PANCYCLICITY IN FAULTY K-ARY 2-CUBES

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ABSTRACT
We prove that a k-ary 2-cube Q_2^k with 3 faulty edges but where every vertex is incident with at least 2 healthy edges is bipancyclic, if $k \geq 3$, and k-pancyclic, if $k \geq 5$ is odd (these results are optimal).

KEY WORDS
Interconnection networks. k-ary n-cubes. Fault-tolerance. Pancyclicity. Bipancyclicity.

1 Introduction

Low-dimensional tori are regularly used as interconnection networks in distributed-memory parallel computers. For example, the Alpha 21364-based HP GS1280 machine [7], the iWarp [4] and the Cray X1E vector computer have a two-dimensional torus as their interconnection networks, while the Cray T3D and T3E [16] have three-dimensional tori as theirs. Furthermore, two-dimensional mesh and torus topologies are popular choices for networks-on-chips [22]. This has motivated a considerable amount of work on the structural aspects of (arbitrary dimensional) tori, and in particular their uniform variants k-ary n-cubes, that are relevant to parallel computing. For example, the k-ary n-cube Q_n^k has the following basic properties: it is vertex- and edge-symmetric [1]; it is Hamiltonian [3, 5]; it has diameter $n\lfloor \frac{k}{2} \rfloor$ [3, 5]; it has a recursive decomposition; and it contains many important interconnection networks such as cycles (of certain lengths) [1], meshes (of certain dimensions) [3] and even hypercubes (of certain dimensions) [5]. Moreover, it has admirable properties in relation to routing, broadcasting and communication in general (see, for example, [1, 5, 8]).

Of particular relevance to us are some recent results concerning paths and cycles embedded within k-ary n-cubes. Paths and cycles are fundamental in parallel computing; for not only is there a multitude of algorithms specifically designed for linear arrays of processors and cycles of processors but paths and cycles appear as data structures in many more algorithms for parallel machines whose processors are interconnected in a variety of topologies. We shall be concerned with questions relating to Hamiltonicity, pancyclicity and bipancyclicity (these concepts are defined in the next section). The existence of these properties in an interconnection network enables a much higher degree of flexibility with regard to the simulation of linear arrays or cycles of processors. Our primary motivation is the results of [18] where earlier results due to Hsieh, Lin and Huang [12] and to Wang, An, Pan, Wang and Qu [21] were extended and the situation as regards the pancyclicity and bipancyclicity of Q_n^k was settled. Amongst other results, it was shown that Q_n^k is bipancyclic, when $n \geq 2$ and $k \geq 3$, and k-pancyclic, when $n \geq 2$ and $k \geq 3$ is odd.

As more and more processors are incorporated into parallel machines, faults become more common, be it faults in the processors or on the connections between processors. Of course, the temporary unavailability of a connection between two processors due to, for example, high traffic can also be regarded as a fault. Given the significant cost of parallel machines, we would prefer to be able to tolerate (small numbers of) faults and still be able to use our parallel machine. Whilst 'static' structural results such as those mentioned above are important, we are interested here in the tolerance of k-ary n-cubes when a (limited) number of edges are faulty (that is, are missing). In particular, we are interested in how many faulty edges Q_n^k can tolerate yet still remain bipancyclic and k-pancyclic.

As the k-ary n-cube Q_n^k has degree $2n$, an immediate upper bound on the number of faulty edges Q_n^k can tolerate and still remain bipancyclic or k-pancyclic is clearly $2n-2$ (for we can make the edges from a vertex to $2n-1$ of its neighbours faulty and there clearly can be no cycle through the vertex). Consequently, many studies assume the conditional fault assumption on the distribution of the faults so that no matter how many faulty edges there are, it is always the case that every vertex is incident with at least 2 healthy edges (the legitimacy of this conditional fault assumption is given credence as there is a very small probability that a configuration of faulty edges will be such as to make a vertex of one of our networks have degree less than 2). For example, under this conditional fault assumption: it was shown in [2] that Q_n^k with $4n-5$ faulty edges still has a Hamiltonian cycle (and that this result is optimal); it was shown in [20] that an n-dimensional alternating group graph with $4n-13$ faulty edges still has a Hamiltonian cycle (and that this result is optimal); it was shown in [15] that an n-dimensional crossed cube with $2n-5$ faulty edges still has a Hamiltonian cycle (and that this result is optimal);

and in [11] a more general consideration of matching composition networks was made with regard to whether they remain Hamiltonian under a limited number of faults. Other Hamiltonicity results under our conditional fault assumption are available in, for example, [6, 9, 10, 13, 14, 19]. As far as we are aware, [19] is the only paper to have considered pancyclicity issues in a family of interconnection networks in the presence of faulty edges and under our conditional fault assumption: in [19] it was proven that, under our conditional fault assumption, an n-dimensional hypercube with $2n-5$ faulty edges remains bipancyclic (and that this result is optimal).

In this paper we begin the consideration of pancyclicity in k-ary n-cubes under our conditional fault assumption by resolving the situation for k-ary 2-cubes. In particular, we prove that a k-ary 2-cube Q_2^k with 3 faulty edges but where every vertex is incident with at least 2 healthy edges is bipancyclic, if $k \geq 3$, and k-pancyclic, if $k \geq 5$ is odd (these results are optimal). Our results can be viewed as providing the base case of any induction which might prove a more general result for k-ary n-cubes where $n \geq 2$ is arbitrary. In the next section we detail the basic definitions relevant to this paper, and in Section 3 we prove our main results. We give our conclusions, a conjecture in relation to Q_n^k when $n > 2$, and directions for further research in Section 4.

