Multiswapped networks and their topological and algorithmic properties

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Abstract

We generalise the biswapped network $Bsw(G)$ to obtain a multiswapped network $Msw(H;G)$, built around two graphs G and H. We show that the network $Msw(H;G)$ lends itself to optoelectronic implementation and examine its topological and algorithmic. We derive the length of a shortest path joining any two vertices in $Msw(H;G)$ and consequently a formula for the diameter. We show that if G has connectivity $\kappa \geq 1$ and H has connectivity $\lambda \geq 1$ where $\lambda \leq \kappa$ then $Msw(H;G)$ has connectivity at least $\kappa + \lambda$, and we derive upper bounds on the $(\kappa + \lambda)$-diameter of $Msw(H;G)$. Our analysis yields distributed routing algorithms for a distributed-memory multiprocessor whose underlying topology is $Msw(H;G)$. We also prove that if G and H are Cayley graphs then $Msw(H;G)$ need not be a Cayley graph, but when H is a bipartite Cayley graph then $Msw(H;G)$ is necessarily a Cayley graph.

Keywords: interconnection networks, hierarchical interconnection networks, OTIS networks, biswapped networks, multiswapped networks, shortest paths, connectivity, Cayley graphs

1. Introduction

Interconnection networks play a fundamental role in computer science. They are the means by which the processors of a distributed-memory multiprocessor computer (such as a Cray Jaguar or an IBM Blue Gene) communicate and also by which I/O devices communicate with processors and memory; they are increasingly used in network switches and routers to replace buses and crossbars; and they are crucial to on-chip networks. They are usually abstracted as directed or undirected graphs, with the vertices representing processors, devices or memory, and the edges the individual communication links. The design of interconnection networks is complex, with topology, flow control, routing and traffic patterns all impacting upon the usefulness of an interconnection network (see, for example, [5] for more details).

The implementation of an interconnection network cannot be overlooked (that is, its layout or packaging: see [5]). Ordinarily, interconnection networks are implemented electronically and the 'two-dimensional nature' of this environment can impose restrictions. Free-space optical interconnect technologies can offer several advantages over electronic implementations. For example, optical signals can pass through one another with little interference, and over a distance of greater than a few millimetres optical connections out-perform electronic connections in terms of power consumption, speed and crosstalk. However, optical connections are not a panacea for they can be difficult to route (the reader is referred to, for example, [4, 10, 13, 14, 31] for further details on the physical properties of optical connections and we no longer concern ourselves with such properties in this paper).

A popular realization of optical communication is the *Optical Transpose Interconnection System* (*OTIS*) (OTIS networks originated in [16] with their study initiated within the computer architecture community in [24] and independently, under the name of swapped networks, in [28, 29, 30]). OTIS networks are designed

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so as to utilize the best aspects of both electronic and optical communication. In essence, OTIS networks are such that the processors are uniformly partitioned into clusters so that the processors in a cluster are co-located in a small neighbourhood (perhaps on the same chip) and interconnected (according to the topology of some base graph) using electronic connections, whilst the (longer) communication links between processors in different clusters (the chip-to-chip connections, for example) are optical. There has been a considerable amount of research undertaken as regards OTIS networks, both with regard to the optoelectronics and also with regard to topological and algorithmic aspects of the networks (it is the latter that are more relevant to this paper and the reader is referred to, for example, [3, 18, 19, 20, 32] for a selection of some recent research in this vein).

One significant drawback of OTIS networks is that no matter what the base graph G, the resulting network OTIS-G is never vertex-symmetric and thus we can lose many of G's 'symmetry properties' when we form OTIS-G (we shall define OTIS networks, and other networks mentioned in this introduction, in detail in the next section and henceforth we consider such networks to be their abstractions as undirected graphs). In particular, if G is a Cayley graph (which is an extremely useful property for an interconnection network to have) then OTIS-G is never a Cayley graph (as every Cayley graph is vertex-symmetric). In order to 'recapture' symmetric aspects of OTIS networks, Xiao, Parhami, Chen, He and Wei [27] have recently proposed *biswapped networks* which, they claim, are 'fully symmetric and have cluster connectivity very similar to OTIS networks'. Like OTIS networks, a biswapped network $Bsw(G)$ also has a base graph G but, unlike OTIS networks, the adopted construction enables one to show that if G is a Cayley graph then $Bsw(G)$ is a Cayley graph too [25, 27]. Biswapped networks involve twice the number of vertices as their OTIS counterparts but, like their OTIS counterparts, still lend themselves to an optoelectronic layout.

In this paper, we generalise the notion of a biswapped network. This generalisation comes about from the simple observation that if one 'concatenates' biswapped networks then one can still obtain graphs that can easily be laid out (in an optoelectronic sense, just as OTIS and biswapped networks can) but where these new graphs have increased flexibility and improved topological and algorithmic properties. Our new graphs are not only parameterized by a base graph G but also by a network graph H which determines the 'pattern of concatenation'; we denote the resulting graph by $Msw(H;G)$ and call it a *multiswapped network*. As such, a graph $Msw(H;G)$ is hierarchical. The biswapped network $Bsw(G)$, with base graph G, from [25, 27] is the graph $Msw(H;G)$ where H consists of a solitary edge.

We go on to exhibit some fundamental algorithmic and topological properties of $Msw(H;G)$ (in terms of those of the two constituent graphs G and H). We establish a formula for the length of a shortest path joining any two vertices in $Msw(H;G)$, in terms of the shortest path between specific pairs of vertices in G and H, and consequently a formula for the diameter $\Delta(Msw(H;G))$ of $Msw(H;G)$ in terms of the diameters $\Delta(G)$ and $\Delta(H)$ of G and H. We show that if G is a graph of connectivity $\kappa \geq 1$ and H is a graph of connectivity $\lambda \geq 1$ where $\lambda \leq \kappa$ then $Msw(H;G)$ is a graph of connectivity at least $\kappa + \lambda$ and that the $(\kappa+\lambda)$-diameter of $Msw(H;G)$, $\Delta_{\kappa+\lambda}(Msw(H;G))$, is at most $\max\{\Delta_\kappa(G)+2\Delta(G)+\Delta_\lambda(H), 3\Delta(G)+5\}$, unless G consists of a solitary edge when it is at most $\max\{\Delta(H)+4, 8\}$, where $\Delta_\kappa(G)$ and $\Delta_\lambda(H)$ are the κ-diameter and λ-diameter of G and H. Our analysis yields distributed routing algorithms for a distributed-memory multiprocessor whose underlying topology is $Msw(H;G)$. Finally, we examine conditions on G and H which imply that $Msw(H;G)$ is a Cayley graph. We show that even if G and H are both Cayley graphs then $Msw(H;G)$ need not be a Cayley graph but that when additionally H is a bipartite Cayley graph, we have that $Msw(H;G)$ is a Cayley graph.

In the next section, we provide the basic definitions relating to this paper and define OTIS networks, biswapped networks and our new multiswapped networks in detail. In Section 3, we examine the construction of shortest paths in $Msw(H;G)$ and in Section 4 we examine the connectivity of $Msw(H;G)$ in relation to the connectivity of G and of H. In Section 5, we examine conditions on G and H which imply that $Msw(H;G)$ is a Cayley graph, and we present our conclusions and directions for further research in Section 6.

2. Background and definitions

In this section, we: detail our graph-theoretic terminology; provide background relating to (optical) interconnection networks from parallel computing; and introduce our new generalisations of the biswapped networks from [25, 27].

2.1. Graph terminology

All graphs $G = (V, E)$ are undirected with vertex set V and edge set E, and for any graph-theoretic terminology not defined here, we refer the reader to [9]. A *path* in a graph is a sequence of distinct vertices so that there is an edge joining consecutive vertices, with the *length* of a path ρ being the number of vertices in the sequence minus 1 and written $|\rho|$. A *cycle* (or *circuit*) is a path of length at least 2 where there is also an edge joining the first and last vertices. A *walk* is a sequence of not necessarily distinct vertices so that there is an edge joining consecutive vertices, and a *walk with repetitions* is a sequence of vertices where for any pair of consecutive vertices, either there is an edge joining them or they are identical. A *Hamiltonian path* in a graph is a path that contains every vertex of the graph exactly once, and a *Hamiltonian cycle* is a Hamiltonian path with an edge from the last vertex of the path to the first. The *internal vertices* of a path from a vertex v to a vertex v' in a graph are those vertices of the path different from v and v'. Two paths are *internally vertex-disjoint* (resp. *vertex-disjoint*) if neither has an internal vertex (resp. a vertex) that appears on the other path, and a set of paths in a graph are *mutually internally vertex-disjoint* (resp. *mutually vertex-disjoint*) if any two distinct paths are internally vertex-disjoint (resp. vertex-disjoint). The length of a shortest path in a graph $G = (V, E)$ between two vertices $v, v' \in V$ is denoted $d_G(v, v')$. The *diameter* of a graph G, denoted $\Delta(G)$, is $\max\{d_G(v, v') : v, v' \in V\}$. We denote by $d^0_G(v, v')$ (resp. $d^1_G(v, v')$) the length of a shortest even-length (resp. odd-length) path in G from v to v'.

A multipath routing algorithm is often associated with mutually internally vertex-disjoint paths of an interconnection network, or, more precisely, of a distributed-memory multiprocessor whose underlying topology is that interconnection network, where a *multipath routing algorithm* is an algorithm that finds mutually internally vertex-disjoint paths joining processors located at any pair of distinct vertices in the network. A multipath routing algorithm is a *source* multipath routing algorithm if the paths are fully computed at the source processor before messages are sent, and a multipath routing algorithm is *deterministic* if the algorithm depends solely upon the vertices at which the source and destination processors are located. Note that many interconnection networks have an exponential number of vertices in terms of the network's degree or connectivity. Consequently, in order for a multipath source routing algorithm to be efficient, one ordinarily wants it to run in time polynomial in the maximal degree of the network and/or the connectivity (this would mean that the lengths of the paths produced should be bounded by some polynomial in the maximal degree of the network and/or the connectivity too).

The *neighbourhood* of a vertex v of a graph $G = (V, E)$ is defined as $N_G(v) = \{v' \in V : (v, v') \in E\}$. The *degree* of a vertex v of the graph G is $|N_G(v)|$, and a graph is *regular of degree* d if every vertex has degree d. An *articulation set* for a graph $G = (V, E)$ is a subset of vertices $U \subseteq V$ so that if we remove every vertex of U from G, along with its incident edges, then the resulting graph has at least 2 connected components. A graph $G = (V, E)$ has *connectivity* $\kappa \geq 1$ if G has more than κ vertices and there is a set of κ vertices forming an articulation set but there exists no articulation set of size smaller than κ. We repeatedly use Menger's Theorem: if a graph $G = (V, E)$ has connectivity κ then given any vertex $v \in V$ and any distinct vertices $v_1, v_2, \ldots, v_\kappa \in V$, different from v, there are κ mutually internally vertex-disjoint paths from v to $v_1, v_2, \ldots, v_\kappa$ (one for each vertex of $\{v_1, v_2, \ldots, v_\kappa\}$). If a graph $G = (V, E)$ has connectivity at least $\kappa \geq 1$ then the κ*-diameter* $\Delta_\kappa(G)$ is the smallest integer such that for every pair of distinct vertices v and v' of V, there are κ mutually internally vertex-disjoint paths from v to v' so that the longest such path has length at most $\Delta_\kappa(G)$. Note that whilst Menger's Theorem relates the connectivity κ of a graph with the existence of κ mutually internally vertex-disjoint paths between vertices, it gives us no information concerning the κ-diameter of the graph. The *wide-diameter* of a graph G of connectivity κ is $\Delta_\kappa(G)$.

A *Cayley digraph* G is defined as follows. Let Γ be a finite group with generating set $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$. The elements of Γ form the vertex set of G and there is a directed edge (γ, γ') in the graph G if $\gamma\gamma_i = \gamma'$, for some $i \in \{1, 2, \ldots, r\}$. A *Cayley graph* is a Cayley digraph where the associated generating set is closed under inverses (and so directed edges can effectively be regarded as undirected edges). A graph $G = (V, E)$ is *vertex-symmetric* if given any two distinct vertices $v, v' \in V$, there is an automorphism of G mapping v to v'. Every Cayley graph is well known to be vertex-symmetric.

