Hamiltonian cycles through prescribed edges in k-ary n-cubes

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Abstract

We prove that if P is a set of at most $2n-1$ edges in a k-ary n-cube, where $k \geq 4$ and $n \geq 2$, then there is a Hamiltonian cycle on which every edge of P lies if, and only if, the subgraph of the k-ary n-cube induced by the edges of P is a vertex-disjoint collection of paths. This answers a question posed by Wang, Li and Wang who proved the analogous result for 3-ary n-cubes.

Keywords: Hamiltonian cycles, k-ary n-cubes, prescribed edges.
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1 Introduction

A whole range of families of graphs have been proposed for use as interconnection networks in the design of distributed-memory multiprocessors, where the vertices of a graph represent the processors of a machine and the edges between vertices the physical interconnections between processors. There are numerous properties such a family of graphs should have in order to be deemed suitable for such a purpose. For example, the graphs of such families should have small diameter (so as to aid message latency), be recursively decomposable (so as to aid scalability), have low degree (so as to lessen communication overheads), have high connectivity (so as to aid fault tolerance or data transfer), possess embeddings of other standard graphs (so as to aid simulations) and so on. The study of families of graphs suitable for use as interconnection networks has motivated new purely graph-theoretic research where the properties under consideration are relevant to the intended usage of these graphs in the context of parallel computation. Unfortunately the properties one might require are so diverse that there does not exist a family of graphs possessing every one of these properties and in practice trade-offs have to be made. Perhaps the most ubiquitous family of graphs in the landscape of interconnection networks are the hypercube Q_n and its close relation the k-ary n-cube Q_n^k (of course, such graphs are also common-place across discrete mathematics in general). The reader is referred to, for example, [6] for more on the relationship between graph theory and interconnection networks.

The study of Hamiltonian cycles in graphs is widespread. The basic problem of deciding whether an arbitrary graph has a Hamiltonian cycle is one of the canonical **NP**-complete problems, and there has been much research into restrictions upon arbitrary graphs under which this problem becomes solvable in polynomial-time. With respect to graphs used as interconnection networks, if such a graph possesses a Hamiltonian cycle then this cycle can, for example, easily be utilized so that all-to-all broadcasts can be accomplished and ring-based simulations undertaken (in the underlying distributed-memory multiprocessor). Many of the families of graphs used as interconnection networks possess Hamiltonian cycles: for instance, it has long been known that hypercubes [7] and k-ary n-cubes do [2].

Whilst the question of existence of Hamiltonian cycles in graphs such as hypercubes and k-ary n-cubes becomes a non-event, the question of efficiently constructing Hamiltonian cycles (especially using distributed-memory multiprocessors whose interconnection network is the graph in question) is still pertinent as is the consideration of additional impositions under which Hamiltonian cycles still exist. These impositions are usually related to avoiding or prescribing specific edges. For example, Chan and Lee [4] showed that an n-dimensional hypercube with at most $2n-5$ 'faulty links' (that is, with at most $2n-5$ edges missing) but where every vertex has degree at least 2, still has a Hamiltonian cycle but that there exist such faulty hypercubes with $2n-4$ faulty links (but so that every vertex has degree at least 2) that do not possess a Hamiltonian cycle. Ashir and Stewart [1] proved an analogous result for k-ary n-cubes with at most $4n-5$ faulty links. Chan and Lee also showed that it is **NP**-complete to decide whether a hypercube with an arbitrary set of faulty links is Hamiltonian [4], with Ashir and Stewart doing likewise for k-ary n-cubes [1].

However, it is with the prescription of specific vertices that we are concerned in this paper, which is, in some sense, complementary to edge avoidance. With regard to hypercubes, Caha and Koubek [3] proved that if the dimension n of a hypercube is at least 3 then for any set P of at most $n-1$ edges, there is a Hamiltonian cycle on which every edge of P lies if, and only if, the subgraph of the hypercube induced by the edges of P is a vertex-disjoint collection of paths. This result was extended by Dvořák [5] who showed that it holds when P consists of $2n-3$ edges and that this result is optimal (in that there are sets of $2n-2$ edges in an n-dimensional hypercube where the subgraph induced by these edges consists of vertex-disjoint paths but where there does not exist a Hamiltonian cycle upon which all of these edges lie). Consequently, Dvořák's result provides a precise classification as to when prescribed edges are guaranteed to lie upon a Hamiltonian cycle in a hypercube. Recently, Wang, Li and Wang [9] have embarked upon classifying when prescribed edges are guaranteed to lie upon a Hamiltonian cycle in a k-ary n-cube. Their result states that if P is a set of at most $2n-1$ edges in a 3-ary n-cube, where $n \geq 2$, then there is a Hamiltonian cycle on which every edge of P lies if, and only if, the subgraph of the 3-ary n-cube induced by the edges of P is a vertex-disjoint collection of paths. They make no comment as regards whether the number of edges in P can be increased so that the statement still holds (though they show that their result is optimal for $n = 2$) and pose the question of what happens in k-ary n-cubes when $k \geq 4$ as an open problem which we answer in this paper. In particular, we prove here that if P is a set of at most $2n-1$ edges in a k-ary n-cube, where $n \geq 2$ and $k \geq 4$, then there is a Hamiltonian cycle on which every edge of P lies if, and only if, the subgraph of the k-ary n-cube induced by the edges of P is a vertex-disjoint collection of paths.

