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New PDE-based methods for image enhancement using SOM and Bayesian inference in various discretization schemes

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Abstract
A novel approach is presented in this paper for improving anisotropic diffusion PDE models, based on the Perona–Malik equation. A solution is proposed from an engineering perspective to adaptively estimate the parameters of the regularizing function in this equation. The goal of such a new adaptive diffusion scheme is to better preserve edges when the anisotropic diffusion PDE models are applied to image enhancement tasks. The proposed adaptive parameter estimation in the anisotropic diffusion PDE model involves self-organizing maps and Bayesian inference to define edge probabilities accurately. The proposed modifications attempt to capture not only simple edges but also difficult textural edges and incorporate their probability in the anisotropic diffusion model. In the context of the application of PDE models to image processing such adaptive schemes are closely related to the discrete image representation problem and the investigation of more suitable discretization algorithms using constraints derived from image processing theory. The proposed adaptive anisotropic diffusion model illustrates these concepts when it is numerically approximated by various discretization schemes in a database of magnetic resonance images (MRI), where it is shown to be efficient in image filtering and restoration applications.

Keywords: anisotropic diffusion PDEs, PDE discretization, image inpainting, Bayesian inference, self-organizing maps, edge-stopping diffusion

1. Introduction

While various PDE models have been used for 15 years and are widely applied nowadays in image processing and computer vision [1], including restoration, filtering, segmentation and object tracking, the perspective adopted in the majority of the relevant reports is the view of an applied mathematician, attempting to prove theorems and devise exact numerical methods for solving them. However, such solutions are exact for the continuous PDEs. The discrete approximations involved in image processing yielded unsatisfactory results. There is a need to investigate, from an engineering perspective, how to incorporate sophisticated image processing algorithms in the discretization schemes used in the numerical analysis of continuous PDE models when applied to image processing tasks in order to achieve good results.

There are only a few reports in the literature presenting the computational perspective of the available PDE models with respect to image processing models suitably incorporated in these PDE methods. The majority of efforts are focused on devising new continuous PDE models and analyzing their solutions in general mathematical spaces, having example image datasets as application, but they pay less attention to enhancing the results of the existing PDE models by incorporating sophisticated image processing algorithms in their numerical solution. To apply PDE models in image processing efficiently it is important to merge the two fields, namely, numerical analysis of PDE models and image processing techniques. The goal of this paper is to illustrate this concept in the application of anisotropic diffusion PDE models in image enhancement. It is shown that such PDE models could be improved if sophisticated edge-stopping diffusion schemes were incorporated in the models and their numerical solutions.

For image enhancement using anisotropic diffusion PDE models, it is necessary to preserve edges [1] and especially textural edges. To this end, a modification of the anisotropic diffusion model of Perona–Malik [4] is presented by adaptively estimating the parameters introduced in the regularization function involved. Such an adaptive estimation is based on Bayesian inference and self-organization schemes. The results achieved, which are illustrated in section 5 of the paper with respect to image filtering and restoration applications, show that many other PDE models for image inpainting [2], attempting to prevent edge blurring, could be enhanced by integrating suitable image processing techniques in their formalism and their discretization and numerical approximation, through adaptive estimation of model parameters. It is, therefore, important to critically overview the principles of these PDE models as well as their associated model parameters. This overview will be naturally focused, however, on anisotropic diffusion PDEs. The proposed adaptive anisotropic diffusion model is numerically approximated using different discretization schemes to investigate their suitability. More specifically, the finite difference method (FDM) and the radial basis function (RBF) discretizing scheme are employed to solve the proposed PDE models.

2. Image inpainting PDE models preventing edge blurring

PDEs have led to an entire new field in image processing and computer vision. They offer several advantages.

- Reinterpretation of traditional techniques under a novel unifying framework. This includes many known techniques such as convolution, filtering and morphological operations of dilation/erosion.
- More invariances could be offered with respect to classical techniques.
- Better mathematical modeling, connection with physical phenomena and better approximation to the geometry (Euclidean or generalized) of the problem. Guaranteed mathematical results with respect to well-posedness are available, such as proving that the numerical algorithms involved are stable.
- Shape recognition, structure-preserving filtering, object segmentation could be performed within a new more intuitive framework.

