,x_1 + \cdots + a_{n+1,\bar{l}}(t)\,x_{k-n} + a_{m\bar{l}}(t)\,x_{k+1-m} \geq b_{\bar{l}}(t)\,, \forall t \in \Pi_n^1 \tag{A1.4}$$

$$\left[\frac{a_{k\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{a_{k\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\right]x_1+\cdots+\left[\frac{a_{n+1,\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{a_{n+1,\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\right]x_{k-n}$$

$$+\left[\frac{a_{m\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{a_{m\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\right]x_{k+1-m}\geq\left[\frac{b_{\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{b_{\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\right],\forall\,(t_2,t_3)\in\Pi_n^2\times\Pi_n^3 \quad (A1.5)$$

where $\Pi_r^1=\left\{t\in\Pi:a_{r\bar{l}}(t)=0\right\}$, $\Pi_r^2=\left\{t\in\Pi:a_{r\bar{l}}(t)>0\right\}$, $\Pi_r^3=\left\{t\in\Pi:a_{r\bar{l}}(t)<0\right\}$ and $r\in\{m,n\}$.

The coefficients and the constant terms of the inequalities (A1.2)-(A1.5) coincide with the corresponding coefficients, as defined in of Definition 2. For example, in the second inequality above we have:

$$a_{k\bar{l}n}(\bar{t})=\frac{a_{k\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{a_{k\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|}\text{ and }b_{\bar{l}n}(\bar{t})=\frac{b_{\bar{l}}(t_2)}{\left|a_{n\bar{l}}(t_2)\right|}+\frac{b_{\bar{l}}(t_3)}{\left|a_{n\bar{l}}(t_3)\right|},\text{ where }\bar{t}=(t_2,t_3).$$

It is also clear that the indices of all the coefficients in a particular inequality of (A1.2)-(A1.5) coincide except for their first index part, they all have the same length and their common second part is exactly the index of the corresponding constant term. Also the first index part of the coefficient of the variable x_r in each one of the above four inequalities is equal to the first index part of the corresponding coefficient in (A1.1) and thus equal to the corresponding coefficient in (5).

Also, it is easily seen in Definition 2 that the sets $S^1_{m\bar{l}}, S^2_{m\bar{l}}, S^3_{m\bar{l}}$ are defined recursively and constitute a natural generalization of S_i^1, S_i^2, S_i^3. Thus, in the above inequalities it holds: $\Pi_m^1\subseteq S^1_{m\bar{l}}$, $\Pi_m^2\subseteq S^2_{m\bar{l}}$, $\Pi_m^3\subseteq S^3_{m\bar{l}}$, $\Pi_n^1\subseteq S^1_{n\bar{l}}$, $\Pi_n^2\subseteq S^2_{n\bar{l}}$ and $\Pi_n^3\subseteq S^3_{n\bar{l}}$, which means that the corresponding functions are properly defined.

It results from the above procedure that the length of the index of a coefficient increases if and only if one of the inequalities (A1.3) or (A1.5) appear; otherwise it remains constant. In addition, each time that some index increases, the increment equals the first index part of the coefficient of the variable, which vanishes in the inequality that appears.

In the whole procedure above, either the first most right coefficient, or the second one from the right, disappears, due to the inequalities (A1.2)-(A1.5). Continuing in this way, it results that all the indices r of $x_r,\ r\in\{1,2,...,k\}$ that appear in a particular iteration are either:

1. successive natural numbers, or

2. successive natural numbers except of the most right one, which can is arbitrary bigger than the others.

Appendix 2: Proof of Lemma 3

In the sequel, we call "parent" the inequality, from which the "present" inequality (19) is derived, after one application of the GM decomposition.

Initial Condition. In the given initial inequality (5), the two most right coefficients are $a_2(t)$ and $a_1(t)$. Thus this inequality is defined on the set $P_{1,2}$. The definition domain for this inequality is the whole T and thus the initial condition is $P_{1,2} = T$. Here $m = 1$, $l = \emptyset$, $n = m + 1 = 2$.

Case 1: $n = m + 1$. The case, where $m = 1$ is the initial condition. Suppose now that $m > 1$. This means that the two most right coefficients are $a_{(m+1)\bar{l}}$ and $a_{m\bar{l}}$. The present inequality (19) is defined on the set $P_{m\bar{l}(m+1)}$. The fact that the two most right coefficients have consecutive $m, n = m + 1$ first index parts, dictates that this inequality can be produced only by (13) or (14).

In the parent inequality, the first index part of the most right coefficient may take the values in $\{1, 2, ..., m - 1\}$, while the first index part of the second coefficient is m. Let $(m - i)$ be the first index part of the first coefficient in the parent inequality, for some $i \in \{1, 2, ..., m - 1\}$.