We give the basic definitions and results relevant to this paper in Section 2 and prove our result for $n = 2$ in Section 3. In Section 4, we prove our result (by induction) for $n \geq 3$ (using the result in Section 3 as the base case). Our conclusions and directions for further research are given in Section 5.

2 Basic definitions and results

For $n \geq 1$ and $k \geq 3$, the *k-ary n-cube* Q_n^k is the graph whose vertex set is $\{(u_n, u_{n-1}, \ldots, u_1) : u_i \in \{0, 1, \ldots, k-1\},\ \text{for } i \in \{1, 2, \ldots, n\}\}$ and whose edge set consists of those pairs $((u_n, u_{n-1}, \ldots, u_1), (v_n, v_{n-1}, \ldots, v_1))$ where there exists some $d \in \{1, 2, \ldots, n\}$ such that $u_i = v_i$, whenever $i \neq d$, and either $u_d = v_d + 1$ or $u_d = v_d - 1$, with addition and subtraction modulo k (throughout this paper we assume that addition and subtraction on the components of vertices of Q_n^k is always modulo k). A path in some graph is a sequence of distinct vertices written $(x_1, x_2, \ldots, x_m)$, for some $m \geq 1$, so that (x_i, x_{i+1}) is an edge, for $i \in \{1, 2, \ldots, m-1\}$. The vertices x_1 and x_m of the path $(x_1, x_2, \ldots, x_m)$ are its *terminal vertices*, with all other vertices (when $m \geq 3$) its *internal vertices*. A path in a graph is *maximal* if it cannot be extended to a longer path in the graph. A cycle is a path $(x_1, x_2, \ldots, x_m)$, for some $m \geq 3$, for which we also have that (x_m, x_1) is an edge. Although a path and the corresponding cycle are written identically as sequences of vertices, it is always apparent as to whether we are referring to the sequence as a path or as a cycle.

Consider Q_n^k, where $n \geq 2$. Fix some $d \in \{1, 2, \ldots, n\}$. For any $i \in \{0, 1, \ldots, k-1\}$, consider those vertices of Q_n^k whose dth component is fixed at i. It is trivial to see that the subgraph of Q_n^k induced by these vertices is isomorphic to Q_{n-1}^k. We denote this subgraph by Q_i (when n, k and d are understood). We say that $Q_0, Q_1, \ldots, Q_{k-1}$ are formed by *partitioning* Q_n^k *over dimension* d. Note that any vertex x of Q_i has a corresponding vertex, denoted $n_j(x)$, in Q_j, for $j \in \{0, 1, \ldots, k-1\}$, where $n_j(x)$ is identical to x as a k-bit-string except that the dth component is equal to j. The vertex $x = n_i(x)$ is a neighbour of $n_{i-1}(x)$ and $n_{i+1}(x)$ in Q_n^k (with addition and subtraction on the indices modulo k), and the subgraph induced by the vertices of $\{x\} \cup \{n_j(x) : j \in \{0, 1, \ldots, k-1\} \setminus \{i\}\}$ is the cycle $(n_0(x), n_1(x), \ldots, n_{i-1}(x), x, n_{i+1}(x), \ldots, n_{k-1}(x))$. Any edge of Q_n^k that is not in $Q_0, Q_1, \ldots, Q_{k-1}$ is said to *lie in dimension* d. Let G be some subgraph of Q_i and let G' be the subgraph of Q_j, where $j \neq i$, induced by the edges of $\{(x, y) : (x, y)$ is an edge of Q_j such that $(n_i(x), n_i(y))$ is an edge of $G\}$. The graph G' is clearly isomorphic to G and is said to be the *isomorphic copy of* G *in* Q_j. Let X be a set of edges of Q_n^k. We write $\langle X \rangle$ to denote the subgraph induced by the edges of X. If every edge of X lies in some Q_i then the *isomorphic copy of* X *in* Q_j, where $i \neq j$, is the set of edges X' so that $\langle X' \rangle$ is the isomorphic copy of $\langle X \rangle$ in Q_j. We shall be interested in specific sets of edges in Q_n^k which we denote by P. We write P_j to denote those edges of Q_j that are in P, for $j \in \{0, 1, \ldots, k-1\}$. Edges in $P \setminus \cup_{i=0}^{k-1} P_i$ clearly lie in dimension d.