2.2. Optical transpose interconnection networks

OTIS networks have a *base graph* G, on n vertices, and consist of n disjoint copies of G. These copies are labelled $G_1, G_2, \ldots, G_n$ and the vertices of any copy are $v_1, v_2, \ldots, v_n$. The edges involved in any one of these copies of G are intended to model (shorter) electronic connections whereas additional edges, where there is an edge from vertex v_i of copy G_j to vertex v_j of copy G_i, for every $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, are intended to model the (longer) optical connections. The resulting OTIS network is denoted by OTIS-G. Of course, an OTIS network is dependent upon its base graph G, and numerous results have been proven for both specific base graphs and classes of base graphs (see, for example, the papers [1, 3, 7, 8, 17, 18, 19, 23] and the references therein).

As remarked earlier, one displeasing aspect of OTIS networks is that no matter what the base graph G is, the corresponding OTIS network OTIS-G cannot be a Cayley graph, or even a vertex-symmetric graph, as an OTIS network is not regular. In general, if the base graph G has some aspect of symmetry then we lose this symmetry in the graph OTIS-G, and as well as losing desirable specific properties, like vertex-symmetry, the loss of this symmetry can make general network analysis more problematic. The *biswapped network* $Bsw(G)$ is defined very similarly to the OTIS network OTIS-G except that instead of having n copies of the base graph G (where G has n vertices), we have $2n$ copies $G_0^1, G_0^2, \ldots, G_0^n, G_1^1, G_1^2, \ldots, G_1^n$ and the 'optical' edges join vertex v_j in G_0^i with vertex v_i in G_1^j, where $i, j \in \{1, 2, \ldots, n\}$. Immediately we see that if G is regular then the biswapped network $Bsw(G)$ is regular and so there is some hope for recapturing any symmetric properties of the base graph G. In [27] some basic properties of biswapped graphs are derived relating to shortest paths and routing algorithms. In [25], it is shown, amongst other things, that: if G is a Hamiltonian graph then $Bsw(G)$ is; and if G is a Cayley graph then $Bsw(G)$ is. Also in [25], a systematic construction of $\kappa + 1$ mutually internally vertex-disjoint paths joining any two distinct vertices in $Bsw(G)$ is derived, where $\kappa \geq 1$ is the connectivity of G. In doing so, an upper bound of $\max\{2\Delta(G) + 5, \Delta_\kappa(G) + \Delta(G) + 2\}$ on the $(\kappa + 1)$-diameter of $Bsw(G)$ is obtained.

It is worth clarifying the results in two other papers relating to biswapped networks that actually predate [25, 27], namely [2, 26], and which were unknown to the authors of [25] when this paper was written (the two papers not having been cited in [27]). The notion of a biswapped network is actually first defined in [26] and a number of properties of biswapped networks are claimed without proof. Although proofs that G is Hamiltonian implies that $Bsw(G)$ is Hamiltonian, and that G is a Cayley graph implies that $Bsw(G)$ is a Cayley graph are given in [26], the proofs of shortest path properties of biswapped networks claimed in [26] actually only appear in [27]. Also, the results stated in [26] relating to the connectivity of $Bsw(G)$ are proven in [2]. Whilst there are minor (but easily surmountable) deficiencies with the proofs from [2], the bounds obtained there are not as good as the ones obtained in [25] where not only are constructions given which relate to vertex degrees (and which are similar to those in [2]) but refined constructions are given which relate to the connectivity of the base graph G. These refined constructions give better bounds on the wide-diameter of $Bsw(G)$ by relating it to the wide-diameter of G and not to the degrees of vertices in G. For example, the hypercube Q_n has diameter n and wide-diameter $n + 1$ [21]. So, by [2], $Bsw(Q_n)$ has wide-diameter at most $3n + 6$ but by [25], $Bsw(Q_n)$ has wide-diameter at most $2n + 5$. All results (many of which have their proofs omitted) from [2, 26] can be found in full in [25, 27] and in a significantly improved form.

2.3. Multiswapped networks

We now generalise the definition of a biswapped network. Unlike the definition of a biswapped network, above, the construction involves two component graphs.

Definition 1. *Let $H = (U, F)$ and $G = (V, E)$ be graphs where U and V both contain at least 2 vertices. The graph $Msw(H; G)$ is known as the* multiswapped graph with network H and base G *and is defined as follows*:

- *$Msw(H;G)$ has vertex set* $\{(u,v,w) : u \in U, v,w \in V\}$
- *$Msw(H;G)$ has edge set consisting of:*
  - $\{((u,v,w),(u,v,w')) : u \in U, v,w,w' \in V, (w,w') \in E\}$*, the* cluster edges*, and*
  - $\{((u,v,w),(u',w,v)) : (u,u') \in F, v,w \in V\}$*, the* swap edges.

We say that the vertices *corresponding* to some vertex $u \in U$ are the vertices of $Msw(H;G)$ whose first component is u, and that a vertex (u,v,w) of $Msw(H;G)$ corresponding to $u \in U$ has *index* $v \in V$. The edges induced by the vertices of $Msw(H;G)$ corresponding to some vertex $u \in U$ are the cluster edges. We denote the copy of G induced by the vertices corresponding to u and indexed by v as G_u^v. We often write a cluster edge of the form $((u,v,w),(u,v,w'))$ as $(u,v,w) \rightarrow_c (u,v,w')$, and a swap edge of the form $((u,v,w),(u,w,v))$ is often written $(u,v,w) \rightarrow_s (u,w,v)$. A path of (possibly no) cluster edges $(u,v,w) \rightarrow_c \ldots \rightarrow_c (u,v,w')$ is often written as $(u,v,w) \rightarrow_c^* (u,v,w')$.

The vertices corresponding to the vertices u and u' of U and the edge (u,u') of F are depicted in two different ways in Fig. 1. In both depictions, the vertices of V are enumerated as $v_1, v_2, \ldots, v_n$. In the top depiction, the vertex (u, v_i, v_j), for example, lies on the row corresponding to vertex $u \in U$, and within this row it is vertex v_j of the cluster indexed by v_i. In the bottom depiction, as regards the vertices corresponding to u', there is one row for the vertices indexed by each $v \in V$, and the vertex (u', v_i, v_j), for example, lies on the row indexed by vertex $v_i \in V$. Note that if H is a solitary edge then $Msw(H;G)$ is identical to the biswapped network $Bsw(G)$ from [27].

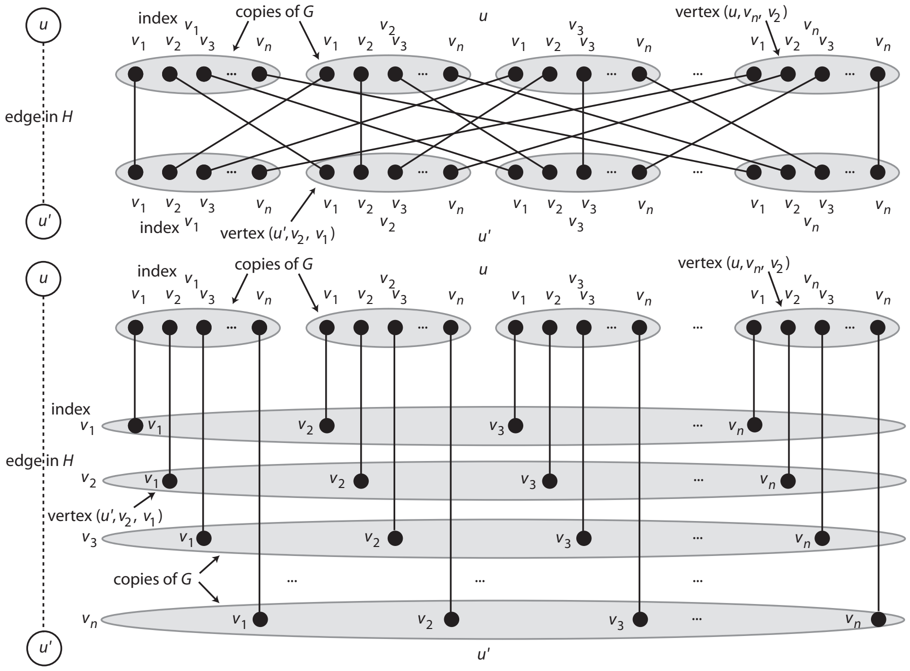

Figure 1. Some edges in $Msw(H;G)$.

As regards the optoelectronic implementation of OTIS (and biswapped) networks, the reader is referred to [31] for a clear explanation of how this might be done. It is remarked in [31] that the OTIS optoelectronic architecture can be 'cascaded' so that successive optoelectronic links can be accommodated; hence, our multiswapped networks can be (potentially) implemented. However, it should be pointed out that the optoelectronic implementation of multiswapped networks will involve additional (though not insurmountable) technological and hardware costs that need to be overcome relating to, for example, the number of beam splitters and transceivers and the consequent extended footprint (see the overview of optical interconnection networks in [4]).

Let us also point out that there exists another extension of biswapped networks in the literature, namely the *generalized biswapped networks* from [12]. These networks share, with multiswapped networks, the property of being built from two basis networks, with these two basis networks being incorporated into a 'biswapped construction'. However, the similarity ends there; generalized biswapped networks have no 'network dimension' and are, to some extent, a modest extension of biswapped networks.

Note that whilst the construction of our multiswapped networks is motivated by the construction of optoelectronic networks, as we shall see these networks are extremely interesting from a combinatorial perspective and they are worthy of study even if one treats them purely as graph-theoretic objects or as topologies for standard (electronic) interconnection networks.

Throughout this paper, the graph H (resp. G) has vertex set U (resp. V) and edge set F (resp. E).

3. The composition of shortest paths

In this section, we start with the comparatively simple task of establishing a formula for the length of a shortest path joining any two vertices of $Msw(H;G)$, where $G = (V,E)$ and $H = (U,F)$ are connected graphs. We also discuss the resulting distributed routing algorithm for $Msw(H;G)$. As regards motivation, the study and usage of shortest paths for routing in interconnection networks (optoelectronic or otherwise) is absolutely fundamental.

Consider any path in $Msw(H;G)$ from vertex (u,v,w) to vertex (u',v',w'). Such a path must be of the following form:

$$\begin{aligned}
(u,v,w) = (u_1,v_1,w_1) &\rightarrow_c^* (u_1,v_1,w_2) \rightarrow_s (u_2,w_2,v_1) \rightarrow_c^* (u_2,w_2,v_2) \rightarrow_s (u_3,v_2,w_2)\\
&\rightarrow_c^* (u_3,v_2,w_3) \rightarrow_s (u_4,w_3,v_2) \rightarrow_c^* (u_4,w_3,v_3) \rightarrow_s (u_5,v_3,w_3)\\
&\rightarrow_c^* (u_5,v_3,w_4) \rightarrow_s (u_6,w_4,v_3) \rightarrow_c^* (u_6,w_4,v_4) \rightarrow_s (u_7,v_4,w_4)\\
&\ldots \rightarrow_s (u_{2p},w_{p+1},v_p)\\
&\text{ or } \rightarrow_c^* (u_{2p-1},v_p,w_{p+1})\\
&\text{ or } \rightarrow_s (u_{2p-1},v_p,w_p)\\
&\text{ or } \rightarrow_c^* (u_{2p-2},w_p,v_p).
\end{aligned}$$

Suppose that $(u',v',w') = (u_{2p},w_{p+1},v_p)$ and so the path from (u,v,w) to (u',v',w') in $Msw(H;G)$ contains an odd number of swap edges. So: there are walks with repetitions in G from $w = w_1$ through w_2 through $\ldots$ to $w_{p+1} = v'$ and from $v = v_1$ through v_2 through $\ldots$ to $v_p = w'$; and $u = u_1, u_2, \ldots, u_{2p} = u'$ is a walk in H. In particular, any such path from (u,v,w) to (u',v',w') has length at least $d_G(w,v') + d_G(v,w') + d_H^1(u,u')$. An identical argument for the case when $(u',v',w') = (u_{2p-2},w_p,v_p)$ yields the same conclusion. Arguing similarly yields that any path from (u,v,w) to (u',v',w') containing a non-zero even number of swap edges has length at least $d_G(v,v') + d_G(w,w') + d_H^0(u,u')$. Hence, if $u \neq u'$ then

$$\begin{aligned}
d_{Msw(H;G)}((u,v,w),(u',v',w')) \geq \quad \min\{ \quad & d_G(w,v') + d_G(v,w') + d_H^1(u,u'),\\
& d_G(v,v') + d_G(w,w') + d_H^0(u,u')\},
\end{aligned}$$

assuming that there are both odd- and even-length paths from u to u' in H; if there is no odd-length path then $d_{Msw(H;G)}((u,v,w),(u',v',w')) \geq d_G(v,v') + d_G(w,w') + d_H^0(u,u')$ and if there is no even-length path then $d_{Msw(H;G)}((u,v,w),(u',v',w')) \geq d_G(w,v') + d_G(v,w') + d_H^1(u,u')$.