Now the following cases are discriminated:

1. Let $(m - i) \in \bar{l}$. The present inequality is produced from (14) and $(m - i)$ is the last integer that occurs in the sequence $\bar{l}$, i.e. $\bar{l} = \bar{l'}(m - i)$ for another sequence $\bar{l'}$. So, in the parent inequality, the common index part is $\bar{l}\setminus(m - i)$, which means that the two most right coefficients are $a_{m\left(\bar{l}\setminus(m-i)\right)}$ and $a_{(m-i)\left(\bar{l}\setminus(m-i)\right)}$. Thus, the parent inequality is defined on the set $P_{(m-i)\left(\bar{l}\setminus(m-i)\right)m}$ and it holds:
$$P_{m\bar{l}n} = R^2_{(m-i)\left(\bar{l}\setminus(m-i)\right)m} \times R^3_{(m-i)\left(\bar{l}\setminus(m-i)\right)m}.$$

2. Let $(m - i) \notin \bar{l}$. The present inequality is produced from (13), which means that in the parent inequality the two most right coefficients are $a_{m\bar{l}}$ and $a_{(m-i)\bar{l}}$. Thus, the parent inequality is defined on the set $P_{(m-i)\bar{l}m}$, which is possible for every $i \in \{1, 2, ..., m - 1\}$ and it holds:
$$P_{m\bar{l}n} = \bigcup_{i=1}^{m-1} R^1_{(m-i)\bar{l}m}.$$

Case 2: $n > m + 1$ In this case the two most right coefficients are $a_{n\bar{l}}$ and $a_{m\bar{l}}$, with $n > m + 1$. The fact that $n > m+1$ dictates that this inequality can be produced from the parent inequality only by (15) or (16).

Now the following cases are discriminated:

1. Let $(n - 1) \in \bar{l}$. The present inequality is produced from (16) and $(n - 1)$ is the last integer that occurs in the sequence $\bar{l}$, i.e. $\bar{l} = \bar{l'}(n - 1)$ for another sequence $\bar{l'}$. So, in the parent inequality, the common index part is $\bar{l}\setminus(n - 1)$, which means that the two most right coefficients are $a_{(n-1)\left(\bar{l}\setminus(n-1)\right)}$ and $a_{m\left(\bar{l}\setminus(n-1)\right)}$. Thus, the parent inequality is defined on the set $P_{m\left(\bar{l}\setminus(n-1)\right)(n-1)}$ and it holds:
$$P_{m\bar{l}n} = \left[P_{m\left(\bar{l}\setminus(n-1)\right)(n-1)} \cap S^2_{(n-1)\left(\bar{l}\setminus(n-1)\right)}\right] \times \left[P_{m\left(\bar{l}\setminus(n-1)\right)(n-1)} \cap S^3_{(n-1)\left(\bar{l}\setminus(n-1)\right)}\right].$$

2. Let $(n-1) \notin \bar{l}$. The present inequality is produced from (15), which means that in the parent inequality the two most right coefficients are $a_{(n-1)\,\bar{l}}$ and $a_{m\,\bar{l}}$. Thus, the parent inequality is defined on the set $P_{m\,\bar{l}\,(n-1)}$ and it holds:

$$P_{m\,\bar{l}\,n} = P_{m\,\bar{l}\,(n-1)} \cap S^1_{(n-1)\,\bar{l}}.$$

Appendix 3: Computation of all possible integer-sequences $\bar{l}$.

At first note that a value l_i in $\bar{l}$ denotes that somewhere during the iterations of the GM decomposition the coefficient with l_i as first index part i.e. the coefficient of x_{k-l_i+1}, has been eliminated from some of the inequalities (A1.3) or (A1.5).

There are two "blocks" of l_i in $\bar{l}$; those, which are smaller than $k-r+1$ and those, which are greater than $k-r+1$, since the value $l_i = k-r+1$ corresponds to an elimination of the coefficient of x_r that did not happen. The block with those l_i's that are smaller than $k-r+1$ appears first in $\bar{l}$ and the other block appears afterwards. Indeed, suppose there are some $l_j > k-r+1 > l_i$ for some $j < i$. This means that the coefficient of x_{k-l_j+1}, with l_j as first index part, has been eliminated in an inequality of the form (A1.3) or (A1.5), while the coefficient with l_i as first index part has not been eliminated yet (since $i > j$). Moreover, when the coefficient having $l_j > k-r+1$ as first index part is eliminated, then at least two coefficients on its right side appeared, each one having smaller first index part (that with l_i and that with $k-r+1$ as first index part respectively). However, this can never happen, due to the structure of the GM decomposition (always either the most right coefficient, or the second one is eliminated). Thus, the first "block" of l_i's comes first in the representation of $\bar{l}$.

Also, obviously, $l_i \neq l_j$ for l_i and l_j in $\bar{l}$. Thus, the length of the first and the second "blocks" of $\bar{l}$ are maximal $k-r$ and $r-1$ respectively and we may write $\bar{l} = \overline{l^a}\,\overline{l^b}$, where:

- $\overline{l^a} = l^a_1 l^a_2 ... l^a_{j_1}$, with: $l^a_1, l^a_2, ..., l^a_{j_1} \in \{1, 2, ..., k-r\}$, $j_1 \in \{0, 1, ..., k-r\}$,
- $\overline{l^b} = l^b_1 l^b_2 ... l^b_{j_2}$, with: $l^b_1, l^b_2, ..., l^b_{j_1} \in \{k-r+2, ..., k\}$, $j_2 \in \{0, 1, ..., r-1\}$.