Suppose that we have partitioned Q_n^k over some dimension d to get

$Q_0, Q_1, \ldots, Q_{k-1}$ and we have a specific set of edges P. If (x, y) is an edge of Q_i then we say that the cycle $(x, n_{i+1}(x), n_{i+1}(y), y)$ is a *bridge joining Q_i and Q_{i+1}*. Of course, $(n_{i+1}(x), n_i(n_{i+1}(x)), n_i(n_{i+1}(y)), n_{i+1}(y)) = (n_{i+1}(x), x, y, n_{i+1}(y))$ is the same bridge. We shall be interested in bridges with certain properties. Let $(x, n_{i+1}(x), n_{i+1}(y), y)$ be a bridge joining Q_i and Q_{i+1}. We say that this bridge is *right-useable* if

1. $(n_{i+1}(x), n_{i+1}(y)) \notin P_{i+1}$
2. $n_{i+1}(x)$ and $n_{i+1}(y)$ are not terminal vertices on some maximal path of $\langle P_{i+1} \rangle$ of length at least 2
3. both $n_{i+1}(x)$ and $n_{i+1}(y)$ are incident with at most 1 edge of P_{i+1}.

Note that if $\langle P_{i+1} \rangle$ consists of a set of vertex-disjoint paths and the bridge $(x, n_{i+1}(x), n_{i+1}(y), y)$ is right-useable then by conditions 2 and 3 of the definition of right-useability, $\langle P_{i+1} \cup \{(n_{i+1}(x), n_{i+1}(y))\} \rangle$ consists of a set of vertex-disjoint paths also. Our bridge is *left-useable* if $(x, y) \notin P_i$, x and y are not terminal vertices on some maximal path of $\langle P_i \rangle$ of length at least 2, and both x and y are incident with at most 1 edge of P_i. Our bridge is *useable* if it is both left-useable and right-useable (the relevance of these definitions will be made clear later). We shall use bridges to build larger cycles out of smaller cycles as follows. Suppose that C_i is a cycle in Q_i and C_{i+1} is a cycle in Q_{i+1} so that (x, y) is an edge of C_i and $(n_{i+1}(x), n_{i+1}(y))$ is an edge of C_{i+1}. The cycle D formed by removing the edges (x, y) and $(n_{i+1}(x), n_{i+1}(y))$ from C_i and C_{i+1}, respectively, and including the edges $(x, n_{i+1}(x))$ and $(y, n_{i+1}(y))$ is said to have been formed by *joining C_i and C_{i+1} using the bridge* $(x, n_{i+1}(x), n_{i+1}(y), y)$. We also say that D has been formed by *extending* C_i or C_{i+1}.

The following result will prove very useful.

Theorem 1 [8] Let $k \geq 3$ be odd and let $n \geq 2$. Given any two distinct vertices x and y of Q_n^k, there exists a Hamiltonian path from x to y.

Our primary motivation in this paper is the following result due to Wang, Li and Wang.

Theorem 2 [9] Let $n \geq 2$ and let P be any set of $2n - 1$ edges in Q_n^3. The 3-ary n-cube Q_n^3 has a Hamiltonian cycle on which every edge of P lies if, and only if, $\langle P \rangle$ consists of pairwise vertex-disjoint paths.

3 The base case

We begin by proving our main result but only for Q_2^k where $k \geq 4$.

Theorem 3 Let $k \geq 4$ and let P be a set of 3 edges in Q_2^k. The k-ary 2-cube Q_2^k has a Hamiltonian cycle on which every edge of P lies if, and only if, $\langle P \rangle$ consists of pairwise vertex-disjoint paths (that is, $\langle P \rangle$ is not a triangle or a star with 3 radii).

Proof W.l.o.g. we may assume that there are at least 2 edges of P in dimension 1. Partition Q_2^k over dimension 2 so as to obtain cycles $Q_0, Q_1, \ldots, Q_{k-1}$, with P_j the set of edges of P that lie in Q_j, for $j \in \{0, 1, \ldots, k-1\}$.

Consider Q_0 and Q_1. Suppose that all 3 edges of P lie in Q_0 or Q_1 or join a vertex in Q_0 to a vertex in Q_1. As we show below, irrespective of where the edges of P lie, as $k \geq 4$ we can obtain a cycle D_{01} spanning the vertices in Q_0 and Q_1 such that D_{01} contains all edges of $P_0 \cup P_1$ as well as any dimension 2 edge joining a vertex in Q_0 and a vertex in Q_1, except in one situation (Case 2.d in Fig. 2) where we have a path as opposed to a cycle. W.l.o.g. the different situations are depicted as follows.

Case 1: There exists 1 edge of P in dimension 2 and $|P_0| = 2$.

The essential cases are illustrated in Fig. 1 (where edges of P are represented using dashed lines).