If $u = u'$ and $v \neq v'$ then any path from (u,v,w) to (u,v',w') must contain vertices not corresponding to u and we can proceed as above so that we obtain

$$\begin{aligned}
d_{Msw(H;G)}((u,v,w),(u,v',w')) \geq \quad \min\{ \quad & d_G(w,v') + d_G(v,w') + c_H^1(u),\\
& d_G(v,v') + d_G(w,w') + 2\}.
\end{aligned}$$

where $c^1_H(u)$ is the shortest odd-length cycle in H that contains u (if no such cycle exists then the corresponding term is omitted). If $u = u'$ and $v = v'$ then we obtain

$$d_{Msw(H;G)}((u,v,w),(u,v,w')) \geq \min\{ \; d_G(w,v) + d_G(v,w') + c^1_H(u), \\ d_G(w,w')\},$$

assuming that there is an odd-length cycle in H that contains u. Clearly, irrespective of whether there is an odd-length cycle in H containing u, we have that $d_{Msw(H;G)}((u,v,w),(u,v,w')) = d_G(w,w')$.

Suppose that $d^1_H(u,u') = r$ (so r is odd) and let $u = u_1, u_2, \ldots, u_{r+1} = u'$ be a shortest odd-length path from u to u' in H. Consider the following path in $Msw(H;G)$:

$$(u,v,w) = (u_1,v,w) \rightarrow_c \ldots \rightarrow_c (u_1,v,v') \rightarrow_s (u_2,v',v) \rightarrow_c \ldots \rightarrow_c (u_2,v',w') \\ \rightarrow_s (u_3,w',v') \rightarrow_s (u_4,v',w') \rightarrow_s \ldots \rightarrow_s (u_{r+1},v',w') = (u',v',w')$$

where the corresponding paths in G from w to v' and from v to w' are as short as possible. This path has length $d_G(w,v') + d_G(v,w') + d^1_H(u,u')$.

Suppose that $d^0_H(u,u') = s > 0$ (so s is even and $u \neq u'$) and let $u = u_1, u_2, \ldots, u_{s+1} = u'$ be a shortest even-length path from u to u' in H. Consider the following path in $Msw(H;G)$:

$$(u,v,w) = (u_1,v,w) \rightarrow_c \ldots \rightarrow_c (u_1,v,w') \rightarrow_s (u_2,w',v) \rightarrow_c \ldots \rightarrow_c (u_2,w',v') \\ \rightarrow_s (u_3,v',w') \rightarrow_s (u_4,w',v') \rightarrow_s \ldots \rightarrow_s (u_{s+1},v',w') = (u',v',w')$$

where the corresponding paths in G from v to v' and from w to w' are as short as possible. This path has length $d_G(v,v') + d_G(w,w') + d^0_H(u,u')$.

Finally, suppose that $u = u'$ and $v \neq v'$. Build the path as we did above but where we substitute a shortest odd-length walk from u to u in H for a shortest odd-length path from u to u' in H; so, the resulting path has length $d_G(w,v') + d_G(v,w') + c^1_H(u,u)$. Define the path

$$(u,v,w) \rightarrow_c \ldots \rightarrow_c (u,v,w') \rightarrow_s (u',w',v) \rightarrow_c \ldots \rightarrow_c (u',w',v') \rightarrow_s (u',v',w'),$$

where $(u,u') \in F$ and where the corresponding paths in G from v to v' and from w to w' are as short as possible; so, this path has length $d_G(v,v') + d_G(w,w') + 2$. Consequently, we obtain the following result.

Theorem 2. *Let G and H be connected graphs and let (u,v,w) and (u',v',w') be vertices of $Msw(H;G)$.*

1. *If $u \neq u'$ and there are paths of both odd- and even-length in H from u to u' then*

$$d_{Msw(H;G)}((u,v,w),(u',v',w')) = \min\{ \; d_G(w,v') + d_G(v,w') + d^1_H(u,u'), \\ d_G(v,v') + d_G(w,w') + d^0_H(u,u')\}.$$

2. *If $u \neq u'$ and there are no paths of odd-length in H from u to u' then $d_{Msw(H;G)}((u,v,w),(u',v',w')) = d_G(v,v') + d_G(w,w') + d^0_H(u,u')$.*
3. *If $u \neq u'$ and there are no paths of even-length in H from u to u' then $d_{Msw(H;G)}((u,v,w),(u',v',w')) = d_G(w,v') + d_G(v,w') + d^1_H(u,u')$.*
4. *If $u = u'$ and $v = v'$ then $d_{Msw(H;G)}((u,v,w),(u,v,w')) = d_G(w,w')$.*
5. *If $u = u'$, $v \neq v'$ and there is an odd-length cycle in H containing u then*

$$d_{Msw(H;G)}((u,v,w),(u,v',w')) = \min \; \{d_G(w,v') + d_G(v,w') + c^1_H(u), \\ d_G(v,v') + d_G(w,w') + 2\}.$$

6. *If $u = u'$, $v \neq v'$ and there is no odd-length cycle in H containing u then $d_{Msw(H;G)}((u,v,w)(u,v',w'))$ $= d_G(v,v') + d_G(w,w') + 2$.*

Of course, Theorem 2 subsumes parts (3) and (4) of Theorem 1 of [27] where the lengths of shortest paths in $Bsw(G)$ are obtained.

Corollary 3. *Let G and H be connected graphs. The diameter $\Delta(Msw(H;G))$ of $Msw(H;G)$ is $2\Delta(G)+\Delta(H)$, unless H is a clique when it is $2\Delta(G)+2$.*

Proof. Let $u,u' \in U$ be such that $d_H(u,u') = \Delta(H)$ and let $v,v' \in V$ be such that $d_G(v,v') = \Delta(G)$.

Suppose that $\Delta(H)$ is even or odd and at least 3. By Theorem 2, we have that $\Delta(Msw(H;G)) \leq 2\Delta(G)+\Delta(H)$. However, by Theorem 2, $d_{Msw(H;G)}((u,v,v),(u',v',v')) = 2\Delta(G)+\Delta(H)$. Hence, $\Delta(Msw(H;G)) = 2\Delta(G)+\Delta(H)$.

Suppose that $\Delta(H) = 1$ (so H is a clique). By Theorem 2, $\Delta(Msw(H;G)) \leq 2\Delta(G)+2$. However, by Theorem 2, $d_{Msw(H;G)}((u,v,v),(u,v',v')) = 2\Delta(G)+2$. Hence, $\Delta(Msw(H;G)) = 2\Delta(G)+2$. □

We now show that if either of G or H is not connected then $Msw(H;G)$ is not connected.

Proposition 4. *Suppose that G and H are graphs at least one of which is not connected. The graph $Msw(H;G)$ is not connected.*

Proof. Suppose that H is not connected. Thus, there is a partition of U into two non-empty subsets of vertices so that no edge of H joins a vertex in one set to a vertex in the other. Consequently, if u and u' lie in different sets of the partition of U then by looking at our description of an arbitrary path from some vertex (u,v,w) to a vertex (u',v',w') in $Msw(H;G)$, we find that such a path cannot exist.

Suppose that G is not connected. Thus, there is a partition of V into two non-empty subsets of vertices so that no edge of G joins a vertex in one set to a vertex in the other. Consequently, if v and v' lie in different sets of the partition of V and $u,u' \in U$ are such that $u \neq u'$ then by looking at our description of an arbitrary path from the vertex (u,v,v) to the vertex (u',v',v') in $Msw(H;G)$, we find that such a path cannot exist. □

In [27], a distributed routing algorithm was developed that routed a message from a source vertex to a destination vertex in $Bsw(G)$ (more precisely, in a distributed-memory multiprocessor whose underlying topology is $Bsw(G)$) along a shortest path, assuming that there is such a distributed routing algorithm for G. Using our discussion above, and in particular our construction of shortest paths from one vertex to another in $Msw(H;G)$, we can obtain an analogous distributed routing algorithm for $Msw(H;G)$, assuming that there are such distributed routing algorithms for G and H. However, a little more sophistication is required for $Msw(H;G)$. Suppose that we wish to route a message from vertex (u,v,w) to vertex (u',v',w'), where $u \neq u'$ and where there are (shortest) odd- and even-length paths ρ and ρ' in H from u to u', respectively. As part of our distributed algorithm, we must decide which of ρ and ρ' yields the shortest path in $Msw(H;G)$ from (u,v,w) to (u',v',w') prior to transmitting the message (see part (1) of Theorem 2); that is, we assume that any processor has available to it a description of the graphs G and H. If this description of H is not available then our distributed algorithm must transmit the message via both potential shortest paths (which involves some redundancy).

4. The composition of connected graphs

In this section, we examine the connectivity of $Msw(H;G)$ in relation to the connectivity of $G=(V,E)$ and of $H=(U,F)$, and also the existence of efficient deterministic multipath source routing algorithms in distributed-memory multiprocessors whose underlying interconnection network is $Msw(H;G)$.

4.1. Connectivity

In the constructions that follow, we explicitly use the proofs of Propositions 4.1–4.3 from [25], which go to providing an upper bound on $\Delta_{\kappa+1}(Bsw(G))$ when G is a graph of connectivity κ. As such, the exposition below should be read in conjunction with the relevant proposition from [25]. The study of connectivity and the lengths of mutually internally vertex-disjoint paths in a general interconnection network is again a fundamental problem as it impacts upon the fault tolerance of the interconnection network and its capacity to route traffic via different paths so as to speed up message transfer.

Proposition 5. *Let G and H be graphs of connectivity κ and λ, respectively, where $1 \leq \lambda \leq \kappa$. Let (u, v, w) and (u, v, w') be distinct vertices of $Msw(H; G)$. There are at least $\kappa + \lambda$ mutually internally vertex-disjoint paths in $Msw(H; G)$ joining the vertices (u, v, w) and (u, v, w'), each of which has length at most* $\max\{\Delta_\kappa(G), \Delta(G) + 6\}$.

Proof. Let $v_1, v_2, \ldots, v_\kappa$ be distinct neighbours of v in G, and let $u_1, u_2, \ldots, u_\lambda$ be distinct neighbours of u in H. With reference to the proof of Proposition 4.1 of [25], let ρ_i be the path ρ of $Msw(H; G)$ as constructed in that proof except that the parameter v_i replaces v^* and the parameter u_i replaces $\bar{u}$, for $i = 1, 2, \ldots, \lambda$ (crucially, each ρ_i contains no internal vertices in G_u^v within $Msw(H; G)$). Along with the κ mutually internally vertex-disjoint paths from (u, v, w) to (u, v, w') in G_u^v within $Msw(H; G)$ (each of which has length at most $\Delta_\kappa(G)$), this yields $\kappa + \lambda$ mutually internally vertex-disjoint paths from (u, v, w) to (u, v, w') in $Msw(H; G)$. The longest such path has length at most $\max\{\Delta_\kappa(G), \Delta(G) + 6\}$. □

Proposition 6. *Let G and H be graphs of connectivity κ and λ, respectively, where $1 \leq \lambda \leq \kappa$. Let (u, v, w) and (u, v', w') be distinct vertices of $Msw(H; G)$, where $v \neq v'$. There are at least $\kappa + \lambda$ mutually internally vertex-disjoint paths in $Msw(H; G)$ joining (u, v, w) and (u, v', w'), each of which has length at most* $\max\{\Delta_\kappa(G) + \Delta(G) + 2, 2\Delta(G) + 5\}$.