2 Basic definitions

For any $k \geq 3$ and $n \geq 1$, a *k-ary n-cube* Q_n^k has vertex set $\{0, 1, \ldots, k-1\}^n$ and there is an edge $((u_n, u_{n-1}, \ldots, u_1), (v_n, v_{n-1}, \ldots, v_1))$ if, and only if, $|u_i - v_i| = 1$ (mod k), for some $i \in \{1, 2, \ldots, n\}$, with $u_j = v_j$, for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$; such an edge is termed as lying in *dimension* i (throughout, all arithmetic on the components of vertices of Q_n^k is modulo k). A $k_1 \times k_2$ *torus* has vertex set $\{(u, v) : 0 \leq u \leq k_1 - 1, 0 \leq v \leq k_2 - 1\}$ and there is an edge $((u_1, u_2), (v_1, v_2))$ if, and only if, $|u_i - v_i| = 1$ (mod k), for some $i \in \{1, 2\}$, with $u_j = v_j$, for $j \neq i$. We consider k-ary n-cubes and tori with *faulty edges*; that is, where certain edges are missing. Thus, a *faulty* k-ary n-cube (resp. torus) is really just a copy of Q_n^k (resp. a torus) where some edges, the faulty edges, are missing, and where we refer to the edges that remain as the *healthy* edges. Even though our faulty edges are regarded as missing edges, we still say, for example, that a vertex v is incident with some faulty edge e when the edge e was originally incident with v before it was removed. On occasion, we temporarily make faulty edges healthy or we want to emphasise that all edges of some graph are healthy and so we say, for example, that a cycle or a path is healthy.

A *conditional fault assumption* is an assumption relating to the faults (in our case, faulty edges) and their distribution within an interconnection network (which for us is always a k-ary n-cube). The conditional fault assumption we make is that the distribution of faults is such that no vertex in any faulty k-ary n-cube is ever incident with less than 2 healthy edges (that is, has degree less than 2 when we regard our faulty k-ary n-cube as being a k-ary n-cube with some edges missing).

A graph on n vertices is: *pancyclic* if it contains a cycle of every length from 3 up to n; and *m-pancyclic* if it contains a cycle of every length from m up to n. Of course, no bipartite graph can be pancyclic (as there can be no odd length cycles); consequently, a notion of pancyclicity has been devised for bipartite graphs. A bipartite graph on n vertices is *bipancyclic* if there is an even length cycle of every even length from 4 up to n. Even though the notion of bipancyclicity has been devised to primarily apply to bipartite graphs, it still makes sense to apply it to non-bipartite graphs too. We shall be building cycles of various lengths in faulty k-ary n-cubes. We say that a cycle C, of length c, say, can be *progressively shortened* to a cycle of length c', say, if starting from C we can iteratively apply the following construction to obtain cycles of all lengths c, $c-2$, $c-4$, down to c': in the current cycle C', replace a sub-path u, v, w, y of length 3 with the edge u, y to obtain a cycle of length 2 less than the length of C' (note that we also describe a cycle in a graph as a sequence of vertices so that consecutive vertices in the sequence are joined by an edge in the cycle, as well as there being an edge from the last vertex of the sequence to the first).

If π is a property of graphs then a graph G is said to be *m-edge-fault-tolerant* π if G still has property π even after the removal of at most m edges from G. Thus, for example, to say that a k-ary n-cube Q_n^k is $(4n-5)$-edge-fault-tolerant bipancyclic under the conditional fault assumption that no vertex is incident with less than 2 healthy edges means that no matter which $4n-5$ edges we remove from Q_n^k, so long as no vertex in the resulting graph has degree less than 2, there is a cycle of every even length m where $4 \leq m \leq k^n$.

A graph G is *vertex-symmetric* if given any two distinct vertices u and v of G, there is an automorphism of G mapping u to v. Similarly, a graph is *edge-symmetric* if given two distinct edges e and f of G (possibly incident), there is an automorphism of G mapping e to f. The k-ary n-cube Q_n^k has a number of automorphisms. For example, the maps $(i, j) \mapsto (i+1, j)$, $(i, j) \mapsto (i, j+1)$, $(i, j) \mapsto (k-1-i, j)$, and $(i, j) \mapsto (i, k-1-j)$ are all automorphisms of Q_n^k.

The vertices of Q_n^k are indexed by n-tuples of elements of $\{0, 1, \ldots, k-1\}$. On occasion, we move from Q_n^k to Q_{n-1}^k and when we do we 'project' vertices along some dimension. We use the notation $(u_{n-1}, \ldots, \hat{u}_i, \ldots, u_1)$ to denote the vertex $(u_{n-1}, \ldots, u_{i+1}, u_{i-1}, \ldots, u_1)$ of Q_{n-1}^k.

3 The k-ary 2-cube Q_2^k

We prove a number of lemmas which, when put together, will yield our main result.

Lemma 1 *Consider a k-ary 2-cube Q_2^k, for some even $k \geq 6$, in which there are at most 3 faulty edges but where every* *vertex is incident with at least 2 healthy edges. There is a cycle of length l for every even l such that $4 \leq l \leq k^2$.*

Proof There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2. As Q_2^k is edge-symmetric [1], w.l.o.g. we may assume that the edge $((0,0),(k-1,0))$ is faulty.