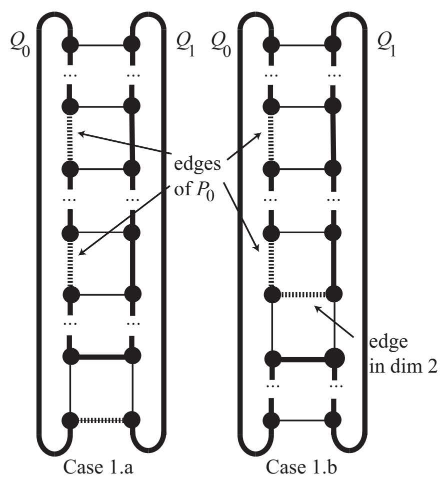

Figure 1: 1 edge in dimension 2 and $|P_0| = 2$.

Case 2: There exists 1 edge of P in dimension 2 and $|P_0| = |P_1| = 1$.

The essential cases are illustrated in Fig. 2. Note the special case, Case 2.d, where the edges of P lie as shown and $k \geq 5$ is odd, where we have a path spanning the vertices of Q_0 and Q_1 (and containing all edges of P) as opposed to a cycle.

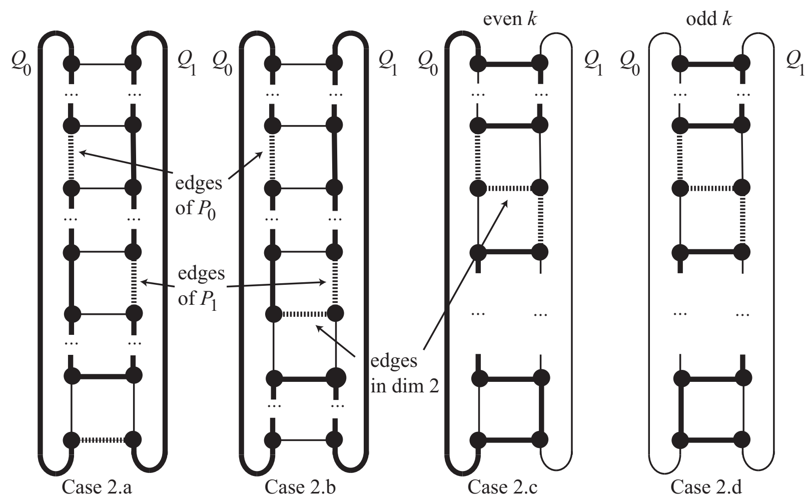

Figure 2: 1 edge in dimension 2 and $|P_0| = |P_1| = 1$.

Case 3: There exists no edge of P in dimension 2.

The essential cases are illustrated in Fig. 3.

Of courses, by symmetry our observations above apply to any pair Q_i and Q_{i+1}, for $i \in \{1, 2, \ldots, k-1\}$, and not just Q_0 and Q_1. Also, we can apply the above constructions even when the number of edges of P in Q_i or Q_{i+1} or joining a vertex of Q_i with a vertex of Q_{i+1} is less than 3.

W.l.o.g. we may suppose that if there is a dimension 2 edge then this edge joins a vertex in Q_0 with a vertex in Q_1.

If $k \geq 4$ is even then we can build cycles as above spanning: Q_0 and Q_1; Q_2 and Q_3; ...; and Q_{k-2} and Q_{k-1}. We can join, for example, a cycle D_{01} spanning the vertices of Q_0 and Q_1 with a cycle D_{23} spanning the vertices of Q_2 and Q_3 using a bridge $(x, n_1(x), n_1(y), y)$ where (x, y) is an edge of D_{01} not in P and $(n_1(x), n_1(y))$ is an edge of D_{23} not in P. It is not difficult to see that no matter what the configuration of edges of P, we can iteratively choose and join our cycles together to obtain a Hamiltonian cycle of Q_2^k on which every edge of P lies.

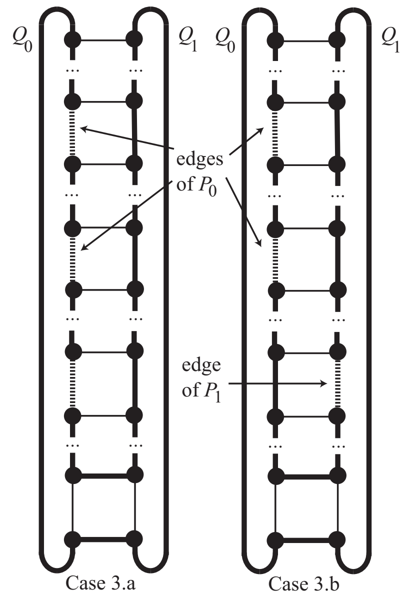

Figure 3: no edges in dimension 2.