Proof. Let $v_1, v_2, \ldots, v_\kappa$ be distinct neighbours of v in G, and let $u_1, u_2, \ldots, u_\lambda$ be distinct neighbours of u in H. If $v' \in \{v_1, v_2, \ldots, v_\kappa\}$ then assume w.l.o.g. that $v' = v_1$. With reference to the proof of Proposition 4.2 of [25], there are two cases. The first is when $w \neq w'$. Build the paths $\sigma_1, \sigma_2, \ldots, \sigma_\kappa, \rho$ in $Msw(H; G)$ as in Case 1 of the proof of Proposition 4.2 of [25] but replace the parameter $\bar{u}$ with the parameter u_1 (note that all vertices involved in these paths correspond to u or u_1). For $j = 2, 3, \ldots, \lambda$, build the path ρ_j as

$$\begin{aligned}(u, v, w) &\rightarrow_s (u_j, w, v) \rightarrow_c (u_j, w, v_j) \rightarrow_s (u, v_j, w) \rightarrow_c^* (u, v_j, w') \\ &\rightarrow_s (u_j, w', v_j) \rightarrow_c^* (u_j, w', v') \rightarrow_s (u, v', w'),\end{aligned}$$

where the path in $G_u^{v_j}$ from (u, v_j, w) to (u, v_j, w') is arbitrary as is the path in $G_{u_j}^{w'}$ from (u_j, w', v_j) to (u_j, w', v') (in future, and as was the case in Propositions 4.1–4.3 of [25], if we write, for example, $(u, v_j, w) \rightarrow_c^* (u, v_j, w')$ then unless we state otherwise the implied path is any path in $G_u^{v_j}$ from (u, v_j, w) to (u, v_j, w')). This yields $\kappa + \lambda$ mutually internally vertex-disjoint paths in $Msw(H; G)$ from (u, v, w) to (u, v', w') so that each path has length at most $\max\{\Delta_\kappa(G) + \Delta(G) + 2, 2\Delta(G) + 5\}$.

The second case is when $w = w'$. Again, build the paths $\sigma_1, \sigma_2, \ldots, \sigma_\kappa, \rho$ in $Msw(H; G)$ as in Case 2 of the proof of Proposition 4.2 of [25] but replace the parameter $\bar{u}$ with the parameter u_1. For $j = 2, 3, \ldots, \lambda$, build the path ρ_j as

$$(u, v, w) \rightarrow_s (u_j, w, v) \rightarrow_c^* (u_j, w, v') \rightarrow_s (u, v', w) = (u, v', w').$$

This yields $\kappa + \lambda$ mutually internally vertex-disjoint paths in $Msw(H; G)$ from (u, v, w) to (u, v', w') so that each path has length at most $\Delta(G) + 4$. □

Proposition 7. *Let G and H be graphs of connectivity κ and λ, respectively, where $1 \leq \lambda \leq \kappa$ and $\kappa \geq 2$. Let (u, v, w) and (u', v', w') be distinct vertices of $Msw(H; G)$, where $u \neq u'$ and where $w \neq w'$. There are at least $\kappa + \lambda$ mutually internally vertex-disjoint paths in $Msw(H; G)$ joining (u, v, w) and (u', v', w'), each of which has length at most* $\max\{\Delta_\kappa(G) + 2\Delta(G) + \Delta_\lambda(H), 3\Delta(G) + 5\}$.

Proof. By hypothesis, there exist λ mutually internally vertex-disjoint paths $\alpha_1, \alpha_2, \ldots, \alpha_\lambda$ in H joining u and u' so that the path α_i has length $d_i \leq \Delta_\lambda(H)$; w.l.o.g. we may assume that $d_1 \leq d_2 \leq \ldots \leq d_\lambda$. Let $\tau_1, \tau_2, \ldots, \tau_\kappa$ be mutually internally vertex-disjoint paths from w to w' in G, where τ_1 is a path of minimal length from amongst these paths and where each path has length at most $\Delta_\kappa(G)$. Let w_i be the neighbour of w on the path τ_i, for $i = 1, 2, \ldots, \kappa$ (note that w_1 might be w'). These assumptions apply throughout this proof.

We begin by detailing some generic constructions of paths in $Msw(H; G)$ that are built around paths in H.

Paths in H of length 2: Suppose that α is the path u, u_1, u' in H. Define the path $\rho_0^2(\alpha)$ in $Msw(H;G)$ as follows:

$$(u,v,w) \to_s (u_1,w,v) \to_c^* (u_1,w,v') \to_s (u',v',w) \to_c^* (u',v',w'),$$

where the path in $G_{u'}^{v'}$ from (u',v',w) to (u',v',w') is isomorphic to the path τ_1. Define the path $\rho_1^2(\alpha)$ in $Msw(H;G)$ as follows:

$$(u,v,w) \to_c^* (u,v,w') \to_s (u_1,w',v) \to_c^* (u_1,w',v') \to_s (u',v',w'),$$

where the path in G_u^v from (u,v,w) to (u,v,w') is isomorphic to the path τ_1. For $i = 2,3,\ldots,\kappa$, define the path $\rho_i^2(\alpha)$ in $Msw(H;G)$ as follows:

$$(u,v,w) \to_c (u,v,w_i) \to_s (u_1,w_i,v) \to_c^* (u_1,w_i,v') \to_s (u',v',w_i) \to_c^* (u',v',w'),$$

where the path in $G_{u'}^{v'}$ from (u',v',w_i) to (u',v',w') is isomorphic to the sub-path of τ_i from w_i to w'. The paths $\rho_0^2(\alpha), \rho_1^2(\alpha), \ldots, \rho_\kappa^2(\alpha)$ are clearly mutually internally vertex-disjoint paths from (u,v,w) to (u',v',w') in $Msw(H;G)$ so that the vertices involved all correspond to an element of $\{u,u_1,u'\}$ and where if a vertex of one of the paths corresponds to u (resp. u') then it is indexed by v (resp. v'). We may assume that each of the paths $\rho_0^2(\alpha), \rho_1^2(\alpha), \ldots, \rho_\kappa^2(\alpha)$ has length at most $\Delta_\kappa(G) + \Delta(G) + 2$. The paths can be visualized as in Fig. 2.

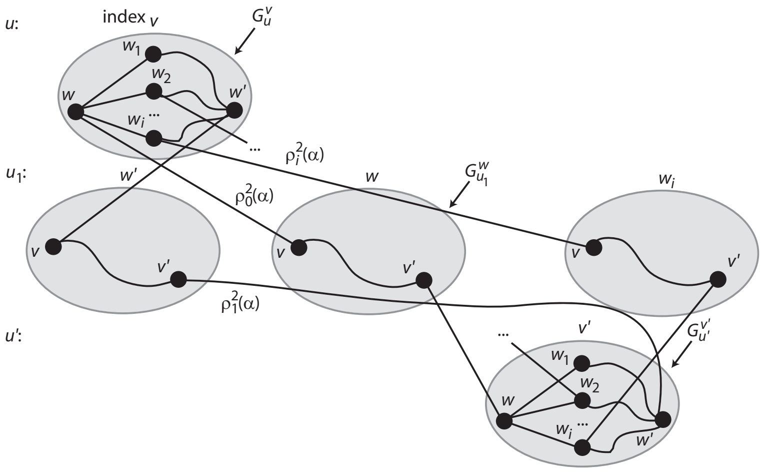

Figure 2. The paths $\rho_0^2(\alpha), \rho_1^2(\alpha), \ldots, \rho_\kappa^2(\alpha)$.

Paths in H of length 3: Suppose that α is the path u, u_1, u_2, u' in H. Define the path $\rho_0^3(\alpha)$ in $Msw(H;G)$ as follows:

$$(u,v,w) \to_s (u_1,w,v) \to_c^* (u_1,w,w) \to_s (u_2,w,w) \to_c^* (u_2,w,v') \to_s (u',v',w) \to_c^* (u',v',w'),$$

where the path in $G_{u'}^{v'}$ from (u',v',w) to (u',v',w') is isomorphic to the path τ_1. Define the path $\rho_1^3(\alpha)$ in $Msw(H;G)$ as follows:

$$(u,v,w) \to_c^* (u,v,w') \to_s (u_1,w',v) \to_c^* (u_1,w',w') \to_s (u_2,w',w') \to_c^* (u_2,w',v') \to_s (u',v',w'),$$

where the path in G_u^v from (u,v,w) to (u,v,w') is isomorphic to the path τ_1. For $i = 2,3,\ldots,\kappa$, define the path $\rho_i^3(\alpha)$ in $Msw(H;G)$ as follows:

$$\begin{aligned}(u,v,w) &\to_c (u,v,w_i) \to_s (u_1,w_i,v) \to_c^* (u_1,w_i,w_i)\\ &\to_s (u_2,w_i,w_i) \to_c^* (u_2,w_i,v') \to_s (u',v',w_i) \to_c^* (u',v',w'),\end{aligned}$$

where the path in $G_{u'}^{v'}$ from (u',v',w_i) to (u',v',w') is isomorphic to the sub-path of τ_i from w_i to w'. The paths $\rho_0^3(\alpha), \rho_1^3(\alpha), \ldots, \rho_\kappa^3(\alpha)$ are clearly mutually internally vertex-disjoint paths from (u,v,w) to (u',v',w')

in $Msw(H;G)$ so that the vertices involved all correspond to an element of $\{u, u_1, u_2, u'\}$ and where if a vertex of one of the paths corresponds to u (resp. u') then it is indexed by v (resp. v'). We may assume that each of the paths $\rho_0^3(\alpha), \rho_1^3(\alpha), \ldots, \rho_\kappa^3(\alpha)$ has length at most $\Delta_\kappa(G) + 2\Delta(G) + 3$. The paths can be visualized as in Fig. 3. Finally, define the path $\rho_*^3(\alpha)$ as follows:

$$(u, v, w) \to_s (u_1, w, v) \to_c^* (u_1, w, w') \to_s (u_2, w', w) \to_c^* (u_2, w', v') \to_s (u', v', w').$$

Note that all internal vertices of $\rho_*^3(\alpha)$ correspond to u_1 or u_2 and that $\rho_*^3(\alpha)$ has length at most $2\Delta(G)+3$.

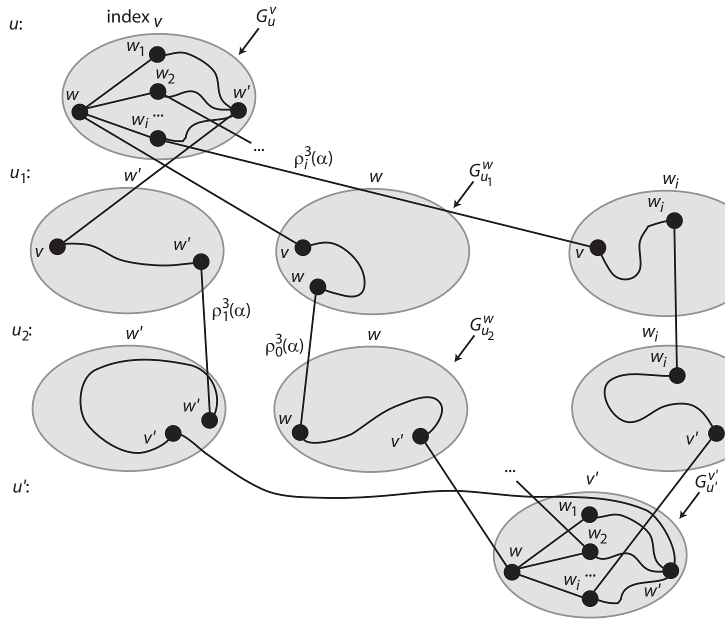

Figure 3. The paths $\rho_0^3(\alpha), \rho_1^3(\alpha), \ldots, \rho_\kappa^3(\alpha)$.

We now show how to 'extend' the above paths.