Case (*i*): all faulty edges lie in dimension 2.

Consider the Hamiltonian cycle of Q_2^k as pictured in Fig. 1 (although we have only drawn a Hamiltonian cycle in Q_2^8, the analogous cycle in Q_2^k, for any even $k \geq 6$, should be clear). This cycle, which we call the E-cycle rooted at $(0,0)$, can be translated by simply increasing the first component of every vertex by 2, and yet another cycle can be obtained by increasing the first component of every vertex by 4. Note that the 3 Hamiltonian cycles so obtained are edge-disjoint when we consider only edges in dimension 2 (as $k \geq 6$). Thus, at least one of these Hamiltonian cycles contains only healthy links; call it C. W.l.o.g. we may assume that C is the E-cycle rooted at $(0,0)$. We remark that throughout the proof of this lemma, we never use edges of the form $((k-1,i),(0,i))$ and so we may simply ignore the faulty edge $((0,0),(k-1,0))$ and assume that we are working in the $k \times k$ grid with wrap-around edges of the form $((i,k-1),(i,0))$, for $i = 0,1,\ldots,k-1$.

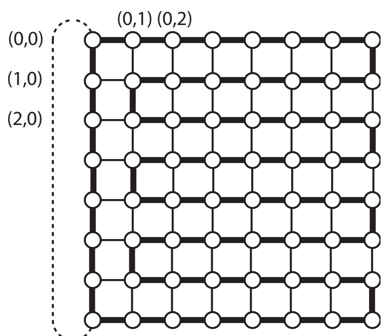

Figure 1. A Hamiltonian cycle in Q_2^8.

If our copy of Q_2^k had no faulty edges then we could clearly progressively shorten C, by at each step removing 3 edges and including 1 new edge, so that we obtain a cycle of every even length l for which $4 \leq l \leq k^2$. However, in the process of progressively shortening our cycle, we might try to include a new edge that is actually faulty (note that we encounter at most 2 faulty edges in our process of progressive shortening). We deal with this situation as follows. Our process of progressive shortening begins by introducing edges in dimension 2; so, with reference to Fig. 1, we shorten our cycle 'from the right-hand side' (note that there are $\frac{k}{2} \geq 3$ ways in which we could do this). We also ensure that we shorten the cycle in this way as much as we can before having to deal with attempting to introduce a faulty edge. If we try to introduce a faulty edge then we simply 'jump' that particular iteration of our process of progressive shortening and instead of shortening the cycle by 2, we shorten the cycle by 4, unless the next edge to be introduced is faulty too, when we shorten the cycle by 6. Note that because of how we have chosen to progressively shorten our cycle up until this point, we can simultaneously lengthen our cycle by 2 or 4 (in a different part of the cycle) to ensure that we obtain cycles of all the required lengths. The process can be visualized in Fig. 2 where we have encountered a faulty edge in Fig. 2(*a*) and 'jumped' over it in Fig. 2(*b*) as well as lengthening our cycle by 2.

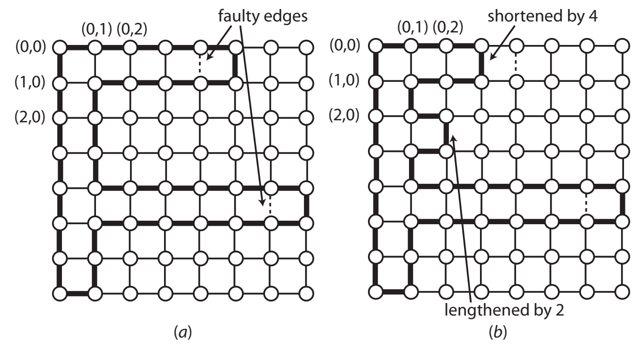

Figure 2. The shortening process in Q_2^8.

Case (*ii*): exactly 2 faulty edges lie in dimension 2.

By translating the E-cycle rooted at $(0,0)$ (as illustrated in Fig. 1) by increasing the first component of every vertex by 1, 2, and so on, w.l.o.g. we may assume that: we have a Hamiltonian cycle C in Q_2^k that is the E-cycle rooted at $(0,0)$; C contains no faulty edge in dimension 2; and C contains a faulty edge in dimension 1 and this faulty edge is one of $\{((i,k-4),(i,k-3)),((i,k-3),(i,k-2)),((i,k-2),(i,k-1))\}$, for some $i \in \{0,1,\ldots,k-1\}$.

Depending upon where the dimension 1 faulty edge lies, we now amend our cycle C analogously to the illustration in Fig. 3(*a*) where: we remove the faulty edge and its 'opposite' on the same 'branch' of the E-cycle; we include the two dimension 2 edges which join the two edges just removed; and we 'join' the resulting disconnected path to the main cycle by removing a dimension 2 edge and including two dimension 1 edges. What results is a Hamiltonian cycle C each of whose edges is healthy, unless one of the dimension 2 edges we have tried to add is the dimension 2 faulty edge (different from $((0,0),(k-1,0))$).

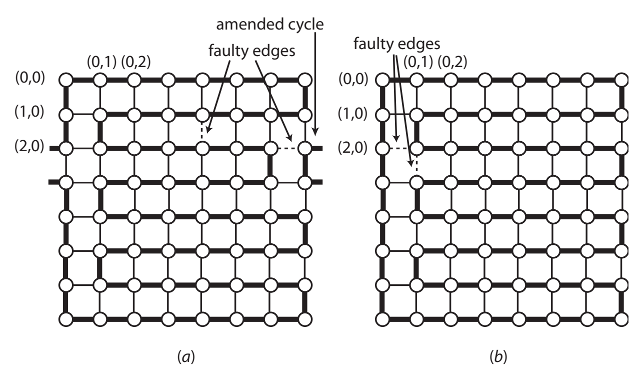

Figure 3. Amending the cycle in Q_2^8.