Suppose that $k \geq 5$ is odd and that the edges of P do not form the configuration as in Case 2.d in Fig. 2. As $k \geq 5$, either: there is no dimension 2 edge and there is some Q_j containing no edges of P, for $j \in \{0, 1, \ldots, k-1\}$; or there is some Q_j containing no edges of P, where $j \in \{2, 3, \ldots, k-1\}$. In both cases, form pairs Q_i and Q_{i+1}, where $i \neq j \neq i+1$, as in the previous paragraph, and build cycles spanning the vertices of Q_i and Q_{i+1} and containing all edges of P that are in Q_i or Q_{i+1} or join a vertex in Q_i with a vertex in Q_{i+1} (in the latter case, we ensure that Q_0 and Q_1 are paired together). In both cases, it is not difficult to see that we can always join these cycles together, as in the previous paragraph, to obtain a cycle spanning all vertices of Q_i, for $i \in \{0, 1, \ldots, k-1\} \setminus \{j\}$, and containing all edges of P. We can then join this cycle to Q_j using an appropriate bridge.

Thus, all that remains is the case when $k \geq 5$ is odd and the edges of P form the configuration in Case 2.d of Fig. 2. Let the path ρ spanning the vertices of Q_0 and Q_1 as constructed in Case 2.d of Fig. 2 go from vertex x of Q_0 to vertex y of Q_1. Concatenate to ρ the edge $(y, n_2(y))$, then the path of length $k-1$ from $n_2(y)$ to $n_2(x)$ in Q_2, then the edge $(n_2(x), n_3(x))$, then the path of length $k-1$ from $n_3(x)$ to $n_3(y)$ in Q_3, …, then the edge $(n_{k-2}(y), n_{k-1}(y))$, then the path of length $k-1$ from $n_{k-1}(y)$ to $n_{k-1}(x)$, and then the edge $(n_{k-1}(x), x)$. The resulting cycle is a Hamiltonian cycle of Q_2^k and contains all edges of P. □

4 The main result

In this section we prove our main result, namely Theorem 4. The proof of Theorem 4 is by induction, with Theorem 3 forming the base case.

Theorem 4 Let $n \geq 2$, let $k \geq 4$ and let P be a set of edges of Q_n^k with $|P| \leq 2n-1$. There exists a Hamiltonian cycle of Q_n^k on which every edge of P lies if, and only if, $\langle P \rangle$ consists of pairwise vertex-disjoint paths.

Proof Note that because $k \geq 4$, we may assume that Q_n^k contains no cycles of length 3. We proceed by induction on n. As our induction hypothesis, suppose that $n \geq 3$ and that whenever we have a set P' of at most $2n-3$ edges of Q_{n-1}^k, there exists a Hamiltonian cycle of Q_{n-1}^k on which every edge of P' lies if, and only if, $\langle P' \rangle$ consists of pairwise vertex-disjoint paths. The base case of our induction follows by Theorem 3. Let P be a set of at most $2n-1$ edges in Q_n^k. If Q_n^k has a Hamiltonian cycle containing every edge of P then trivially $\langle P \rangle$ consists of pairwise vertex-disjoint paths. So, assume that $\langle P \rangle$ consists of pairwise vertex-disjoint paths.

There exists some dimension d such that at most 1 edge of P lies in dimension d. Partition Q_n^k over dimension d to obtain the k-ary $(n-1)$-cubes $Q_0, Q_1, \ldots, Q_{n-1}$, with P_j consisting of those edges of P that lie in Q_j, for $j \in \{0, 1, \ldots, k-1\}$. W.l.o.g. assume that $|P_0| \geq |P_j|$, for $j \in \{1, 2, \ldots, k-1\}$. There are two essential cases: when there is 1 edge in dimension d; and when there are no edges in dimension d. However, we begin with two useful lemmas.

Lemma 5 Let X be a set of edges in Q_i where $\langle X \rangle$ consists of a set of vertex-disjoint paths or cycles. Let the set of edges X' be the set of edges of Q_{i+1} isomorphic to X. The number of right-useable bridges of the form $(x, n_{i+1}(x), n_{i+1}(y), y)$ joining Q_i and Q_{i+1}, where $(x, y) \in X$, is at least $\max\{|X| - |X' \cap P_{i+1}| - 2|P_{i+1}|, 0\}$.

Proof Throughout this proof, by a bridge we mean a bridge of the form $(x, n_{i+1}(x), n_{i+1}(y), y)$, where $(x, y) \in X$.

Consider some edge $f \in P_{i+1}$. If $f \in X'$ then this makes the bridge containing f not right-useable: so, $|X' \cap P_{i+1}|$ bridges are not right-useable because condition 1 of the definition of right-useability fails.

Suppose that the edge $f \in X' \setminus P_{i+1}$ is such that its 2 incident vertices are the terminal vertices of a maximal path of $\langle P_{i+1} \rangle$ of length at least 2. As there are no cycles of length 3, this path must have length at least 3. Thus, if α is the number of vertex-disjoint maximal paths in $\langle P_{i+1}\rangle$ of length at least 3 then at most α bridges are not right-useable because condition 2 of the definition of right-useability fails.