Regarding image inpainting [2], there are a variety of available PDE models proposed mainly to smooth and denoise images. In the last two decades, the use of nonlinear PDEs for image smoothing and denoising has met with tremendous success [1]. Before nonlinear PDEs were introduced, images were denoised by linear filtering, which is equivalent to using a noisy image as an initial condition for the heat equation. Although this method removes high frequency noise, it also badly blurs edges. To prevent blurring, a number of authors suggested using a nonlinear diffusion equation or a variational PDE model. Since the present paper investigates a novel way of preserving edges when the anisotropic diffusion PDE models are applied, by integrating an adaptive strategy for model parameter estimation, it is important to summarize both diffusion and variational approaches.

Among the equations belonging in the first category the most famous example is the Perona–Malik equation [3, 4]. On the other hand, among the most famous examples of PDEs belonging in the second category is the variational Mumford–Shah model [1, 3, 5] as well as the total variation (TV) model [6, 7]. Although effective, the methods produce piecewise constant images, often giving 'blocky' results. The Perona–Malik equation for instance behaves as a backward heat equation and instantly creates jumps (i.e. shocks) in unpredictable locations [8]. Such results occur either because of the PDE model, which might not be representative of the image dynamical system, or because of the discretization and numerical approximation schemes involved.

In an attempt to improve upon the piecewise constant images resulting from second-order image diffusions, fourth-order diffusions have been mainly suggested for image denoising. Examples include the 'low curvature image simplifier' (LCIS) equation of Tumblin and Turk [8, 9] as well as similar higher order PDE models [10]. Other attempts include imposing constraints in the diffusion PDE models [11]. The adaptive anisotropic diffusion scheme proposed herein could be viewed as belonging in such a line of research.

The main trend is, therefore, investigation of nonlinear PDE models, although it might be difficult or even impossible to analyze. Moreover, the majority of and the most useful image analysis techniques are nonlinear, which is due to the inability of linear systems to successfully model important problems. The best known vision problem modeled via PDEs is that of multiscale analysis, which is a useful and often required framework for many tasks such as feature/object detection, motion detection, stereo and multi-band frequency analysis or even image enhancement as is the case in this study.

Consider a multiscale operator, T_t, mapping an input image f to an output image $T_t(f)$, which results from the interaction of f with some kernel function dependent on a continuous scale parameter $t \geqslant 0$, i.e. $T_t(f)(x, y) = u(x, y, t)$. The scale-space function $u(x, y, t)$ holds all the history of transforming f through all the scales and can be viewed as the output of T_t at any fixed scale t. The evolution of u in scale space as a continuous dynamical system can be modeled by evolution PDEs of the type $u_t(x, y, t) = \text{function}(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y, t)$ and u can be viewed as the solution of the PDE with the initial condition $u(x, y, 0) = f(x, y)$. This multiscale analysis offers a unified framework for applying diffusion models to image processing tasks and, more specifically, to image enhancement. The most important PDE models and their variations for solving image enhancement problems are summarized below, taking into account their strategies in preventing edge blurring.

2.1. Linear heat-diffusion PDE

The most investigated PDE method for smoothing images is to apply a linear diffusion process for modeling the Gaussian scale space [12]. The convolution of an image with a Gaussian function of increasing variance is equivalent from a physical point of view to linear diffusion filtering. The connection of Gaussian convolution and linear diffusion filtering extends its limits to *multiscale analysis*. When it is not clear in advance what is the right scale, it is desirable to have an image representation at multiple scales. Diffusion could be thought of as a physical process that equilibrates concentration differences without creating or destroying the mass. The equilibration property is expressed by *Fick's* law:

$$j = -D \cdot \nabla u, \tag{1}$$

which states that the concentration gradient ∇u causes a flux j, aiming to compensate the gradient. The relation between ∇u and j is described by the *diffusion tensor D*, a positive definite symmetric matrix. The case where ∇u and j are parallel is called isotropic. Then the diffusion tensor may be replaced by a positive scalar-valued *diffusivity g*. In the general anisotropic case, ∇u and j are not parallel.

The observation that diffusion only transports mass without destroying it or creating a new mass is expressed by the *continuity equation*:

$$\partial_t u = -\mathrm{div}(j), \tag{2}$$

where t denotes the time. By connecting Fick's law with the continuity equation the formalism ends up with the *diffusion equation*:

$$\partial_t u = \mathrm{div}(D \cdot \nabla u). \tag{3}$$

Equation (3) appears in many physical transport processes. In image processing concentration can be associated with the gray value at a certain location.