Assuming, for the moment, that C consists entirely of healthy edges, it can be progressively shortened just as we did in Case (*i*), above (and possibly 'jumping' over the faulty dimension 2 edge, should it be encountered), so that we obtain a cycle of length l for every even l for which $4 \leq l \leq k^2$.

Alternatively, consider when one of the dimension 2 edges introduced in the previous paragraph causes our construction to fail. For the construction to fail, the dimension 1 faulty edge must be of the form $((i,j),(i,j+1))$ and the dimension 2 faulty edge must be $((i,j),(i+1,j))$ or $((i,j+1),(i+1,j+1))$, if i is even, or $((i,j),(i-1,j))$ or $((i,j+1),(i-1,j+1))$, if i is odd. W.l.o.g. we may assume that the dimension 1 faulty edge is $((i,0),(i,1))$, where i is even, and the dimension 2 faulty edge is $((i,1),(i+1,1))$ (by applying appropriate automorphisms of the $k \times k$ grid with wrap-around edges). Hence, the E-cycle rooted at $(0,0)$ consists entirely of healthy edges and can be progressively shortened just as we did in Case (*i*), above, so that we obtain a cycle of length l for every even l for which $4 \leq l \leq k^2$ (see Fig. 3(*b*)). □

Lemma 2 *Consider a* 4*-ary* 2*-cube* Q_2^4 *in which there are at most* 3 *faulty edges but where every vertex is incident with at least* 2 *healthy edges. There is a cycle of length* l *for every even* l *such that* $4 \leq l \leq 16$.

Proof There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2.

Case (*i*): all faulty edges lie in dimension 2.

For $i = 0,1,2,3$, let C_i be the cycle $((i,0),(i,1),(i,2),(i,3))$. Both edges of at least one of the edge-pairs

- $\{((0,0),(1,0)),((0,1),(1,1))\}$
- $\{((0,0),(3,0)),((0,1),(3,1))\}$
- $\{((0,2),(1,2)),((0,3),(1,3))\}$
- $\{((0,2),(3,2)),((0,3),(3,3))\}$

are healthy; thus, we can 'join' C_0 to C_1 or C_3, as appropriate and using these edges, to obtain a cycle of length 8. We can also 'join' C_0 to an edge of C_1 or C_3, as appropriate and using these edges, to obtain a cycle of length 6 (see Fig. 4 for an illustration of these constructions). By continuing in the same way and using the same reasoning with the resulting cycle of length 8, we can ultimately obtain cycles of every even length from 4 up to 16 as required.

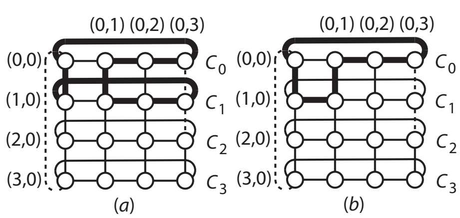

Figure 4. Joining cycles in a faulty Q_2^4.

Case (*ii*): exactly 2 faulty edges lie in dimension 2.

As Q_2^4 is edge-symmetric [1], w.l.o.g. we may assume that the edge $((0,0),(0,1))$ is faulty. If either $((0,0),(1,0))$ and $((0,1),(1,1))$ are both healthy or $((0,0),(3,0))$ and $((0,1),(3,1))$ are both healthy then we may proceed as we did in Case (*i*) and iteratively obtain cycles of all the required lengths. Thus, suppose that either $((0,0),(1,0))$ or $((0,1),(1,1))$ is faulty and either $((0,0),(3,0))$ or $((0,1),(3,1))$ is faulty. W.l.o.g. we may suppose that the 3 faulty edges are $((0,0),(0,1))$, $((0,0),(1,0))$, and $((0,1),(3,1))$. The Hamiltonian cycle in Fig. 5 can clearly be progressively shortened so that we obtain cycles of lengths 14, 12, 10, and 8, and it is trivial to find cycles of lengths 6 and 4. The result follows. □

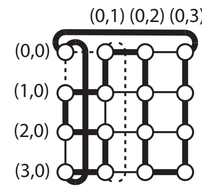

Figure 5. A Hamiltonian cycle in a faulty Q_2^4.

Lemma 3 *Consider a* k*-ary* 2*-cube* Q_2^k*, for some odd* $k \geq 7$*, in which there are at most* 3 *faulty edges but where every vertex is incident with at least* 2 *healthy edges. There is a cycle of length* l *for every even* l *such that* $4 \leq l \leq k^2 - 1$.

Proof There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2. As Q_2^k is edge-symmetric [1], w.l.o.g. we may assume that the edge $((0,0),(k-1,0))$ is faulty.

We begin by noting that the constructions in the proof of Lemma 1 suffice to prove that in a $k_1 \times k_2$ grid with wrap-around edges of the form $((i,k_2-1),(i,0))$, for $i = 0,1,\ldots,k_1-1$, where: $k_1 \geq 6$ is even; $k_2 \geq 6$ (with k_2 odd or even); and in which there are 2 faulty edges, we have cycles of all even lengths l where $4 \leq l \leq k_1 \cdot k_2$. Note also that according to our constructions, if there is no dimension 1 faulty edge of the form $((0,i),(0,i+1))$ (resp. $((k-1,i),(k-1,i+1))$) then the Hamiltonian cycle of the $k_1 \times k_2$ grid with wrap-around edges contains a sub-path of length $k-1$ of vertices all of which are of the form $(0,j)$ (resp. $(k-1,j)$).