Consider some edge $f \in X' \setminus P_{i+1}$ where its 2 incident vertices are not the terminal vertices of a maximal path of $\langle P_{i+1}\rangle$ of length at least 2. However, suppose that the bridge involving f is still not right-useable. So, one of its incident vertices is incident with 2 edges of P_{i+1}: that is, this vertex is an internal vertex of some maximal path in $\langle P_{i+1}\rangle$. As $\langle X'\rangle$ is such that every vertex has degree at most 2, any such internal vertex renders at most 2 bridges not right-useable. Hence, the maximum number of bridges rendered not right-useable because condition 3 of the definition of right-useability fails is at most $2(|P_{i+1}| - \beta)$, where β is the number of vertex-disjoint maximal paths in $\langle P_{i+1}\rangle$.

Consequently, the total number of bridges rendered not right-useable is at most $|X' \cap P_{i+1}| + \alpha + 2(|P_{i+1}| - \beta) \leq |X' \cap P_{i+1}| + 2|P_{i+1}|$. □

Lemma 6 Let D be a cycle spanning all vertices of $Q_i, Q_{i+1}, \ldots, Q_l$, for some i and l (with possibly $i = l$), where

- $|P_j| \leq 2n - 4$, for $j \in \{0, 1, \ldots, k-1\} \setminus \{i, i+1, \ldots, l\}$
- D contains all edges of $P_i \cup P_{i+1} \cup \ldots \cup P_l$ as well as any dimension d edge of P that might happen to join vertices of $Q_i, Q_{i+1}, \ldots, Q_l$
- there are no edges of P lying in dimension d and incident with a vertex from Q_j, for $j \in \{0, 1, \ldots, k-1\} \setminus \{i, i+1, \ldots, l\}$
- the number of edges of D lying in Q_i is greater than $6n - 9$.

The cycle D can be extended to a Hamiltonian cycle of Q_n^k containing every edge of P.

Proof Let X be the set of edges of D lying in Q_i and let X' be the isomorphic copy of X in Q_{i-1}. By Lemma 5, there are at least $\max\{|X| - |X' \cap P_{i-1}| - 2|P_{i-1}|, 0\}$ left-useable bridges joining D and Q_{i-1}. Moreover, at least $\max\{|X| - |X' \cap P_{i-1}| - |X \cap P_i| - 2|P_{i-1}|, 0\} = \max\{|X| - (2n-1) - 2(2n-4), 0\} = \max\{|X| - 6n + 9, 0\} > 0$ of these bridges are such that the edge of the bridge lying in Q_i is not in P_i.

Let (u, u', v', v) be such a left-useable bridge joining the edge (u, v) of D lying in Q_i to the edge (u', v') of Q_{i-1}. By the induction hypothesis applied to $(P_{i-1} \cup \{(u', v')\}, Q_{i-1})$ (see the remark immediately after the definition of right-useability), there exists a Hamiltonian cycle C_{i-1} in Q_{i-1} containing every edge of P_{i-1} as well as the edge (u', v'). Join D and C_{i-1} using the bridge (u, u', v', v) to obtain a cycle spanning the vertices of $Q_{i-1}, Q_i, \ldots, Q_l$ and containing every edge of $P_{i-1} \cup P_i \cup \ldots \cup P_l$ as well as any dimension d edge of P that might happen to join vertices of $Q_i, Q_{i+1}, \ldots, Q_l$. Proceeding iteratively in this way (noting that $k^{n-1} - 1 > 6n - 9$ and repeatedly applying Lemma 5) yields the result (the process can be visualised as in Fig. 4). □

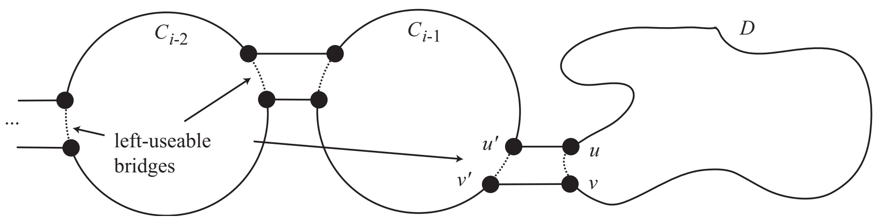

Figure 4: Extending the cycle D.

Case (a): $|P \setminus \cup_{i=0}^{k-1} P_i| = 1$.

Let the edge of P that is not in $\cup_{i=0}^{k-1} P_i$ be $e = (x, y)$.

Lemma 7 Suppose that x lies in Q_i, y lies in Q_{i+1} and both $|P_i|$ and $|P_{i+1}|$ are at most $2n - 4$. There exists a cycle D spanning all vertices of Q_i and Q_{i+1} that contains every edge of $P_i \cup P_{i+1} \cup \{e\}$ and is such that only 2 edges of D do not lie in Q_i or Q_{i+1} (one of which is e) with these 2 edges being part of a bridge joining Q_i and Q_{i+1}.