2.2. Anisotropic diffusion PDE models

Since linear filtering causes edge blurring and linear shifting, the development of anisotropic nonlinear diffusion PDEs for multiscale directional image smoothing and edge detection was motivated. Perona and Malik [4] proposed a nonlinear diffusion method for avoiding the blurring and localization problems of linear diffusion filtering (hence T_t nonlinear). This Perona–Malik scheme appears to be the finite difference discretization of a nonlinear PDE not followed by a theory of well-posedness. It was known that, despite its success at its intended purpose, the scheme is very sensitive to noise and the choice of parameters such as the resolution of the digital image—a fact intimately connected with the lack of a continuum PDE theory. The work of Lions *et al* [13] replaced the Perona–Malik scheme with one that has all the desirable characteristics of the original, as well as a rigorously established continuum limit. A scale space is an image representation at a continuum of scales, embedding the image into its family of gradually simplified versions [13]. The practical implication is much more stable behavior with respect to the presence of noise and different resolutions.

Computationally, solving the modified Perona–Malik anisotropic diffusion equation, mainly following Rothe's approximation in time and the finite element method in space, involves the PDE

$$\partial u/\partial t - \mathrm{div}(g(|\nabla G_\sigma \otimes u|)\nabla u) = f(u_0 - u), \quad \text{with} \quad u(x, y, 0) = u_0(x, y) \tag{4}$$

together with zero Neumann boundary conditions and initial condition representing the processed image. Here, $g(s)$ tends to 0 for s tending to infinity. It causes the selective smoothing of the image regions and retention of the edges on which the 'Gaussian gradient' is large (G_σ is the smoothing kernel of the convolution denoted by the operator $\otimes$ in equation (4)). Such image analysis is included in the so-called nonlinear scale-space theory.

2.3. Variational PDE models

The variational approach to the image denoising problem seeks to exhibit the 'restored' image as the minimizer of a functional defined over the space of all images. The first task is clearly to decide which space of functions to take images from. For example, Sobolev spaces are ill-suited for this purpose since their elements cannot have discontinuities. Such discontinuities need to be allowed because one of the most important features of images, namely 'edges', corresponds squarely to this type of behavior.

A variational approach has been proposed [5] for the solution of the image segmentation problem, where the segmentation is obtained by finding the minimizer of an energy function, given an original image. The correct space of functions for energy minimization turns out to be a subset of functions of bounded variations. The Mumford–Shah model is a non-typical variational problem, whose analysis led to a wealth of new mathematics. Numerical implementation of the Mumford–Shah model has also been a subject of intense mathematical research. The energy is very difficult to handle since it requires minimization over subsets. The work of Ambrosio and Tortorelli [1] has rigorously shown how to approximate it in the sense of Gamma convergence by elliptic functionals. In a different vein, the work of Chan and Vese [1] has shown how the level set method of Osher and Sethian can be effectively utilized in the minimization of these types of energies.

Another successful example of the variational and PDE methods is the TV minimization [6]. An improved version of the latter technique that is based on the connectivity principle is the curvature-driven diffusion (CCD) inpainting scheme [3, 14].

3. Methods for anisotropic diffusion PDE numerical solution and discretization

The numerical solution of PDEs has been dominated by FDM, finite element methods (FEM) and finite volume methods (FVM). These methods can be derived from the assumptions of the local interpolation schemes. These methods require a mesh to support the localized approximations. The construction of a mesh in three or more dimensions is a non-trivial problem. Typically with only these methods the function is continuous across meshes, but not its partial derivatives.

In practice, only low-order approximations are used because of the polynomial snaking problem. While higher order schemes are necessary for more accurate approximations of the spatial derivatives, they are not sufficient without monotonicity constraints. Because of the low-order schemes typically employed, the spatial truncation errors can only be controlled by using progressively smaller meshes. The mesh spacing, h, must be sufficiently fine to capture the functions of the partial derivative behavior and to avoid unnecessarily large amounts of numerical artifacts contaminating the solution. Spectral methods while offering very high order spatial schemes typically depend upon tensor product grids in higher dimensions [15].

In the last decade the idea of using *mesh-free* methods for the numerical solution of PDEs has received much attention [16, 17] including RBF-based methods and wavelets [18, 19]. In this paper, the solutions offered by the proposed adaptive anisotropic diffusion scheme are compared when the relevant PDEs are numerically solved with the FDM method as well as with the RBF mesh-free technique.