Paths in H of even length at least 2: Suppose that α is the path u, u_1, u_2, u_3, u' in H and that α' is the sub-path u, u_1, u_2. Consider the path $\rho_0^2(\alpha')$ in $Msw(H;G)$. Truncate this path at (u_2, v', w) and then extend it by the path:

$$(u_2, v', w) \to_s (u_3, w, v') \to_s (u', v', w) \to_c^* (u', v', w'),$$

where the path in $G_{u'}^{v'}$ from (u', v', w) to (u', v', w') is isomorphic to the path τ_1. Denote the resulting path by $\rho_0^4(\alpha)$. Consider the path $\rho_1^2(\alpha')$ in $Msw(H;G)$. Extend it by the path:

$$(u_2, v', w') \to_s (u_3, w', v') \to_s (u', v', w')$$

and denote the resulting path by $\rho_1^4(\alpha)$. Consider the path $\rho_i^2(\alpha')$ in $Msw(H;G)$, where $i \in \{2, 3, \ldots, \kappa\}$. Truncate this path at (u_2, v', w_i) and then extend it by the path:

$$(u_2, v', w_i) \to_s (u_3, w_i, v') \to_s (u', v', w_i) \to_c^* (u', v', w'),$$

where the path in $G_{u'}^{v'}$ from (u', v', w_i) to (u', v', w') is isomorphic to the sub-path of τ_i from w_i to w'. Denote the resulting path by $\rho_i^4(\alpha)$. The paths $\rho_0^4(\alpha), \rho_1^4(\alpha), \ldots, \rho_\kappa^4(\alpha)$ are clearly mutually internally vertex-disjoint paths from (u, v, w) to (u', v', w') in $Msw(H;G)$ so that the vertices involved all correspond to an element of $\{u, u_1, u_2, u_3, u'\}$ and where if a vertex of one of the paths corresponds to u (resp. u') then it is indexed by v (resp. v'). We may assume that each of the paths $\rho_0^4(\alpha), \rho_1^4(\alpha), \ldots, \rho_\kappa^4(\alpha)$ has length at most $\Delta_\kappa(G) + \Delta(G) + 4$.

Define the path $\rho_*^4(\alpha)$ as follows:

$$(u,v,w) \to_s (u_1,w,v) \to_s (u_2,v,w) \to_c^* (u_2,v,w') \to_s (u_3,w',v) \to_c^* (u_3,w',v') \to_s (u',v',w').$$

Note that all internal vertices of $\rho_*^4(\alpha)$ correspond to one of $\{u_1,u_2,u_3\}$ and that the path $\rho_*^4(\alpha)$ has length at most $2\Delta(G)+4$.

Now suppose that α is the path $u,u_1,u_2,\ldots,u_{d-1},u'$ in H, where $d \geq 4$ is even. We can repeatedly apply the above construction to obtain paths $\rho_0^d(\alpha),\rho_1^d(\alpha),\ldots,\rho_\kappa^d(\alpha)$ in $Msw(H;G)$ that are mutually internally vertex-disjoint paths from (u,v,w) to (u',v',w') so that the vertices involved all correspond to an element of $\{u,u_1,u_2,\ldots,u_{d-1},u'\}$ and where if a vertex of one of the paths corresponds to u (resp. u') then it is indexed by v (resp. v'). We may assume that each of the paths $\rho_0^d(\alpha),\rho_1^d(\alpha),\ldots,\rho_\kappa^d(\alpha)$ has length at most $\Delta_\kappa(G)+\Delta(G)+d$. We can also analogously obtain a path $\rho_*^d(\alpha)$ so that all internal vertices correspond to one of $\{u_1,u_2,\ldots,u_{d-1}\}$ and so that the length of this path is at most $2\Delta(G)+d$.

<u>Paths in H of odd length at least 3</u>: In the same way we can extend paths of odd length. Suppose that α is the path $u,u_1,u_2,\ldots,u_{d-1},u'$ in H, where $d \geq 3$ is odd. By proceeding analogously to as we did above, but starting from the paths $\rho_0^3(\alpha'),\rho_1^3(\alpha'),\ldots,\rho_\kappa^3(\alpha')$ in $Msw(H;G)$, where α' is the path u,u_1,u_2,u_3 in H, we can obtain paths $\rho_0^d(\alpha),\rho_1^d(\alpha),\ldots,\rho_\kappa^d(\alpha)$ in $Msw(H;G)$ that are mutually internally vertex-disjoint paths from (u,v,w) to (u',v',w') so that the vertices involved all correspond to an element of $\{u,u_1,u_2,\ldots,u_{d-1},u'\}$ and where if a vertex of one of the paths corresponds to u (resp. u') then it is indexed by v (resp. v'). We may assume that each of the paths $\rho_0^d(\alpha),\rho_1^d(\alpha),\ldots,\rho_\kappa^d(\alpha)$ has length at most $\Delta_\kappa(G)+2\Delta(G)+d$. We can also similarly obtain a path $\rho_*^d(\alpha)$ so that all internal vertices correspond to one of $\{u_1,u_2,\ldots,u_{d-1}\}$ and so that the length of this path is at most $2\Delta(G)+d$.

Now we are in a position to prove the result.

<u>Case 1</u>: $d_1 \geq 3$.

The paths $\rho_0^{d_1}(\alpha_1),\rho_1^{d_1}(\alpha_1),\ldots,\rho_\kappa^{d_1}(\alpha_1),\rho_*^{d_2}(\alpha_2),\rho_*^{d_3}(\alpha_3),\ldots,\rho_*^{d_\lambda}(\alpha_\lambda)$ are mutually internally vertex-disjoint paths from (u,v,w) to (u',v',w') in $Msw(H;G)$ so that the lengths of the paths are bounded as follows:

- if d_1 is odd then $|\rho_i^{d_1}(\alpha_1)| \leq \Delta_\kappa(G)+2\Delta(G)+d_1$ and if d_1 is even then $|\rho_i^{d_1}(\alpha_1)| \leq \Delta_\kappa(G)+\Delta(G)+d_1$, for $i=0,1,\ldots,\kappa$;
- $|\rho_*^{d_i}(\alpha_i)| \leq 2\Delta(G)+d_i$, for $i=2,3,\ldots,\lambda$.

Thus, the length of any of these paths is at most $\Delta_\kappa(G)+2\Delta(G)+\Delta_\lambda(H)$. (The crucial point to note is that the paths $\rho_*^{d_2}(\alpha_2),\rho_*^{d_3}(\alpha_3),\ldots,\rho_*^{d_\lambda}(\alpha_\lambda)$ all start with a vertex corresponding to u, end with a vertex corresponding to u' and contain no internal vertices that correspond to u or u'.)

<u>Case 2</u>: $d_1 = 2$.

This situation is slightly more complicated as if $d_2 = 2$ and α_2 is the path u,u_2,u' then we do not have a path $\rho_*^2(\alpha_2)$ whose internal vertices do not correspond to u or u' (and so it is not immediate that we can build a path whose vertices correspond to one of $\{u,u_2,u'\}$ and that is internally vertex-disjoint with each of $\rho_0^2(\alpha_1),\rho_1^2(\alpha_1),\ldots,\rho_\kappa^2(\alpha_1)$).

Let α be the path $u,\bar{u},u'$ in H and let $v^* \in V \setminus \{v\}$. We define a new path in $Msw(H;G)$ built around α and v^*. Define the path $\bar{\rho}_{v^*}^2(\alpha)$ in $Msw(H;G)$ as follows:

$$(u,v,w) \to_s (\bar{u},w,v) \to_c^* (\bar{u},w,v^*) \to_s (u,v^*,w) \to_c^* (u,v^*,w') \to_s (\bar{u},w',v^*) \to_c^* (\bar{u},w',v') \to_s (u',v',w').$$

Suppose that $d_2 = d_3 = \ldots = d_m = 2$, where $1 \leq m \leq \lambda$, with either $m=\lambda$ or $d_{m+1} \geq 3$, and that for $i=2,3,\ldots,m$, the path α_i is u,u_i,u'. By hypothesis, if $v \neq v'$ then there are $m-1$ mutually internally vertex-disjoint paths $\tau_2',\tau_3',\ldots,\tau_m'$ from v to v' in G so that each path has length at least 2 and at most $\Delta_\kappa(G)$. For $i=2,3,\ldots,m$, let v_i be the neighbour of v on the path τ_i'. Alternatively, if $v=v'$ then let $v_2,v_3,\ldots,v_m$ simply be distinct neighbours of v and for $i=2,3,\ldots,m$, define τ_i' to be the path v_i,v. Regardless, we have that neither v nor v' is in $\{v_2,v_3,\ldots,v_m\}$.

For $i=2,3,\ldots,m$, define the path $\bar{\rho}_{v_i}^2(\alpha_i)$ but so that the path in $G_{u_i}^w$ is the edge $(u_i,w,v) \to_c (u_i,w,v_i)$ and so that the path in $G_{u_i}^{w'}$ from (u_i,w',v_i) to (u_i,w',v') is isomorphic to the sub-path of τ_i' from v_i to v'.

The paths $\bar{\rho}^2_{v_2}(\alpha_2), \bar{\rho}^2_{v_3}(\alpha_3), \ldots, \bar{\rho}^2_{v_m}(\alpha_m)$ all have length at most $\Delta_\kappa(G) + \Delta(G) + 4$. Moreover, any internal vertex of $\bar{\rho}^2_{v_2}(\alpha_2), \bar{\rho}^2_{v_3}(\alpha_3), \ldots, \bar{\rho}^2_{v_m}(\alpha_m)$ corresponding to u has index from $\{v_2, v_3, \ldots, v_m\}$ and there are no internal vertices corresponding to u'.

Clearly the paths $\rho^2_0(\alpha_1), \rho^2_1(\alpha_1), \ldots, \rho^2_\kappa(\alpha_1), \bar{\rho}^2_{v_2}(\alpha_2), \bar{\rho}^2_{v_3}(\alpha_3), \ldots, \bar{\rho}^2_{v_m}(\alpha_m)$ in $Msw(H;G)$ are mutually internally vertex-disjoint paths from (u, v, w) to (u', v', w'), as are the paths $\rho^2_0(\alpha_1), \rho^2_1(\alpha_1), \ldots, \rho^2_\kappa(\alpha_1)$, $\bar{\rho}^2_{v_2}(\alpha_2), \bar{\rho}^2_{v_3}(\alpha_3), \ldots, \bar{\rho}^2_{v_m}(\alpha_m), \rho^{d_{m+1}}_*(\alpha_{m+1}), \rho^{d_{m+2}}_*(\alpha_{m+2}), \ldots, \rho^{d_\lambda}_*(\alpha_\lambda)$. Each path has length as follows:

- $|\rho^2_i(\alpha_1)| \leq \Delta_\kappa(G) + \Delta(G) + 2$, for $i = 0, 1, \ldots, \kappa$;
- $|\bar{\rho}^2_{v_i}(\alpha_i)| \leq \Delta_\kappa(G) + \Delta(G) + 4$, for $i = 2, 3, \ldots, m$;
- $|\rho^{d_i}_*(\alpha_i)| \leq 2\Delta(G) + d_i$, for $i = m+1, m+2, \ldots, \lambda$.

Thus, the length of any of these paths is at most $\Delta_\kappa(G) + \Delta(G) + \Delta_\lambda(H)$.

<u>Case 3</u>: $d_1 = 1$.

We can easily dispense with the case when $d_2 \geq 3$ as follows. Construct $\kappa + 1$ mutually internally vertex-disjoint paths $\rho'_0, \rho'_1, \ldots, \rho'_\kappa$ from (u, v, w) to (u', v', w') in $Msw(H;G)$ using Proposition 4.3 of [25] so that all paths involve only vertices corresponding to u and u'. Note that all paths have length at most $2\Delta(G)+5$. The paths $\rho'_0, \rho'_1, \ldots, \rho'_\kappa, \rho^{d_2}_*(\alpha_2), \rho^{d_3}_*(\alpha_3), \ldots, \rho^{d_\lambda}_*(\alpha_\lambda)$ are mutually internally vertex-disjoint paths from (u, v, w) to (u', v', w') in $Msw(H;G)$ so that each path has length as follows:

- $|\rho'_i| \leq 2\Delta(G) + 5$, for $i = 0, 1, \ldots, \kappa$;
- $|\rho^{d_i}_*(\alpha_i)| \leq 2\Delta(G) + d_i$, for $i = 2, 3, \ldots, \lambda$.

Thus, the length of any of these paths is at most $\max\{2\Delta(G) + \Delta_\lambda(H), 2\Delta(G) + 5\}$.