We can consider such a wrap-around grid embedded within Q_2^k by working with the subgraph induced by the vertices of $\{(i,j) : 0 \leq i \leq k-2, 0 \leq j \leq k-1\}$ or the subgraph induced by the vertices of $\{(i,j) : 1 \leq i \leq k-1, 0 \leq j \leq k-1\}$. W.l.o.g., we may assume that our wrap-around grid G is the subgraph induced by the vertices of $\{(i,j) : 0 \leq i \leq k-2, 0 \leq j \leq k-1\}$ and that: if there is a faulty edge in dimension 1 then this faulty edge is of the form $((i,j),(i,j+1))$, where $i \leq \frac{k-1}{2}$; and if there is not a faulty edge in dimension 1 then there is at least one faulty edge of the form $((i,j),(i+1,j))$, where $i \leq \frac{k-1}{2}$. Thus, we have cycles in Q_2^k of all even lengths l where $4 \leq l \leq (k-1)k$. All we need to do is to build cycles of all even lengths from $(k-1)k+2$ up to k^2-1.

There are two cases. First, suppose that both faulty edges in Q_2^k lie entirely within G. As noted above, there is a sub-path of length $k-1$ in the Hamiltonian cycle C

of G consisting entirely of vertices of the form $(k-2,j)$. W.l.o.g., suppose that this sub-path is $(k-2,0),(k-2,1),\ldots,(k-2,k-1)$. Let $1 \leq l \leq \frac{k-1}{2}$. In order to obtain a cycle of length $(k-1)k+2l$ we 'join' the edges $((k-1,0),(k-1,1)),((k-1,2),(k-1,3)),\ldots,((k-1,2l-2),(k-1,2l-1))$ to the cycle C just as we joined an edge to a cycle in the proof of Lemma 2.

Alternatively, suppose that there is a dimension 2 faulty edge of the form $((k-2,j),(k-1,j))$. This faulty edge could invalidate the construction in the previous paragraph. However, in this case our grid G contains only one faulty edge, and the constructions of Lemma 1 can be used to obtain a Hamiltonian cycle C in G so that if the sub-path of C consisting entirely of vertices of the form $(k-2,i)$ is, w.l.o.g., $(k-2,0),(k-2,1),\ldots,(k-2,k-1)$ then j is even. In such a situation we can proceed as we did in the previous paragraph to obtain cycles of the required lengths (the situation can be illustrated in Fig. 6). The result follows. □

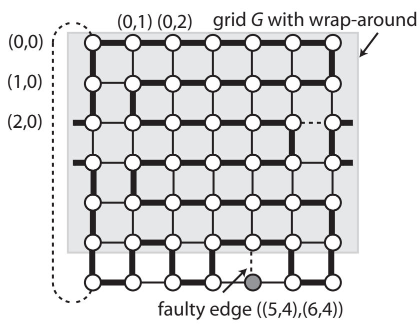

Figure 6. Extending a cycle in Q_2^7.

Lemma 4 *Consider a 5-ary 2-cube Q_2^5 in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every even l such that $4 \leq l \leq 24$.*

Proof There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2.

Case (*i*): all faulty edges lie in dimension 2.

As Q_2^5 is edge-symmetric [1], w.l.o.g. we may assume that the edge $((0,0),(4,0))$ is faulty. Suppose that 2 faulty edges are of the form $((0,i),(1,i))$; it is trivial to see that in the subgraph G of Q_2^5 induced by the vertices of $\{(0,i),(1,i) : 0 \leq i \leq 4\}$ there are cycles of lengths 4, 6, 8, and 10. Suppose that 1 faulty edge is of the form $((0,i),(1,i))$; it is trivial to see that in the subgraph G there are 3 different cycles of lengths 4, 6, 8, and 10, respectively. Of course, the same holds true were we to consider G to be the subgraph of Q_2^5 induced by the vertices of $\{(j,i),(j+1,i) : 0 \leq i \leq 4\}$, for any $j \in \{0,1,2,3\}$. Armed with these observations, it is easy to 'join' cycles and edges together, in the style of the proof of Lemma 2, to obtain cycles of all even lengths l where $4 \leq l \leq 24$.

Case (*ii*): exactly 2 faulty edges lie in dimension 2.

As Q_2^5 is edge-symmetric [1], w.l.o.g. we may assume that the edge $((4,0),(4,4))$ in dimension 1 is a faulty edge. If none of the dimension 2 faulty edges is of the form $((3,i),(4,i))$ then we can proceed as in Case (*i*) to construct our required cycles.

Suppose that exactly 1 of the dimension 2 faulty edges is of the form $((3,i),(4,i))$. If this faulty edge is not incident with the faulty edge $((4,0),(4,4))$ then we can start with the cycle $(3,0),(3,1),(3,2),(3,3),(3,4),(4,4),(4,3),(4,2),(4,1),(4,0)$ of length 10 and extend this cycle, as we did in the proof of Case (*i*), to obtain cycles of all even lengths up to 24. Trivially, we can find cycles of lengths 4, 6, and 8 in Q_2^5. So, suppose w.l.o.g. that the edge $((3,0),(4,0))$ is faulty. We can now proceed as we did in Case (*i*) to obtain cycles of the required lengths.