3.1. Solving PDEs with radial basis functions (RBFs)

The idea of numerically solving PDEs based on RBFs mostly deals with elliptic problems, although some efforts have been made to solve time-dependent parabolic or hyperbolic problems. RBFs were first introduced to scattered data fitting and to the numerical solution of PDEs [15]. This was done in the form of globally supported RBFs and specifically of *multiquadrics (MQ)* $\phi(r) = \sqrt{r^2 + c^2}$ of thin plate splines $\phi(r) = r^2 \log r$, or Gaussians $\phi(r) = \mathrm{e}^{-c^2 r^2}$, where $r = \|x\|$ with $c \neq 0$ a parameter.

For the solution of the scattered data fitting problem [20] an RBF-based expansion of the form

$$s(x) = \sum_{j=1}^{n} c_j \phi(\|x - x_j\|_2),$$

is involved and then the coefficients c_j are determined by satisfying the interpolation conditions

$$s(x_i) = f(x_i), \qquad i = 1, 2, \ldots, n,$$

where f is a known function that generates the data to be fitted. There exists a trade-off that the spectral convergence is achieved at the cost of instability.

Galperin and Zheng [21] argue that all collocation methods are intrinsically ill-conditioned. Ill-posed and badly formulated problems can possess equivalent solutions that represent physical reality despite the mathematical nonexistence of an exact solution. Only Galperin, Pan and Zheng [22] have used global optimization on a few limited problems, with extraordinary results.

Although it is clear that the numerical solutions of PDE, ODE, integral and integro-differential equations would greatly benefit from the global optimization, the major implementation impediment is the lack of robust multi-parameter global optimization software. Unfortunately, gradient-based methods are ill-conditioned, and converge rapidly only under certain restricted conditions. In addition, gradient methods pose the risk of being trapped in a local minimum, rather than in the global minimum. Ferrari and Galperin [23] have published a software package of a fast one-dimensional adaptive cubic algorithm. It is hopeful that fast multi-dimensional global optimization software packages would be developed soon.

Therefore, there are serious limitations to the applicability of global methods, and for large dimension problems the solution should employ localization schemes [24]. The localization of the basis functions leads to locally (compactly) supported RBFs. One of the most popular compactly supported RBFs has been proposed by Wendland [24] to use compactly supported RBFs in the context of scattered data fitting or to solve PDEs; a hierarchical strategy has been developed leading to a multilevel algorithm, in which residuals are fitted iteratively and are used to update the solution. More recently, co-volume methods for solving PDE-based diffusion models for noise removal, with applications to 3D scanners and object recognition, have been proposed [28]. The discretization of these PDE models is numerically improved by using a higher order optimal recovery based on RBFs. However, RBF-based discretization of PDE models in image processing is a research area that still has not been investigated in depth, especially for image denoising.

4. The proposed adaptive anisotropic diffusion edge preserving PDE model

The nonlinear PDE equation of Perona and Malik [4] is considered,

$$\partial u/\partial t = \mathrm{div}(g(|\nabla u|)\nabla u), \quad \text{with} \quad u(x, y, 0) = u_0(x, y) \tag{5}$$

as well as the modified anisotropic diffusion model, proposed in [13] to remove the inconsistencies of the Perona–Malik model,

$$\partial u/\partial t = \mathrm{div}(g(|\nabla G_\sigma \otimes u|)\nabla u), \quad \text{with} \quad u(x, y, 0) = u_0(x, y). \tag{6}$$

The latter anisotropic diffusion model is a special case of equation (4) and it is the model investigated and improved by incorporating an adaptive scheme for estimating its parameters. In the above equations (5) and (6), g is a smooth non-increasing function with $g(0) = 1$, $g(s) \geqslant 0$ and $\lim g(s) = 0$ for $s \to \infty$. These properties of g function show that the diffusion is a conditional process such that when $|\nabla u|$ is large, the diffusion is small and the position of edges is kept. If, on the other hand, $|\nabla u|$ is small, then the diffusion is large and the image in the neighborhood of the point (x, y) is smoothed. The g functions most widely used are

$$g(s) = 1/(1 + s^2/\lambda^2) \quad \text{and} \quad g(s) = 1/e^{\lambda s}. \tag{7}$$

The former of these functions is employed in this paper, that is, $g(s) = 1/(1 + s^2/\lambda^2)$ in the proposed adaptive anisotropic diffusion model. The parameter of this model is, therefore, the coefficient λ, and an adaptive scheme is suggested for its estimation to improve the corresponding anisotropic diffusion model in terms of better preserving edges.