So, we may assume that $d_2 = d_3 = \ldots = d_m = 2$, where $2 \leq m \leq \lambda$, with either $m = \lambda$ or $d_{m+1} \geq 3$, and that for $i = 2, 3, \ldots, m$, the path α_i is u, u_i, u'. Suppose further that $w \neq v'$ and $w' \neq v$. Proceed as in Case 2 with the paths $\alpha_2, \alpha_3, \ldots, \alpha_\lambda$ so as to obtain $\kappa+\lambda-1$ mutually internally vertex-disjoint paths from (u, v, w) to (u', v', w') in $Msw(H;G)$ so that each path has length at most $\Delta_\kappa(G)+\Delta(G)+\Delta_\lambda(H)$ and so that: any internal vertex involved in one of these paths that corresponds to u has index from $\{v, v_3, v_4, \ldots, v_m\}$; and all internal vertices corresponding to u' have index v'. Note that we can choose the paths $\tau'_3, \tau'_4, \ldots, \tau'_m$, as in Case 2, so that none of the resulting vertices $v_3, v_4, \ldots, v_m$ is equal to w' (which is also different from v). Thus, the path

$$(u, v, w) \to_s (u', w, v) \to^*_c (u', w, w') \to_s (u, w', w) \to^*_c (u, w', v') \to_s (u', w', v')$$

is internally vertex-disjoint with each of the $\kappa + \lambda - 1$ paths constructed above. All paths have length at most $\max\{\Delta_\kappa(G) + \Delta(G) + \Delta_\lambda(H), 2\Delta(G) + 3\}$.

Hence, all that remains to deal with is the case when either $w = v'$ or $w' = v$.

<u>Case 3.1</u>: $v = w'$ and $v' \neq w$ (the situation when $v' = w$ and $v \neq w'$ is symmetrically equivalent).

We shall construct below $\kappa + 1$ mutually internally vertex-disjoint paths from (u, v, w) to (u', v', w') in $Msw(H;G)$ using Proposition 4.3 of [25] so that all paths involve only vertices corresponding to u and u'. As we shall amend some of these paths, and also the actual paths we construct depend upon various circumstances, we give the explicit constructions of these paths (as described in the proof of Proposition 4.3 of [25]) in full below.

We begin by building some paths and choosing some vertices in G. There exist κ mutually internally vertex-disjoint paths $\tau_1, \tau_2, \ldots, \tau_\kappa$ from w to v' in G so that each of these paths has length at most $\Delta_\kappa(G)$ and the shortest of these paths is τ_1. Let $w_2, w_3, \ldots, w_\kappa$ be neighbours of w so that w_i lies on τ_i, for $i = 2, 3, \ldots, \kappa$ (note that no w_i is equal to v'). There are two possibilities: either we can find distinct neighbours $w'_1, w'_2, \ldots, w'_\kappa$ of w' in G so that $w \neq w'_i \neq w_i$, for $i = 2, 3, \ldots, \kappa$; or $\kappa = 2$ and (w, w') is an edge of G.

<u>Case 3.1.1</u>: $w'_1, w'_2, \ldots, w'_\kappa$ are distinct neighbours of w' in G so that $w \neq w'_i \neq w_i$, for $i = 2, 3, \ldots, \kappa$.

We use our paths and chosen vertices in G, above, to build paths in $Msw(H;G)$. Define the path σ_0 in $Msw(H;G)$ as follows:

$$(u,v,w) \to_s (u',w,v) \to_c (u',w,w'_1) \to_s (u,w'_1,w) \to_c^* (u,w'_1,v') \to_s (u',v',w'_1) \to_c (u',v',w').$$

The path σ_1 in $Msw(H;G)$ is defined as follows:

$$(u,v,w) \to_c^* (u,v,v') \to_s (u',v',v) = (u',v',w'),$$

where the path in G_u^v from (u,v,w) to (u,v,v') is isomorphic to τ_1. For $i = m+1, m+2, \ldots, \kappa$, define the path σ_i in $Msw(H;G)$ as follows:

$$\begin{aligned}(u,v,w) &\to_c (u,v,w_i) \to_s (u',w_i,v) \to_c (u',w_i,w'_i) \to_s (u,w'_i,w_i)\\ &\to_c^* (u,w'_i,v') \to_s (u',v',w'_i) \to_c (u',v',w').\end{aligned}$$

The paths $\sigma_0, \sigma_1, \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_\kappa$ are mutually internally vertex-disjoint and the length of any of these paths is at most $\max\{\Delta(G)+6, \Delta_\kappa(G)+1\}$.

For each $j \in \{2,3,\ldots,m\}$, corresponding to the path α_j in H is the path σ_j in $Msw(H;G)$ defined as follows.

- If $w_j \neq w'$ $(= v)$ then define σ_j as:

$$\begin{aligned}(u,v,w) &\to_c (u,v,w_j) \to_s (u_j,w_j,v) \to_c (u_j,w_j,w'_j) \to_s (u,w'_j,w_j)\\ &\to_c^* (u,w'_j,w') \to_s (u_j,w',w'_j) \to_c^* (u_j,w',v') \to_s (u',v',w').\end{aligned}$$

- If $w_j = w'$ $(= v)$ then define σ_j as:

$$(u,v,w) \to_c (u,v,w_j) \to_s (u_j,w_j,v) \to_c^* (u_j,w_j,v') \to_s (u',v',w_j) = (u',v',w').$$

The paths $\sigma_2, \sigma_3, \ldots, \sigma_m$ all have length at most $2\Delta(G)+6$.

The paths $\sigma_0, \sigma_1, \ldots, \sigma_\kappa$ can be visualized as in Fig. 4, where the path σ_i, as shown, is such that $i \in \{m+1, m+2, \ldots, \kappa\}$ and the path σ_j, as shown, is such that $j \in \{2,3,\ldots,m\}$ and $w' \neq w_j$. The paths $\sigma_0, \sigma_1, \ldots, \sigma_\kappa$ are mutually internally vertex-disjoint and, moreover: of the vertices in these paths, none both corresponds to u' and is indexed by any of $w_2, w_3, \ldots, w_m$; and the vertices $(u',v',w'_2), (u',v',w'_3), \ldots, (u',v',w'_m)$ do not appear on any of these paths. Thus, we may use the vertices of the sub-graphs $G_{u'}^{w_2}, G_{u'}^{w_3}, \ldots, G_{u'}^{w_m}$ of $Msw(H;G)$ and also the vertices of $\{(u',v',w'_2), (u',v',w'_3), \ldots, (u',v',w'_m)\}$ when we build additional paths that are intended to be mutually internally vertex-disjoint with $\sigma_0, \sigma_1, \ldots, \sigma_\kappa$.

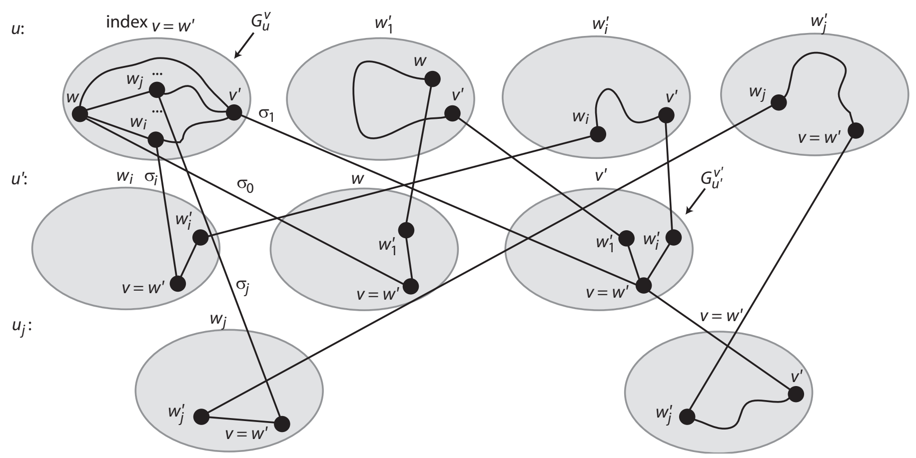

Figure 4. The paths $\sigma_0, \sigma_1, \ldots, \sigma_\kappa$ in Case 3.1.1.

Let us now build some additional paths. For $j = 2, 3, \ldots, m$, define the path ρ_j in $Msw(H;G)$ as:

$$(u,v,w) \to_s (u_j,w,v) \to_c^* (u_j,w,w_j) \to_s (u',w_j,w) \to_c^* (u',w_j,w'_j)$$
$$\to_s (u_j,w'_j,w_j) \to_c^* (u_j,w'_j,v') \to_s (u',v',w'_j) \to_c (u',v',w').$$

The paths $\rho_2, \rho_3, \ldots, \rho_m$ all have length at most $3\Delta(G) + 5$. By our choice of w_j and w'_j and by our construction of σ_j, for $j = 2, 3, \ldots, m$ (which varies according to whether $w_j = w'$ or not), the paths $\rho_2, \rho_3, \ldots, \rho_m$ are all well defined and σ_j and ρ_j are internally vertex-disjoint, for $j = 2, 3, \ldots, m$. Indeed, the paths $\sigma_0, \sigma_1, \ldots, \sigma_\kappa, \rho_2, \rho_3, \ldots \rho_m$ are mutually internally vertex-disjoint.

Finally, for $j = m+1, m+2, \ldots, \lambda$, build the path $\rho_*^{d_j}(\alpha_j)$ in $Msw(H;G)$. For $j = m+1, m+2, \ldots, \lambda$, $|\rho_*^{d_j}(\alpha_j)| \leq 2\Delta(G) + d_j$. This results in $\kappa + \lambda$ mutually internally vertex-disjoint paths from (u,v,w) to (u',v',w') in $Msw(H;G)$ so that all paths have length at most $\max\{3\Delta(G)+5, \Delta_\kappa(G)+1, 2\Delta(G)+\Delta_\lambda(H)\}$.

<u>Case 3.1.2</u>: $\kappa = 2$ and (w,w') is an edge of G.

If $\lambda = 1$ then Proposition 4.3 of [25] yields 3 paths in $Msw(H;G)$ from (u,v,w) to (u',v',w'), each of length at most $2\Delta(G) + 5$; so, we may suppose that $\lambda = 2$ and that α_2 is the path u, u_2, u' in H. Recall that w_2 is the neighbour of w on the path τ_2 in G. Let $\tilde{w}$ and $\bar{w}$ be distinct neighbours of w'. Define the 3 paths $\sigma_1, \sigma_2, \sigma_3$ in $Msw(H;G)$ as follows:

$$(u,v,w) \to_s (u',w,v) \to_c (u',w,\tilde{w}) \to_s (u,\tilde{w},w) \to_c (u,\tilde{w},v') \to_s (u',v',\tilde{w}) \to_c (u',v',w');$$
$$(u,v,w) \to_c^* (u,v,v') \to_s (u',v',v) = (u',v',w');$$
$$(u,v,w) \to_s (u_2,w,v) \to_c (u_2,w,\bar{w}) \to_s (u,\bar{w},w) \to_c^* (u,\bar{w},v') \to_s (u',v',\bar{w}) \to_c (u',v',w'),$$

where the sub-path of σ_2 in G_u^v from (u,v,w) to (u,v,v') is isomorphic to τ_1. If $w_2 \neq w'$ $(= v)$ then define the path σ_4 in $Msw(H;G)$ as:

$$(u,v,w) \to_c (u,v,w_2) \to_s (u_2,w_2,v) \to_c^* (u_2,w_2,w_2) \to_s (u',w_2,w_2)$$
$$\to_c^* (u',w_2,w') \to_s (u_2,w',w_2) \to_c^* (u_2,w',v') \to_s (u',v',w'),$$

otherwise define σ_4 as:

$$(u,v,w) \to_c (u,v,w_2) \to_s (u_2,w_2,v) \to_c^* (u_2,w_2,v') \to_s (u',v',w_2) = (u',v',w').$$

The paths $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are mutually internally vertex-disjoint and all have length at most $\max\{\Delta_2(G) + 1, 3\Delta(G) + 5\}$.

<u>Case 3.2</u>: $v = w'$ and $v' = w$.

There exist κ mutually internally vertex-disjoint paths $\tau_1, \tau_2, \ldots, \tau_\kappa$ from $v' = w$ to $v = w'$ in G so that each of these paths has length at most $\Delta_\kappa(G)$ and the shortest of these paths is τ_1. Let $w_1, w_2, \ldots, w_\kappa$ be neighbours of w so that w_i lies on τ_i, for $i = 1, 2, \ldots, \kappa$ (note that no w_i is equal to w', for $i = 2, 3, \ldots, m$, although it might be the case that $w_1 = w'$).