Suppose that both the dimension 2 faulty edges are of the form $((3,i),(4,i))$. If neither faulty edge is incident with the faulty edge $((4,0),(4,4))$ then we can proceed as in the previous paragraph (that is, start from the cycle of length 10 described there and extend it, and find cycles of lengths 4, 6 and 8 elsewhere). So, w.l.o.g. suppose that $((3,0),(4,0))$ is a faulty edge. No matter where the other dimension 2 faulty edge lies, we can obtain a cycle of length 24 in Q_2^5 as illustrated in Fig. 7 (the cycles corresponding to the other 3 possibilities for the location of the second dimension 2 faulty edge are constructed very similarly). This cycle can be progressively shortened so that we obtain cycles of all even lengths from 10 to 24 in Q_2^5, and it is trivial to find cycles of lengths 4, 6, and 8 elsewhere in Q_2^5. The result follows. □

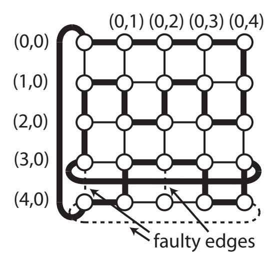

Figure 7. A cycle of length 24 in Q_2^5.

The proof of the following lemma is omitted as it can easily be obtained by hand by a case-by-case analysis (and the use of appropriate automorphisms).

Lemma 5 *Consider a 3-ary 2-cube Q_2^3 in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every even l such that $4 \leq l \leq 8$.*

We draw the lemmas of this section together in the following corollary.

Corollary 6 *Let $k \geq 3$. Consider a k-ary 2-cube Q_2^k in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every even l such that $4 \leq l \leq k^2$.*

When k is even then Corollary 6 is the best we can do, in the sense that as Q_2^k is bipartite, there cannot be any cycle of odd length. However, when k is odd we can say more.

Lemma 7 *Consider a k-ary 2-cube Q_2^k, where $k \geq 7$ is odd, in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every odd l such that $k \leq l \leq k^2$.*

Proof Let us return to the proof of Lemma 3. The scenario here is identical to as it was in Lemma 3. Consequently, w.l.o.g. we can assume that:

- there are at least 2 faulty edges in dimension 2, one of which is $((0,0),(k-1,0))$;
- if there is a faulty edge in dimension 1 then this faulty edge is of the form $((i,j),(i,j+1))$, where $i \leq \frac{k-1}{2}$;
- if there is not a faulty edge in dimension 1 then there is at least one (dimension 2) faulty edge of the form $((i,j),(i+1,j))$, where $i \leq \frac{k-1}{2}$.

By following the constructions of Lemma 1, there is a healthy cycle C of length $(k-1)k$ including every vertex of $\{(i,j) : 0 \leq i \leq k-2, 0 \leq j \leq k-1\}$, as well as a sub-path of length $k-1$ containing all vertices of $\{(k-2,j) : 0 \leq j \leq k-1\}$, that can be progressively shortened, with 'jumps' over any dimension 2 faulty edges encountered (as in the proof of Lemma 1), so that a cycle C_m of any even length m for which $4 \leq m \leq (k-1)k$ is constructed. W.l.o.g. we may suppose that C contains the path $(k-2,0),(k-2,1),\ldots,(k-2,k-1)$ and that the resulting cycle C_4 of length 4, obtained after progressively shortening C, is either:

- $(k-3,0),(k-3,1),(k-2,1),(k-2,0)$ with the edges $((k-1,0),(k-1,1))$, $((k-2,0),(k-1,0)$, and $((k-2,1),(k-1,1))$ all healthy,

  or

- $(k-3,2),(k-3,3),(k-2,3),(k-2,2)$ with the edges $((k-1,2),(k-1,3))$, $((k-2,2),(k-1,2)$, and $((k-2,3),(k-1,3))$ all healthy,

Take any cycle C_m of even length m, where $4 \leq m \leq (k-1)k$, as just constructed. No matter which case above occurs, we can clearly 'join' (in the sense of the proof of Lemma 2) this cycle C_m with the cycle D of length k defined as $(k-1,0),(k-1,1),\ldots,(k-1,k-1)$ to obtain a cycle of length $m+k$. Similarly, we can join the cycle D with either the edge $((k-1,0),(k-1,1))$ or the edge $((k-1,2),(k-1,3))$, as appropriate. The result now follows. □

Lemma 8 *Consider a 5-ary 2-cube Q_2^5 in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every odd l such that $5 \leq l \leq 25$.*

Proof The proof is similar to that of Lemma 4. There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2.

Case (*i*): all faulty edges lie in dimension 2.

As Q_2^5 is edge-symmetric [1], w.l.o.g. we may assume that the edge $((0,0),(4,0))$ is faulty. Let C_0 be the cycle $(0,0),(0,1),(0,2),(0,3),(0,4)$. As noted in the proof of Lemma 4, in the subgraph G_1 (resp. G_2) of Q_2^5 induced by the vertices of $\{(1,i),(2,i) : 0 \leq i \leq 4\}$ (resp. $\{(3,i),(4,i) : 0 \leq i \leq 4\}$) there are cycles of lengths 4, 6, 8, and 10, irrespective of where the faulty edges lie, and if at most one faulty edge lies in G_1 (resp. G_2) then there are 3 distinct cycles of lengths 4, 6, 8, and 10. Just as in the proof of Lemma 4, we can easily join C_0 to appropriate cycles or edges in G_1 and G_2 to obtain a cycle of odd length l whenever $5 \leq l \leq 25$.