In the proposed adaptive scheme, the coefficient $\lambda(x, y)$ is adaptive and could be defined as the inverse probability that the point (x, y) belongs to an image u edge. If such a probability $p(x, y)$ is small and $p(x, y) \to 0$, then the function $g(s)$ takes on its largest value and $g(s) \to 1$. Therefore, the diffusion increases. On the other hand, if $p(x, y) \to 1$, then the function $g(s)$ takes on smaller values and the diffusion is small. That is, a definition of $\lambda(x, y) = 1/p(x, y)$ is reasonable in terms of achieving an edge-preserving diffusion model. Such an enhancement is important for the anisotropic diffusion model since $|\nabla u|$ values are only noisy estimates of edge existence, being the only guidance for the diffusion process. Therefore, the basic idea underlying the proposed adaptive scheme is that edge preservation could be enhanced by incorporating more guidance in the diffusion process with regard to edge detection.

To estimate $p(x, y)$ in the proposed adaptive anisotropic diffusion model, a two-stage methodology is involved. The self-organizing feature map (SOM) of Kohonen [25] is the first step while the Bayesian inference procedure is the second stage performed in order to fine tune SOM probability estimates.

First, regarding SOM application for estimating edge probabilities $p(x, y)$, the image $u(x, y,t)$ is considered for each iteration t of the anisotropic diffusion PDE solution process, and the next steps are followed.

(1) The $N \times N$ image $u(x, y,t)$ is raster scanned by $M \times M$ sliding windows having central points (x, y). For each such point (x, y) its gradient $|\nabla u|_{(x,y)}$ is estimated to become an input characteristic of the SOM network. The output SOM map consists of $K \times K$ processing elements, that is, $K \times K$ codebook vectors of $M \times M$ dimensions each. The goal of SOM is to cluster $|\nabla u|_{(x,y)}$ space in edge and nonedge points providing a measure of such a probability. SOM is known as a topology-preserving map [25], having the capability to cluster input space keeping its probability distribution. It is not known, however, how to extract such *a posteriori* probability distribution measures from winning codebook vectors. *A posteriori* probability estimation is known only in multilayer perceptron (MLP) neural networks. In the following it is demonstrated how a solution could be provided for such a SOM network too.

(2) After the SOM map convergence process is finished following the known Kohonen algorithm [25], its codebook vectors encode the topological space of the $|\nabla u|_{(x,y)}$ domain by preserving the input vectors' probability distribution and are estimated as the weight vectors associated with the SOM map. Let $\mathrm{Cb}_1(I_{1\times1},\ldots,I_{x,y},\ldots I_{M\times M})$, $\mathrm{Cb}_2(I_{1\times1},\ldots,I_{x,y},\ldots I_{M\times M})$,. $\mathrm{Cb}_{K\times K}(I_{1\times1},\ldots,I_{x,y},\ldots I_{M\times M})$ stand for these codebook vectors and $(I_{1\times1},\ldots,I_{x,y},\ldots I_{M\times M})$ for the inputs of the SOM network, belonging to the $M \times M$ sliding window on the $|\nabla u|_{(x,y)}$ domain. (x, y) is the central point of this window with the input value $|\nabla u|_{(x,y)}$.

(3) The SOM weight $\mathrm{Cb_{win}}(I_{x,y})$ corresponding to the winning element/codebook $\mathrm{Cb_{win}}$, when an input vector $(I_{1\times1},\ldots,I_{x,y},\ldots I_{M\times M})$ is presented to the network, is a measure of $|\nabla u|_{(x,y)}$ in terms of topology-preserving clustering. It could be considered as a good representative of $|\nabla u|_{(x,y)}$, with its noisy characteristics removed. This follows from the encoding properties of SOM [25] codebook vectors and it is important to derive a less noisy estimate of $|\nabla u|_{(x,y)}$, since such an estimate only offers the capability to identify edge areas. Therefore, if $|\mathrm{Cb_{win}}(I_{x,y})|$ is large with respect to all other $\mathrm{Cb}_r(I_{x,y})$ codebook vectors weights, then it is more probable that (x, y) belongs to an edge area.