For $j = 2, 3, \ldots, m$, let α_j be the path u, u_j, u' in H (with α_1 the path u, u'). We begin by constructing paths from (u,v,w) to (u',v',w') in $Msw(H;G)$ so that all vertices involved correspond to one of u, u_2, u'. Let σ_0 be the path in $Msw(H;G)$ defined as follows:

$$(u,v,w) \to_s (u_2,w,v) \to_c^* (u_2,w,w) \to_s (u',w,w) \to_c^* (u',w,w') = (u',v',w'),$$

where the path in $G_{u'}^w$ from (u',w,w) to (u',v',w') is isomorphic to τ_1. Let σ_1 be the path in $Msw(H;G)$ defined as follows:

$$(u,v,w) \to_c^* (u,v,w') \to_s (u_2,w',v) \to_c^* (u_2,w',v') \to_s (u',v',w'),$$

where the path in G_u^v from (u,v,w) to (u,v,w') is isomorphic to τ_1. For $i = 2, 3, \ldots, \kappa$, let σ_i be the path in $Msw(H;G)$ defined as follows:

$$(u,v,w) \to_c (u,v,w_i) \to_s (u_2,w_i,v) \to_c^* (u_2,w_i,v') \to_s (u',v',w_i) \to_c^* (u',v',w'),$$

where the path in $G_{u'}^{v'}$ from (u', v', w_i) to (u', v', w') is isomorphic to the sub-path of τ_i from w_i to w'. The paths $\sigma_0, \sigma_1, \ldots, \sigma_\kappa$ are mutually internally vertex-disjoint and all have length at most $\Delta_\kappa(G) + \Delta(G) + 2$.

Let ρ_0 be the path in $Msw(H; G)$ defined as $(u, v, w) \rightarrow_s (u', v', w')$. For $j = 3, 4, \ldots, m$, let ρ_j be the path in $Msw(H; G)$ defined as:

$$(u, v, w) \rightarrow_s (u_j, w, v) \rightarrow_c^* (u_j, w, w_j) \rightarrow_s (u', w_j, w) \rightarrow_c^* (u', w_j, w') \\ \rightarrow_s (u_j, w', w_j) \rightarrow_c^* (u_j, w', v') \rightarrow_s (u', v', w').$$

The paths $\rho_0, \sigma_0, \sigma_1, \ldots, \sigma_\kappa, \rho_3, \rho_4, \ldots, \rho_m$ from (u, v, w) to (u', v', w') are mutually internally vertex-disjoint and all have length at most $\max\{\Delta_\kappa(G) + \Delta(G) + 2, 3\Delta(G) + 4\}$.

Finally, for $j = m+1, m+2, \ldots, \lambda$, build the path $\rho_*^{d_j}(\alpha_j)$ in $Msw(H; G)$. For $j = m+1, m+2, \ldots, \lambda$, $|\rho_*^{d_j}(\alpha_j)| \leq 2\Delta(G) + d_j$. This results in $\kappa + \lambda$ mutually internally vertex-disjoint paths from (u, v, w) to (u', v', w') in $Msw(H; G)$ so that all paths have length at most $\max\{\Delta_\kappa(G) + \Delta(G) + 2, 3\Delta(G) + 4, 2\Delta(G) + \Delta_\lambda(H)\}$. □

Proposition 8. *Let G and H be two graphs, both of connectivity 1. Let (u, v, w) and (u', v', w') be distinct vertices of $Msw(H; G)$, where $u \neq u'$ and where $w \neq w'$. There are at least 2 internally vertex-disjoint paths in $Msw(H; G)$ joining (u, v, w) and (u', v', w'), each of which has length at most* $\max\{3\Delta(G) + \Delta(H), 2\Delta(G) + 5\}$.

Proof. Suppose that $d_H(u, u') \in \{0, 1\}$. By [25], there are at least 2 internally vertex-disjoint paths in $Msw(H; G)$ joining (u, v, w) and (u', v', w'), each of which has length at most $2\Delta(G) + 5$. Suppose that $d_H(u, u') = d \geq 2$ and α is a shortest path from u to u' in H. The paths $\rho_0^d(\alpha)$ and $\rho_1^d(\alpha)$ as constructed in the proof of Proposition 7 are internally vertex-disjoint paths in $Msw(H; G)$ joining (u, v, w) and (u', v', w') and the lengths of these paths are at most $3\Delta(G) + \Delta(H)$. □

Proposition 9. *Let G and H be graphs of connectivity κ and λ, respectively, where $1 \leq \lambda \leq \kappa$. Let (u, v, w) and (u', v', w) be distinct vertices of $Msw(H; G)$, where $u \neq u'$. There are at least $\kappa + \lambda$ mutually internally vertex-disjoint paths in $Msw(H; G)$ joining (u, v, w) and (u', v', w), each of which has length at most* $\max\{2\Delta(G) + 5, 2\Delta(G) + \Delta_\lambda(H) + 2\}$.

Proof. By hypothesis, there exist λ mutually internally vertex-disjoint paths $\alpha_1, \alpha_2, \ldots, \alpha_\lambda$ in H joining u and u' so that the path α_i has length $d_i \leq \Delta_\lambda(H)$; w.l.o.g. we may assume that $d_1 \leq d_2 \leq \ldots \leq d_\lambda$. Let $w_1, w_2, \ldots, w_\kappa$ be distinct neighbours of w in G. As before, we begin by detailing some generic constructions.

Paths in H of length 2: Suppose that α is the path u, u_1, u' in H. Define the path $\pi_0^2(\alpha)$ in $Msw(H; G)$ as follows:

$$(u, v, w) \rightarrow_s (u_1, w, v) \rightarrow_c^* (u_1, w, v') \rightarrow_s (u', v', w).$$

For $i = 1, 2, \ldots, \kappa$, define the path $\pi_i^2(\alpha)$ in $Msw(H; G)$ as follows:

$$(u, v, w) \rightarrow_c (u, v, w_i) \rightarrow_s (u_1, w_i, v) \rightarrow_c^* (u_1, w_i, v') \rightarrow_s (u', v', w_i) \rightarrow_c (u', v', w).$$

The paths $\pi_0^2(\alpha), \pi_1^2(\alpha), \ldots, \pi_\kappa^2(\alpha)$ are clearly mutually internally vertex-disjoint paths from (u, v, w) to (u', v', w) in $Msw(H; G)$ so that the vertices involved all correspond to an element of $\{u, u_1, u'\}$ and where if a vertex of one of the paths corresponds to u (resp. u') then it is indexed by v (resp. v'). We may assume that each of the paths $\pi_0^2(\alpha), \pi_1^2(\alpha), \ldots, \pi_\kappa^2(\alpha)$ has length at most $\Delta(G) + 4$.

Paths in H of length 3: Suppose that α is the path u, u_1, u_2, u' in H. Define the path $\pi_0^3(\alpha)$ in $Msw(H; G)$ as follows:

$$(u, v, w) \rightarrow_s (u_1, w, v) \rightarrow_c^* (u_1, w, w) \rightarrow_s (u_2, w, w) \rightarrow_c^* (u_2, w, v') \rightarrow_s (u', v', w).$$

For $i = 1, 2, \ldots, \kappa$, define the path $\pi_i^3(\alpha)$ in $Msw(H; G)$ as follows:

$$(u, v, w) \rightarrow_c (u, w, w_i) \rightarrow_s (u_1, w_i, w) \rightarrow_c^* (u_1, w_i, w_i) \rightarrow_s (u_2, w_i, w_i) \\ \rightarrow_c^* (u_2, w_i, v') \rightarrow_s (u', v', w_i) \rightarrow_c (u', v', w).$$

The paths $\pi_0^3(\alpha), \pi_1^3(\alpha), \ldots, \pi_\kappa^3(\alpha)$ are clearly mutually internally vertex-disjoint paths from (u,v,w) to (u',v',w) in $Msw(H;G)$ so that the vertices involved all correspond to an element of $\{u, u_1, u_2, u'\}$ and where if a vertex of one of the paths corresponds to u (resp. u') then it is indexed by v (resp. v'). We may assume that each of the paths $\pi_0^3(\alpha), \pi_1^3(\alpha), \ldots, \pi_\kappa^3(\alpha)$ has length at most $2\Delta(G) + 5$.

It is straightforward to 'extend' the above paths so that if α is a path of length $d \geq 2$ in H from u to u' then there are mutually internally vertex-disjoint paths $\pi_0^d(\alpha), \pi_1^d(\alpha), \ldots, \pi_\kappa^d(\alpha)$ in $Msw(H;G)$ from (u,v,w) to (u',v',w) so that the vertices involved all correspond to an element of $\{u, u_1, u_2, \ldots, u_{d-1}, u'\}$ and where if a vertex of one of the paths corresponds to u (resp. u') then it is indexed by v (resp. v'). Moreover, the path $\pi_0^d(\alpha)$ is such that it has no internal vertices corresponding to u or u'. We may assume that the paths $\pi_0^d(\alpha), \pi_1^d(\alpha), \ldots, \pi_\kappa^d(\alpha)$ all have length at most $\Delta(G) + d + 2$ (resp. $2\Delta(G) + d + 2$) if d is even (resp. odd).

There are 2 cases, depending upon the value of d_1.

Case 1: $d_1 = 1$.

By [25], there are $\kappa + 1$ paths in $Msw(H;G)$ from (u,v,w) to (u',v',w) such that all vertices involved correspond to u or u' and such that the length of any of these paths is at most $2\Delta(G) + 5$. Build the paths $\pi_0^{d_2}(\alpha_2), \pi_0^{d_3}(\alpha_3), \ldots, \pi_0^{d_\lambda}(\alpha_\lambda)$. These paths are mutually internally vertex-disjoint and every one of these paths is trivially internally vertex-disjoint with every one of the $\kappa + 1$ paths just constructed above. The length of $\pi_0^{d_i}(\alpha_i)$ is at most $\Delta(G) + d_i$ (resp. $2\Delta(G) + d_i$) if d_i is even (resp. odd), for $i = 1, 2, \ldots, \lambda$.

Case 2: $d_1 \geq 2$.

Build the paths $\pi_0^{d_1}(\alpha_1), \pi_1^{d_1}(\alpha_1), \ldots, \pi_\kappa^{d_1}(\alpha_1), \pi_0^{d_2}(\alpha_2), \pi_0^{d_3}(\alpha_3), \ldots, \pi_0^{d_\lambda}(\alpha_\lambda)$ in $Msw(H;G)$. These paths are clearly mutually internally vertex-disjoint paths from (u,v,w) to (u',v',w) in $Msw(H;G)$ so that the length of any path is at most $2\Delta(G) + \Delta_\lambda(H) + 2$. □

We can draw the results of this section together in the following theorem.

Theorem 10. *Let G and H be graphs of connectivity κ and λ, respectively, where $1 \leq \lambda \leq \kappa$. Given any two distinct vertices (u,v,w) and (u',v',w') of $Msw(H;G)$, there are at least $\kappa + \lambda$ mutually internally vertex-disjoint paths from (u,v,w) to (u',v',w') in $Msw(H;G)$ such that the length of any of these paths is at most* $\max\{\Delta_\kappa(G) + 2\Delta(G) + \Delta_\lambda(H), 3\Delta(G) + 5\}$*, unless G consists of a solitary edge when it is at most* $\max\{\Delta(H) + 4, 8\}$.

4.2. Multipath source routing

Finally, let us comment as regards converting the constructions of this section into a multipath routing algorithm in $Msw(H;G)$ (or, more precisely, in a distributed-memory multiprocessor whose underlying interconnection network is the graph $Msw(H;G)$). We make the following assumptions. The graph G is a graph of connectivity $\kappa \geq 1$ and the graph H is a graph of connectivity $\lambda \geq 1$ where $\lambda \leq \kappa$. The graph G has maximal degree δ_G and the graph H has maximal degree δ_H. Both G and H are represented according to an adjacency list representation (this is because it is often the case that graphs used as interconnection networks have maximal degree that is logarithmic in the number of vertices) and the name of any vertex occupies $O(1)$ space.

- We have a deterministic multipath source routing algorithm R_G for G so that given any two distinct vertices of G: R_G outputs κ mutually vertex-disjoint paths joining these two vertices so that each path has length at most D_G; and R_G runs in $p_G(D_G, \kappa, \delta_G)$ time.

- We have a deterministic multipath source routing algorithm R_H for H so that given any two distinct vertices of H: R_H outputs λ mutually vertex-disjoint paths joining these two vertices so that each path has length at most D_H; and R_H runs in $p_H(D_H, \lambda, \delta_H)$ time.