Proof Let x' be a neighbour of x in Q_i with y' the corresponding neighbour of y in Q_{i+1} (so $y' = n_{i+1}(x')$). Consider the bridge (x, x', y', y). Suppose that this bridge is useable. We can apply the induction hypothesis to $(P_i \cup \{(x, x')\}, Q_i)$ and to $(P_{i+1} \cup \{(y, y')\}, Q_{i+1})$ and obtain Hamiltonian cycles C_i and C_{i+1} of Q_i and Q_{i+1}, respectively, so that C_i contains every edge of P_i, as well as (x, x'), and C_{i+1} contains every edge of P_{i+1}, as well as (y, y'). We can join C_i and C_{i+1} using the bridge (x, x', y', y) to obtain a cycle D as in the statement of the lemma.

Suppose that the bridge (x, x', y', y) is not useable. So, as x is incident with at most 1 edge of P_i and at most 1 edge of P_{i+1}, we must have (at least) one of the following six occurrences: $(x, x') \in P_i$; x and x' are the terminal vertices of some maximal path in $\langle P_i \rangle$ of length at least 2; x' is incident with at least 2 edges of P_i; $(y, y') \in P_{i+1}$; y and y' are the terminal vertices of some maximal path in $\langle P_{i+1} \rangle$; y' is incident with at least 2 edges of P_{i+1}. Let us count the maximal number of bridges joining Q_i and Q_{i+1} involving x that are rendered not useable due to the edges of P_i, *i.e.*, not left-useable. As x is incident with at most 1 edge of P_i, at most 1 bridge of the form (x, x'', y'', y) is rendered not useable because $(x, x'') \in P_i$ or because x and x'' are the terminal vertices of some maximal path in $\langle P_i \rangle$ of length at least 2. Alternatively, if a bridge of the form (x, x', y', y) is rendered not useable because x' is incident with at least 2 edges of P_i then x' must be some internal vertex of a maximal path of $\langle P_i \rangle$.

- Suppose that there exists an edge $(x, x'') \in P_i$. This leaves at most $|P_i| - 2$ internal vertices of paths in $\langle P_i \rangle$ that might be used to render bridges involving x not useable.

- Suppose that x and x'' are the terminal vertices of some maximal path in $\langle P_i \rangle$ of length at least 2. Consider the vertex z of this path adjacent to x''. The vertex z cannot be adjacent to x as Q_n^k has no cycles of length 3. Thus, the (internal) vertex z (of our path) cannot be used to render a bridge involving x not useable, and this leaves at most $|P_i| - 2$ internal vertices of maximal paths in $\langle P_i \rangle$ that can.

- Suppose that there does not exist an edge $(x, x'') \in P_i$ nor is it the case that there exists a neighbour x'' of x in Q_i such that x and x'' are the terminal vertices of some maximal path in $\langle P_i \rangle$ of length at least 2. The maximal number of internal vertices of paths in $\langle P_i \rangle$ that can be used to render a bridge involving x not useable is at most $|P_i| - 1$.

Consequently, the maximal number of bridges involving x rendered not useable because of edges of P_i is at most $|P_i| - 1$. The same goes for the edges of P_{i+1}. Thus, as: there are $2n - 2$ bridges involving x; at most $|P_i| + |P_{i+1}| - 2$ of these bridges are not useable; and $|P_i| + |P_{i+1}| \leq 2n - 1$, at least one bridge must be useable. The result follows. □

Suppose that $|P_0| \leq 2n - 4$. By Lemmas 6 and 7 (noting that $k^{n-1} - 1 > 6n - 9$), there is a Hamiltonian cycle in Q_n^k containing all edges of P. Henceforth, we assume that $|P_0| \geq 2n - 3$.

Lemma 8 Suppose that x is a vertex of Q_0, y is a vertex of Q_1 and $|P_0| = 2n-3$. There is a Hamiltonian cycle in Q_n^k containing all edges of P.

Proof By the induction hypothesis applied to (P_0, Q_0), there is a Hamiltonian cycle in Q_0 containing all edges of P_0. Let x' and x'' be the neighbours of x on C_0, with y' and y'', respectively, their neighbours in Q_1. W.l.o.g. $(x, x') \notin P_0$ (as $(x, y) \in P$). Consider the bridge (x, y, y', x'). If $(y, y') \notin P_1$ then apply the induction hypothesis to $(P_1 \cup \{(y, y')\}, Q_1)$ to obtain a Hamiltonian cycle C_1 in Q_1 that contains all edges of P_1 as well as (y, y'). We can join C_0 and C_1 using the bridge (x, y, y', x') to obtain a cycle spanning all vertices of Q_0 and Q_1 and containing all edges of $P_0 \cup P_1 \cup \{e\}$. The result follows by Lemma 6.

So, suppose that $(y, y') \in P_1$: that is, $P_1 = \{(y, y')\}$ and $\cup_{j=2}^{k-1} P_j = \emptyset$. Let C_1' be the isomorphic copy of C_0 in Q_1. In particular, C_1' contains (y, y'). Consider the path obtained by starting from y'' and traversing C_1' to y (omitting (y'', y)), taking the edge (y, x), and then traversing C_0 to x' (omitting (x, x')). Clearly this path contains every vertex of Q_0 and Q_1 as well as every edge of $P_0 \cup P_1 \cup \{e\}$, *i.e.*, P. Extend this path by the edges $(y'', n_2(x''))$ and $(x', n_{k-1}(x'))$ to obtain the path ρ.