(4) Therefore, if

$$p_0(x, y) = [Cb_r(I_{x,y}) - \min\{Cb_r(I_{x,y})\}]/ [\max\{Cb_r(I_{x,y})\} - \min\{Cb_r(I_{x,y})\}], \tag{8}$$

then $p_0(x, y)$ could be a measure of the probability that (x, y) belongs to an edge area of image $u(x, y)$. That is, $p_0(x, y)$ over the image space $u(x, y, t)$ is an initial estimate of the probability distribution $p(x, y)$ of edge areas in the image space $u(x, y)$. It has been demonstrated, by the above-described scheme, how a SOM network could provide a measure of the probability distribution of its input space.

Regarding the second stage of the proposed adaptive anisotropic diffusion scheme, it is based on Bayesian inference [25]. The proposed algorithm amounts to minimizing the following objective function, and it is similar to the one proposed in [26]:

$$|\underline{\mathbf{P}}_0 - \underline{\mathbf{P}}|^2/(2\sigma^2) + (3/2)\sum_{x,y} \log\{\alpha^2 + ({}^{x}\Delta_{xy})^2 + ({}^{y}\Delta_{xy})^2\} \tag{9}$$

with regard to $\underline{\mathbf{P}}$, which is the unknown probability distribution to be reconstructed from the initial estimates $p_0(x, y)$ given in $\underline{\mathbf{P}}_0$. The first term comes from the likelihood term and the second one from the prior knowledge term of the well-known Bayesian formulation. The second term symbols arise from the imposed 2D Lorentzian prior knowledge. ${}^{x}\Delta_{xy}$ and

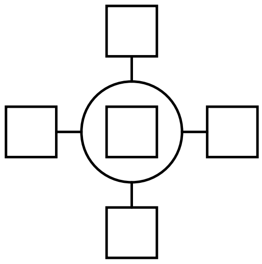

Figure 1. The cross-neighborhood used in the SOM network.

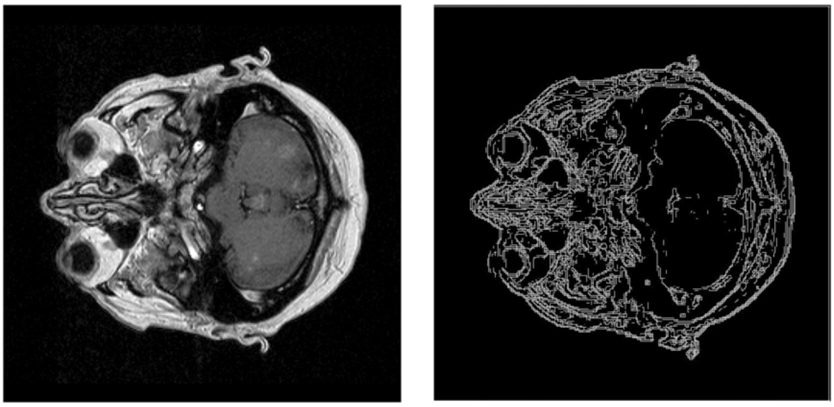

Figure 2. A gray-scale MRI image and its edges detected by applying a Sobel filter to the original image. There is no Gaussian noise applied.

Table 1. Quantitative results with regards to reconstruction/filtering performance of the various methodologies involved, under different Gaussian noise levels imposed on the original images. 45 images randomly chosen from the collection of [27] have been used.

Discretization scheme/numerical approximation method	Proposed adaptive parameter estimation anisotropic diffusion PDE model		Standard modified Perona–Malik anisotropic diffusion PDE model of equation (6)	
Gaussian noise level (SNR = 50 dB)	Mean SSE	Mean dB	Mean SSE	Mean dB
FDM (Crank–Nicholson)	2.45E3	19.60	3.62E3	16.26
RBF mesh-free (Wendland [24])	2.31E3	20.56	3.41E3	16.97
Gaussian noise level (SNR = 20 dB)	Mean SSE	Mean dB	Mean SSE	Mean dB
FDM (Crank–Nicholson)	4.08E3	14.30	4.82E3	12.22
RBF mesh-free (Wendland [24])	3.52E3	15.24	4.11E3	13.1
Gaussian noise level (SNR = 10 dB)	Mean SSE	Mean dB	Mean SSE	Mean dB
FDM (Crank–Nicholson)	4.96E3	13.60	5.72E3	11.73
RBF mesh-free (Wendland [24])	4.67E3	14.1	5.53E3	12.68

${}^{y}\Delta_{xy}$ are the probability differences in the x- and y-directions respectively and α is a Lorentz distribution-width parameter. If it is assumed that PP(p) is the prior, which imposes prior knowledge conditions about the edge probability distribution on the reconstruction algorithm, then the second term of (9) comes as follows.