- We have a deterministic source routing algorithm S_G for G so that given any two distinct vertices of G: S_G finds a shortest path joining these two vertices; and S_G runs in $q_G(\Delta(G), \delta_G)$ time.

- We have a deterministic source routing algorithm S_H for H so that given any two distinct vertices of H: S_H finds a shortest path joining these two vertices; and S_H runs in $q_H(\Delta(H), \delta_H)$ time.

First, we remark that in [25] a deterministic multipath source routing algorithm for $Bsw(G)$ was described (that finds $\kappa+1$ mutually internally vertex-disjoint paths joining any two distinct vertices). Whilst the time complexity of this algorithm was not explicitly detailed in [25], it is easy to see from the proofs of the results in that paper that:

- we have a deterministic multipath source routing algorithm $R_{Bsw(G)}$ for $Bsw(G)$ so that given any two distinct vertices of $Bsw(G)$: $R_{Bsw(G)}$ outputs $\kappa+1$ mutually internally vertex-disjoint paths joining these two vertices so that each path has length at most $\max\{2\Delta(G)+5, D_G+\Delta(G)+2\}$; and $R_{Bsw(G)}$ runs in $O(p_G(D_G,\kappa,\delta_G)+\kappa q_G(\Delta(G),\delta_G))$ time.

If one consults the proofs of the various cases in the various results in this section then one can easily see that we have a deterministic multipath source routing algorithm R for $Msw(H;G)$ so that given any two distinct vertices of $Msw(H;G)$: the algorithm R outputs $\kappa+\lambda$ mutually internally vertex-disjoint paths joining these two vertices so that each path has length at most $\max\{D_G+2\Delta(G)+D_H, 3\Delta(G)+5\}$, unless G consists of a solitary edge when each path has length at most $\max\{\Delta(H)+4, 8\}$, and the algorithm R runs in $O(p_G(D_G,\kappa,\delta_G)+p_H(D_H,\lambda,\delta_H)+\kappa q_G(\Delta(G),\delta_G)+\lambda q_H(\Delta(H),\delta_H))$ time.

As an application of the above, suppose that G is the n-dimensional hypercube Q_n, where $n\geq 2$, and that H is the m-dimensional hypercube Q_m, where $m\leq n$. By [22], there is a deterministic multipath source routing algorithm R_{Q_n} for Q_n so that given any two distinct vertices of Q_n: R_{Q_n} outputs n mutually internally vertex-disjoint paths joining these two vertices so that each path has length at most $n+2$; and R_{Q_n} runs in $O(n^2)$ time. Trivially, Q_n has a shortest path algorithm that runs in $O(n)$ time. So, we have a deterministic multipath source routing algorithm R for $Msw(Q_m;Q_n)$ so that given any two distinct vertices of $Msw(Q_m;Q_n)$: the algorithm R outputs $n+m$ mutually internally vertex-disjoint paths joining these two vertices so that each path has length at most $(n+2)+2n+(m+2)=3n+m+4$; and the algorithm R runs in $O(n^2)$ time.

5. The composition of Cayley graphs

In this section, we examine conditions on graphs G and H which imply that $Msw(H;G)$ is a Cayley graph. We know from [25] that there exist graphs G and H for which $Msw(H;G)$ is a Cayley graph: this is the case when G is a Cayley graph and H is a solitary edge. If an arbitrary interconnection network is a Cayley graph then not only does this significantly aid programming distributed-memory multiprocessor machines based on this underlying interconnection network but it makes the analysis of topological and algorithmic aspects of such machines much easier to undertake.

We begin by showing that there are Cayley graphs G and H for which the graph $Msw(H;G)$ is not a Cayley graph; even further, for which $Msw(H;G)$ is not even vertex-symmetric. Let H be a cycle of length 3 and let G be a solitary edge; so, H is the Cayley graph of the cyclic group of order 3, generated by its 2 non-identity elements, and G is the Cayley graph of the cyclic group of order 2, generated by its 1 non-identity element. Suppose that the vertex set of H is $\{1,2,3\}$ and that the vertex set of G is $\{a,b\}$. The vertices of $\{(1,a,a),(2,a,a),(3,a,a)\}$ in $Msw(H;G)$ form a cycle of length 3 but the vertex $(1,a,b)$ does not lie on any cycle of length 3. Hence, there can be no automorphism of $Msw(H;G)$ mapping $(1,a,a)$ to $(1,a,b)$, and $Msw(H;G)$ is not vertex-symmetric (and so not a Cayley graph). The same argument can essentially be used whenever G does not contain a cycle of length 3 and H does.

Now we prove a more positive result.

Theorem 11. *Let H be the Cayley graph of the group Π with generating set X and let G be the Cayley graph of the group Γ with generating set Y. If H is bipartite then the graph $Msw(H;G)$ is a Cayley graph.*

Proof. We denote the underlying set of any group by the name of the group too. As H is bipartite, the elements of Π can be partitioned into 2 non-empty disjoint sets Π_0 and Π_1, with Π_0 containing the identity element and with Π_1 containing X, so that if $\pi_0, \pi'_0 \in \Pi_0$ and $\pi_1, \pi'_1 \in \Pi_1$ then $\pi_0\pi_1, \pi_1\pi_0 \in \Pi_1$ and $\pi_0\pi'_0, \pi_1\pi'_1 \in \Pi_0$ (Π_0 consists exactly of those elements of Π that can be written as a product of an even number of elements of X).

Let Π act on the set $\Gamma \times \Gamma$ via:

$$(\gamma, \gamma')^\pi = (\gamma, \gamma') \qquad \text{if } \pi \in \Pi_0$$
$$(\gamma, \gamma')^\pi = (\gamma', \gamma) \qquad \text{if } \pi \in \Pi_1.$$

Define the following multiplication on elements of $\Pi \times \Gamma \times \Gamma$:

$$(\pi, \gamma_1, \gamma_2)(\pi', \gamma'_1, \gamma'_2) = (\pi\pi', (\gamma_1, \gamma_2)^{\pi'}(\gamma'_1, \gamma'_2)),$$

(where the 'internal' multiplications are those of the groups Π and $\Gamma \times \Gamma$). That is:

$$(\pi, \gamma_1, \gamma_2)(\pi', \gamma'_1, \gamma'_2) = (\pi\pi', \gamma_1\gamma'_1, \gamma_2\gamma'_2) \qquad \text{if } \pi' \in \Pi_0$$
$$(\pi, \gamma_1, \gamma_2)(\pi', \gamma'_1, \gamma'_2) = (\pi\pi', \gamma_1\gamma'_2, \gamma_2\gamma'_1) \qquad \text{if } \pi' \in \Pi_1.$$

It is trivial to verify that this multiplication on elements of $\Pi \times \Gamma \times \Gamma$ is associative, that there is an identity and that every element has an inverse. Hence, we have a group which we denote Σ.

Define the set $Z \subseteq \Sigma$ as:

$$\{(\pi, 1_\Gamma, 1_\Gamma) : \pi \in Y\} \cup \{(1_\Pi, 1_\Gamma, \gamma) : \gamma \in X\},$$

where 1_Π and 1_Γ are the identity elements of Π and Γ, respectively. Henceforth, we denote both 1_Π and 1_Γ by 1 (this does not cause any confusion). We claim that Z generates Σ. Clearly, we can generate all elements of the form $(\pi, 1, 1)$, where $\pi \in \Pi$, and all elements of the form $(1, 1, \gamma)$, where $\gamma \in \Gamma$. Let $\pi_1 \in \Pi_1$ and $\gamma \in \Gamma$. We have that $(1, 1, \gamma)(\pi_1, 1, 1) = (\pi_1, \gamma, 1)$ and that $(\pi_1^{-1}, 1, 1)(\pi_1, \gamma, 1) = (1, \gamma, 1)$. Hence, we can generate all elements of the form $(1, \gamma, 1)$, for $\gamma \in \Gamma$. Finally, if $\gamma_1, \gamma_2 \in \Gamma$ and $\pi \in \Pi$ then $(\pi, 1, 1)(1, \gamma_1, 1)(1, 1, \gamma_2) = (\pi, \gamma_1, \gamma_2)$. It is trivial to see that $Msw(H; G)$ is isomorphic to the Cayley graph of Σ with generating set Z and the result follows. □

Note that the condition on H in Theorem 11, that it be bipartite, is equivalent to there being a homomorphism from the group Π to the cyclic group of order 2 so that no element of X lies in the kernel of the homomorphism. Irrespective of the formulation, this condition is not too severe in the context of interconnection networks. For example, the n-dimensional hypercube Q_n is a bipartite Cayley graph as is the k-ary n-cube Q_n^k when k is even.

6. Conclusions

We have shown that our multiswapped networks are much more flexible than existing networks that are used to implement interconnection networks optoelectronically (that is, OTIS networks and biswapped networks) and we have ascertained some key properties of our networks. It is important to remember that $Msw(H; G)$ has been devised so as to be implemented optoelectronically and thus it is not particularly instructive to compare its properties with those of a standard interconnection network like, say, a hypercube. Nevertheless, even if one does such a comparison, $Msw(H; G)$ does quite well. For example, a $(a + 2b)$-dimensional hypercube Q_{a+2b} has 2^{a+2b} vertices and connectivity $a + 2b$ whilst $Msw(Q_a; Q_b)$ has 2^{a+2b} vertices and connectivity $a + b$; that is, we forsake half the connectivity of the network graph for the privilege of having an optoelectronic design. However, it is when we compare our multiswapped networks with standard biswapped networks that we make gains. Suppose that the base graph of a biswapped network is Q_n; thus, $Bsw(Q_n)$ has 2^{2n+1} vertices and connectivity $n + 1$. Let a and b be such that $a + 2b = 2n + 1$. The multiswapped network $Msw(Q_a; Q_b)$ has $2^{a+2b} = 2^{2n+1}$ vertices and connectivity $a + b$. If we make, for example, $b = n - 2$ and $a = 5$ then $Msw(Q_a; Q_b)$ has connectivity $a + b = n + 3$ which is an improvement over that of $Bsw(Q_n)$, yet we have the same number of vertices, and we can implement the network optoelectronically (assuming we can do likewise for $Bsw(Q_n)$).

Not only do our new constructions result in (topological and algorithmic) improvements on current optoelectronic networks, our graph $Msw(H;G)$ is worthy of study purely as a combinatorial object. The basic construction can be generally applied and the structural properties of $Msw(H;G)$ proven here hint that other interesting combinatorial properties might be forthcoming. Of course, the construction can be iterated so that we might obtain graphs such as $Msw(Msw(H;G),K)$ and $Msw(K,Msw(H;G))$.

$Msw(H;G)$ can also be viewed as a hierarchical interconnection network. Such networks are, roughly speaking, networks whose edges are partitioned into hierarchies, with each hierarchy defined according to some specific (previously studied) interconnection network. As such, they usually involve a mix of concepts relating to different existing interconnection networks. For example: in [6] the two-level binary hypercube-based hierarchical interconnection network is defined where there are 2^D collections of d-dimensional hypercubes with unique vertices in each hypercube forming a set of vertices that are interconnected as a D-dimensional hypercube; in [11] the hierarchical cubic network is defined where 2^n n-dimensional hypercubes are joined so that each vertex in an n-dimensional hypercube is joined to exactly one vertex from some other n-dimensional hypercube; and in [15] the hierarchical crossed cube $HCC(k,n)$ was studied where 2^{k+n} copies of an n-dimensional crossed cube are joined in the 'shape' of various k-dimensional hypercubes. Hierarchical interconnection networks hold much promise as the systematic composition of various networks can often yield new interconnection networks with attractive properties.

Finally, there are some obvious directions for further research on our multiswapped networks. These directions include the study of other topological and algorithmic properties of these networks (in relation to their usage as interconnection networks) including, for example, their ability to tolerate faults, the existence of one-to-all and all-to-all broadcast algorithms, and structural properties such as path and cycle embeddings (upcoming work will report that our multiswapped networks are indeed Hamiltonian, for example, when the constituent graphs are). Also, it would be useful to ascertain more conditions under which $Msw(H;G)$ is a Cayley graph and to better understand the connectivity of $Msw(H;G)$ when the connectivity of H is not necessarily bounded above by the connectivity of G.

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