Case (*ii*): exactly 2 faulty edges lie in dimension 2.

As Q_2^5 is edge-symmetric [1], w.l.o.g. we may assume that the edge $((4,0),(4,4))$ in dimension 1 is a faulty edge. Suppose that neither $((3,0),(4,0))$ nor $((3,4),(4,4))$ is a faulty edge. By proceeding as in Case (*i*), we can clearly build cycles of all odd lengths from 5 up to 17 in the subgraph of Q_2^5 induced by the vertices of $\{(i,j) : 0 \leq i \leq 3, 0 \leq j \leq 4\}$. For longer cycles, we can join C_0, the cycle $(3,0),(3,1),(3,2),(3,3),(3,4),(4,4),(4,3),(4,2),(4,1),(4,0)$, and appropriate cycles in the subgraph G_1 (defined in Case (*i*)) to yield cycles of all required lengths.

W.l.o.g. we may assume that either $((3,0),(4,0))$ or $((3,4),(4,4))$ is a faulty edge and that either $((0,0),(4,0))$ or $((0,4),(4,4))$ is a faulty edge. Further, w.l.o.g., we may assume that $((3,0),(4,0))$ and $((0,4),(4,4))$ are faulty edges. Consider the Hamiltonian cycle in Q_2^5 as depicted in Fig. 8. This cycle can clearly be progressively shortened to obtain cycles of all odd lengths from 7 to 25. The result follows. □

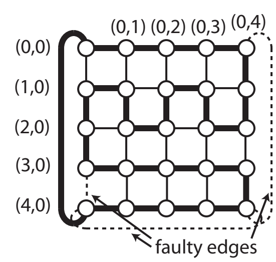

Figure 8. A cycle of length 25 in Q_2^5.

We bring all the results of this section together in the following theorem.

Theorem 9 *Consider a k-ary 2-cube Q_2^k in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges.*

(*a*) *If $k \geq 3$ then Q_2^k is bipancyclic.*

(*a*) *If $k \geq 5$ is odd then Q_2^k is k-pancyclic.*

An equivalent formulation of the above result is that Q_2^k is 3-edge-fault-tolerant bipancyclic, when $k \geq 3$, and 3-edge-fault-tolerant k-pancyclic, when $k \geq 5$ is odd, with both results assuming the conditional fault assumption that no vertex is incident with less than 2 healthy edges.

Theorem 9 cannot be improved when k is odd, for it is not difficult to see that when $n \geq 2$, Q_n^k has no odd length cycles of length less than k (see also [18]). Also, in Q_2^3 there are configurations of 3 faulty edges so that even though every vertex is incident with at least 2 healthy edges, no Hamiltonian cycle exists (one of these configurations is when the edges $((0,0),(0,1))$, $((0,1),(0,2))$, and $((0,2),(0,0))$ are faulty edges). We also note that (as was explained in [2]) Corollary 6 is optimal in the sense that there are configurations of 4 faults in Q_2^k for which a Hamiltonian circuit does not exist, no matter what the value of k (one such configuration is the set of faults $\{((0,0),(0,k-1)),((0,0),(k-1,0)),((1,1),(1,2)),((1,1),(2,1))\}$).

4 Conclusion

In this paper we have established exactly when a k-ary 2-cube Q_2^k remains bipancyclic or k-pancyclic in the presence of 3 faulty edges and under the conditional fault assumption that every vertex is incident with at least 2 healthy edges. We conjecture that under our conditional fault assumption, for any $n \geq 2$, a k-ary n-cube Q_n^k with $4n-5$ faulty edges is bipancyclic, if $k \geq 3$, and k-pancyclic, if $k \geq 5$ is odd. Thus, we have proven in this paper the base case of any induction of this proof, and we hope to be able to deal with the full induction in the near future.

Note that we have been solely concerned with existence proofs with regard to our properties of pancyclicity or bipancyclicity in our faulty k-ary 2-cubes. We have not considered the actual construction of any cycles, either via a centralized sequential algorithm or a distributed algorithm (implemented for a parallel machine whose underlying topology is a k-ary 2-cube; we refer the reader to [17], and the references therein, for more on such constructions). We believe it would be beneficial to consider the actual construction of such cycles by an appropriate algorithm. We note, however, that our constructions are very uniform (with a lot of progressive shortening) and we conjecture that it will not be too difficult to devise efficient (centralized and distributed) algorithms for cycle construction.

Related to pancyclicity and bipancyclicity are panconnectivity and bipanconnectivity. A graph of size n is *panconnnected* if given any two distinct vertices u and v at a distance $d(u,v)$ apart, there are paths of all lengths from $d(u,v)$ up to $n-1$ joining u and v. A graph of size n is *bipanconnnected* if given any two distinct vertices u and v at a distance $d(u,v)$ apart, there are paths of all lengths $d(u,v)+2i$ joining u and v where $i = 0,1,\ldots,\lfloor\frac{n-d(u,v)}{2}\rfloor$. Often, pancyclicity and bipancyclicity results are immediate corollaries of panconnectivity and bipanconnectivity results; however, that is not the case here. It would be interesting to examine the panconnectivity and bipanconnectivity of k-ary n-cubes under our conditional fault assumption.