The starting point is that PP(p) could be obviously expanded into $\mathrm{PP}(p) = \mathrm{PP}(p_{0,0})\ \mathrm{PP}(p_{1,0}|p_{0,0})\ \mathrm{PP}(p_{2,0}|p_{0,0}, p_{1,0})\ldots$. If it is assumed that the probability $p_{x,y}$ depends only on its left neighbor ($p_{x-1,y}$), then the previous PP(p) expansion takes on the form $\mathrm{PP}(p) = \Pi_{(x,y)}\ \mathrm{PP}(p_{x,y}|p_{x-1,y})$, provided that the boundaries are ignored. Next, it is considered that $\mathrm{PP}(p_{x,y}|\ p_{x-1,y})$ is a function only of the difference between the corresponding probabilities. This difference is written down as ${}^{x}\Delta_{xy} = p_{x,y} - p_{x-1,y}$. It has been shown that the probability density function of ${}^{x}\Delta_{xy}$ is Lorentzian shaped (see [26]). These assumptions and calculations lead directly to compute the prior knowledge in the Bayesian reconstruction approach as in (9) and the herein proposed objective function of (9) is minimized using the conjugate gradients optimization methodology.

Therefore, during the first stage of the proposed scheme the initial estimates $p_0(x, y)$ of the edge area probability distribution are obtained and during the second stage the edge probability distribution space $p(x, y)$ is reconstructed. In the following, $\lambda(x, y)$ could be defined as $\lambda(x, y) = 1/p(x, y)$, as already mentioned. This final step concludes the proposed adaptive parameter estimation scheme as a novel modification of the anisotropic diffusion model under investigation.

5. Experimental evaluation of the proposed adaptive diffusion model

An experimental study has been conducted in order to evaluate the proposed adaptive anisotropic diffusion PDE model described in section 4 as compared to the standard modified Perona–Malik anisotropic diffusion PDE model [13] of equation (6), and numerically solved using the discretization schemes outlined in section 3 above, namely, the finite difference method (involving the Crank–Nicholson scheme) as well as the RBF mesh-free technique (involving the scheme proposed by Wendland [24]). All simulations have been performed in the MATLAB version 6.5 system.

The methods involved have been applied to an MRI image database which has been downloaded from the Internet [27]. These images have 256 by 256 dimensions and the present difficulties for image analysis are due to their fine edges. This is why MRI image databases have been selected as the basis