In the sequel the possible values of l^a_i, $1 \leq i \leq j_1$ and l^b_i, $1 \leq i \leq j_2$ will be determined.

At first, consider $\overline{l^a} = l^a_1 l^a_2 ... l^a_{j_1}$. As the GM decomposition evolves, the inequalities (A1.2) or (A1.4) can appear many times, until either (A1.3) or (A1.5) occur. When one of the inequalities (A1.3) or (A1.5) occurs for first time, the coefficient having l^a_1 as first index part is eliminated. Then, the next time that one of the inequalities (A1.3) or (A1.5) occurs, the coefficient having l^a_2 as first index part is eliminated. There are two possibilities for l^a_1; the coefficient with l^a_1 as first index part is either the first, or the second one from the right in the inequality, at which one of the inequalities (A1.3) or (A1.5) occurs for first time. If it is the first one (so there are no other coefficients at its right side), then l^a_2 can be at least equal to $l^a_1 + 1$. If it is the second one (so there is exactly one other coefficient at its right side), then l^a_2 can be at least $l_1^a - 1$ (i.e. l_1^a has been produced from the inequality (A1.5) and l_2^a from the inequality (A1.3)). Summarizing, l_2^a can take any value of the set $\{l_1^a - 1, ..., k-r\} \setminus \{0, l_1^a\}$, since l_1^a and l_2^a are distinct and different from zero. Continuing in a similar way, it results:

$$l_j^a \in \left(\left\{l_{j-1}^a - 1, ..., k-r\right\} \cap \left\{l_{j-2}^a - 1, ..., k-r\right\} \cap ... \cap \left\{l_1^a - 1, ..., k-r\right\}\right) \setminus \left\{0, l_1^a, ..., l_{j-1}^a\right\},$$

or equivalently:

$$l_1^a \in \{1, ..., k-r\} \text{ and } l_j^a \in \left(\bigcap_{i=1}^{j-1} \{l_i^a - 1, ..., k-r\}\right) \setminus \left\{0, l_1^a, ..., l_{j-1}^a\right\}, 2 \le j \le j_1.$$

Now, consider $\overline{l^b} = l_1^b l_2^b ... l_{j_2}^b$. After eliminating all coefficients at the right side of the coefficient of x_r, no elimination from (A1.2) or (A1.3) is possible, since the most right coefficient is the coefficient having $k-r+1$ as first index part and can not be eliminated. Therefore, only an elimination from (A1.4) or (A1.5) is possible each time. Specifically, l_1^b can be produced only from the inequality (A1.5), while any number of eliminations from (A1.4) can be applied, until (A1.5) occurs. Thus, l_1^b can be any number greater than $k-r+1$. Now, with the same argumentation as before, we conclude that l_2^b can be only greater than l_1^b. Continuing in a similar way, it results:

$$l_1^b \in \{k-r+2, ..., k\} \text{ and } l_j^b \in \left\{l_{j-1}^b + 1, ..., k\right\}, 2 \le j \le j_2.$$

Summarizing, all possible integer-sequences $\overline{l}$ that may appear in $P_{(k-r+1)\,\overline{l}\,(k+1)}$ are exactly those of the form $\overline{l} = \overline{l^a}\,\overline{l^b} \in \mathrm{N}^{j_1+j_2}$, with $\overline{l^a} = l_1^a l_2^a ... l_{j_1}^a \in \mathrm{N}^{j_1}$, $\overline{l^b} = l_1^b l_2^b ... l_{j_2}^b \in \mathrm{N}^{j_2}$, $j_1 \in \{0, 1, ..., k-r\}$, $j_2 \in \{0, 1, ..., r-1\}$, such that:

$$\begin{aligned}
&l_1^a \in \{1, ..., k-r\}, \quad l_1^b \in \{k-r+2, ..., k\}, \\
&l_j^a \in \left(\bigcap_{i=1}^{j-1} \{l_i^a - 1, ..., k-r\}\right) \setminus \left\{l_1^a, ..., l_{j-1}^a\right\}, for\ 2 \le j \le j_1, \\
&l_j^b \in \left\{l_{j-1}^b + 1, ..., k\right\}, for\ 2 \le j \le j_2.
\end{aligned}$$

5 Conclusions

The necessary and sufficient conditions for the existence of the solution of LMIs $\mathbf{A}(t)\,\mathbf{x} \ge \mathbf{b}(t), \forall t \in T$, where T is a finite, infinite, or even super countable set and $\mathbf{A}(t) \in \mathrm{R}^{N \times N}$ is a given triangular Toeplitz Matrix, have been presented. Also the restrictions of this solution, if such exists, have been derived using appropriate successive decompositions of the given inequalities into simpler ones. The above results may be extended in the more general case, where $\mathbf{A}(t) \in \mathrm{R}^{N \times N}$ is an arbitrary square matrix.

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