We have a closing remark in relation to the optimality of results such as ours (whether there is a conditional fault assumption or not). The optimality arguments come from the fact that there are certain fault configurations which prohibit a Hamiltonian cycle (or, at least, a cycle close to Hamiltonian). When dealing with the fault-tolerance of an interconnection network with respect to pancyclicity or bipancyclicity, one could look to tolerate higher numbers of faulty edges at the expense of not having long cycles. For example, one could develop a notion of (m_1,m_2)-pancyclicity (or (m_1,m_2)-bipanyclicity, for that matter) where it is known that there are cycles of all lengths from m_1 up to m_2. It would be interesting to investigate how the parameters m_1 and m_2 fluctuate as more and more faulty edges are introduced, both in the presence and absence of our conditional fault assumption.

References

[1] Y.A. Ashir and I.A. Stewart, On embedding cycles in k-ary n-cubes, *Parallel Processing Letters* 7 (1997) 49–55.

[2] Y.A. Ashir and I.A. Stewart, Fault-tolerant embeddings of Hamiltonian circuits in k-ary n-cubes, *SIAM Journal of Discrete Mathematics* 15 (2002) 317–328.

[3] S. Bettayeb, On the k-ary hypercube, *Theoretical Computer Science* 140 (1995) 333–339.

[4] S. Borkar, R. Cohn, G. Cox, S. Gleason, T. Gross, H. T. Kung, M. Lam, B. Moore, C. Peterson, J. Pieper, L. Rankin, P. S. Tseng, J. Sutton, J. Urbanski, and J. Webb, iWarp: an integrated solution to high-speed parallel computing, *Proc. IEEE Supercomputing* 88, IEEE Computer Society Press (1988) 330–339.

[5] B. Bose, B. Broeg, Y. Kwon and Y. Ashir, Lee distance and topological properties of k-ary n-cubes, *IEEE Transations on Computers* 44 (1995) 1021–1030.

[6] M.Y. Chang and S.-J. Lee, On the existence of Hamiltonian circuits in faulty hypercubes, *SIAM Journal on Discrete Mathematics* 4 (1991) 511–527.

[7] Z. Cvetanovic, Performance analysis of the Alpha 21364-based HP GS1280 multiprocessor, *Proc. of* 30*th Ann. Int. Symp. on Computer Architecture*, ACM Press (2003) 218–228.

[8] W. Dally, Performance analysis of k-ary n-cube interconnection networks, *IEEE Transaction on Computers* 39 (1990) 775–785.

[9] J.-S. Fu, Conditional fault-tolerant Hamiltonicity of star graphs, *Parallel Computing* 33 (2007) 488–496.

[10] J.-S. Fu, Fault-free Hamiltonian cycles in twisted cubes with conditional link faults, *Theoretical Computer Science* 407 (2008) 318–329.

[11] S.-Y. Hsieh and C.-W. Lee, Conditional edge-fault Hamiltonicity of matching composition networks, *IEEE Transactions on Parallel and Distributed Systems* 20 (2009) 581–592.

[12] S.-Y. Hsieh, T.-J. Lin and H.-L. Huang, Panconnectivity and edge-pancyclicity of 3-ary n-cubes, *Journal of Supercomputing* 42 (2007) 225–233.

[13] S.-Y. Hsieh and C.-D. Wu, Conditional edge-fault-tolerant Hamiltonian cycle embedding of star graphs, *Proc.* 13*th Int. Conf. on Parallel and Distributed Systems* (*ICPADS* 07), IEEE Computer Society Press (2007) 1–8.

[14] S.-Y. Hsieh and C.-Y. Wu, Edge-fault-tolerant Hamiltonicity of locally twisted cubes under conditional edge faults, *Journal of Combinatorial Optimization*, to appear.

[15] H.-S. Hung, J.-S. Fu and G.-H. Chen, Fault-free Hamiltonian cycles in crossed cubes with conditional link faults, *Information Sciences* 177 (2007) 5664–5674.

[16] S.L. Scott and G.M. Thorson, The Cray T3E network: adaptive routing in a high performance 3D torus, *Proc. Symp. High Performance Interconnects* (*Hot Interconnects IV*) (1996) 147–156.

[17] I.A. Stewart, Distributed algorithms for building Hamiltonian cycles in k-ary n-cubes and hypercubes with faulty links, *Journal of Interconnection Networks* 8 (2007) 253–284.

[18] I.A. Stewart and Y. Xiang, Bipanconnectivity and bipancyclicity in k-ary n-cubes, *IEEE Transactions on Parallel and Distributed Systems* 20 (2009) 25–33.

[19] C.-H. Tsai, Linear array and ring embeddings in conditional faulty hypercubes, *Theoretical Computer Science* 314 (2004) 431–443.

[20] P.-Y. Tsai, J.-S. Fu and G.-H. Chen, Embedding Hamiltonian cycles in alternating group graphs under conditional fault model, *Information Sciences* 179 (2009) 851–857.

[21] D. Wang, T. An, M. Pan, K. Wang and S. Qu, Hamiltonian-like properties of k-ary n-cubes, *Proc. of* 6*th Int. Conf. on Parallel and Distributed Computing, Applications and Technologies* (*PDCAT* 05), IEEE Computer Society Press (2005) 1002–1007.

[22] H. Wang, L.-S. Peh and S. Malik, A technology-aware and energy-oriented topology exploration for on-chip networks, *Proc. of Conf. on Design, Automation and Test in Europe*, IEEE Computer Society Press (2005) 1238–1243.