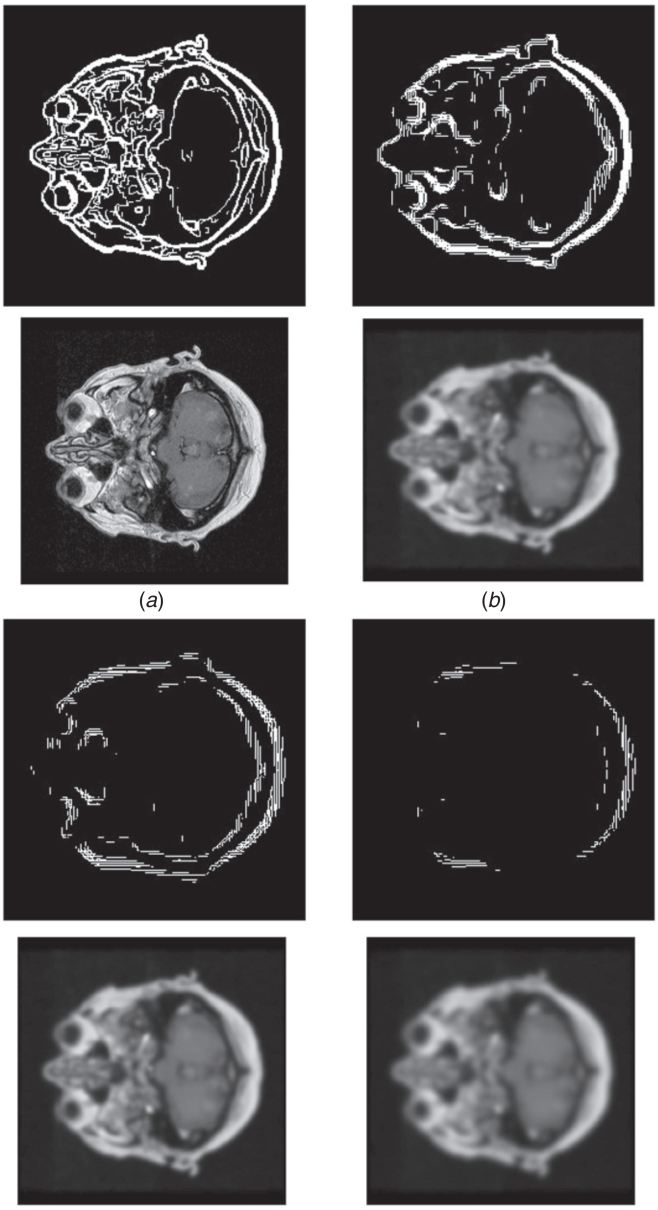

Figure 3. The edges of the MRI image of figure 2 (and the corresponding reconstructions), reconstructed by applying: (*a*) proposed adaptive anisotropic diffusion model in the original image under noise level with SNR = 50 dB; (*b*) standard modified Perona–Malik anisotropic diffusion model of equation (6) in the original image under noise level with SNR = 50 dB; (*c*) proposed adaptive anisotropic diffusion model in the original image under noise level with SNR = 10 dB; (*d*) standard modified Perona–Malik anisotropic diffusion model of equation (6) in the original image under noise level with SNR = 10 dB. In all these cases, the FDM method of the Crank–Nicholson scheme is used to numerically approximate the PDE models, and the edges are detected using a standard Sobel filter on the best outcome of each corresponding PDE model, and a standard contour following method.

of this experimental study. The sliding window used by the SOM network in the proposed adaptive scheme has 5 × 5 dimensions, while the neighborhood used in all numerical approximations is the usual cross-neighborhood of four points as shown next.

Concerning the measures involved to quantitatively compare the performance of the various models in terms of image reconstruction/filtering, the usually used sum of squared errors (SSE) between the original MRI image pixel intensities and the corresponding pixel intensities of the reconstructed/filtered image has been employed as well as the RMS error in dB [26].

The quantitative results obtained by the different PDE models involved are outlined in table 1. These results show a superiority of the proposed adaptive parameter estimation anisotropic diffusion PDE model in terms of image reconstruction performance. The results illustrated in table 1 have been derived using 45 images randomly selected from the above-mentioned database. The best mean values of the reconstruction errors obtained from this sample of images, after a certain number of iterations in the numerical solution, different for each PDE model (so as to obtain best performance for each model) are reported herein under different noise levels.

The above results show that the proposed anisotropic diffusion scheme presents better performance compared to the standard anisotropic diffusion model (modified Perona–Malik model of equation (6)), in terms of image reconstruction under different noise levels and under different discretization schemes. Moreover, qualitative results obtained involving the methods outlined in table 1, regarding edge preservation, are outlined in figure 3.

Figure 2 presents an MRI image and its edges detected involving the standard Sobel algorithm, under no noise conditions.

Figure 3 clearly illustrates the edge preservation properties of the proposed PDE model when its results are compared with the ones obtained by applying the standard anisotropic PDE model of equation (6).

6. Conclusions

A novel methodology is presented in this paper for improving anisotropic diffusion PDE models, based on the Perona–Malik equation, suggesting an image-analysis-derived scheme to adaptively estimate the parameters of the regularizing function involved in this equation. The goal of such a new adaptive diffusion scheme is to better preserve edges when the anisotropic diffusion PDE models are applied to image enhancement. The proposed adaptive parameter estimation in the anisotropic diffusion PDE model involves self-organizing maps and Bayesian inference to define edge probabilities more accurately. As illustrated in the extensive experimental study conducted, the proposed modifications achieve capturing not only of simple edges but also of the more difficult textural edges. In the context of PDE models application to image processing of such adaptive schemes is closely related to the discrete image representation problem and the investigation of more suitable discretization algorithms using constraints derived from image processing theory. It is necessary to investigate, from an engineering perspective, how to incorporate sophisticated image processing algorithms in the discretization schemes used in the numerical analysis of continuous PDE models when applied to image processing to achieve good results. The proposed adaptive anisotropic diffusion model illustrates these concepts in a set of MRI images, although it is a first step in the improvement of anisotropic diffusion PDE models of Perona–Malik type. Another important step might be the application of more robust discretization schemes, like the MD-WDF approach in numerically approximating the involved PDEs [29]. Finally, since the proposed scheme increases computational needs, it is important to investigate faster algorithms in achieving similar edge preserving results.

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