If $k = 5$ then we can, first, apply Theorem 1 three times to obtain Hamiltonian paths in Q_2, Q_3 and Q_4 from $n_2(x'')$ to $n_2(x')$, from $n_3(x'')$ to $n_3(x')$, and from $n_4(x'')$ to $n_4(x')$, respectively, and, second, easily compose these paths with ρ to obtain a Hamiltonian cycle in Q_n^5 containing all edges of P. Indeed, by proceeding similarly, we can obtain a Hamiltonian cycle in any Q_n^k containing all edges of P whenever $k \geq 5$ is odd.

Suppose that $k = 4$. Applying the induction hypothesis to $(\{(n_2(x''), n_2(x))\}, Q_2)$ and to $(\{(n_3(x'), n_3(x))\}, Q_3)$ yields Hamiltonian cycles C_2 and C_3 in Q_2 and Q_3, respectively: that is, Hamiltonian paths in Q_2 from $n_2(x'')$ to $n_2(x)$ and in Q_3 from $n_3(x')$ to $n_3(x)$. These paths can easily be composed with ρ to obtain a Hamiltonian cycle in Q_n^4 containing all edges of P. When $k \geq 6$ is even, we proceed similarly by applying the induction hypothesis to $(\{(n_2(x''), n_2(x))\}, Q_2)$ and to $(\{(n_j(x'), n_j(x))\}, Q_j)$, if $3 \leq j \leq k-1$, before composing the resulting paths. The result follows. □

Lemma 9 Suppose that x is a vertex of Q_i, y is a vertex of Q_{i+1} and $|P_0| = 2n-3$, where $i \neq 0 \neq i+1$. There is a Hamiltonian cycle in Q_n^k containing all edges of P.

Proof As in the proof of Lemma 7, there is a useable bridge (x, y, y', x') joining Q_i and Q_{i+1}. Let the edge of $P \setminus (P_0 \cup \{e\})$ be (p, q). By the induction hypothesis applied to (P_0, Q_0), there is a Hamiltonian cycle C_0 in Q_0 containing every edge of P_0. The cycle C_0 contains $k^{n-1} - (2n-3) \geq 13$ edges not in P_0.

Choose two non-incident edges (u, v) and (s, t) of C_0 that are not in P_0 where $\{u, v, s, t\} \cap \{n_0(x), n_0(x'), n_0(p), n_0(q)\} = \emptyset$. Applying the induction hypothesis to $(P_i \cup \{(x, x'), (n_i(u), n_i(v))\}, Q_i)$ yields a Hamiltonian cycle in Q_i containing all edges of P_i as well as (x, x') and $(n_i(u), n_i(v))$. Applying the induction hypothesis to $(P_j \cup \{(n_j(u), n_j(v)), (n_j(s), n_j(t))\}, Q_j)$, for $j \in \{2, 3, \ldots, i-1\}$, yields a Hamiltonian cycle C_j in Q_j containing all edges of P_j as well as $(n_j(u), n_j(v))$ and $(n_j(s), n_j(t))$. We can join C_i and C_{i-1} using the bridge $(n_i(u), n_{i-1}(u), n_{i-1}(v), n_i(v))$, and then using the bridge $(n_j(u), n_{j-1}(u), n_{j-1}(v), n_j(v))$ or $(n_j(s), n_{j-1}(t), n_{j-1}(t), n_j(s))$, for $j \in \{1, 2, \ldots, i-1\}$, as appropriate, we can join the Hamiltonian cycles $C_0, C_1, \ldots, C_i$ so as to obtain a cycle D spanning all vertices of $Q_0, Q_1, \ldots, Q_i$ and containing all edges of $P_0 \cup P_1 \cup \ldots \cup P_i$.

Applying the induction hypothesis to $(P_{i+1} \cup \{(y, y')\}, Q_{i+1})$ yields a Hamiltonian cycle in Q_{i+1} containing all edges of P_{i+1} as well as (y, y'). We can join D and C_{i+1} using the bridge (x, y, y', x') to obtain a cycle spanning all vertices of $Q_0, Q_1, \ldots, Q_{i+1}$ and containing all edges of $P_0 \cup P_1 \cup \ldots \cup P_{i+1} \cup \{e\}$. This cycle can be extended to a Hamiltonian cycle in Q_n^k containing all edges in P by Lemma 6. $\square$

Suppose that $|P_0| = 2n - 3$. By Lemmas 8 and 9, there is a Hamiltonian cycle in Q_n^k containing all edges of P. Henceforth, we assume that $|P_0| = 2n - 2$.

There are two possibilities: x is a vertex of Q_0 and y is a vertex of Q_1; x is a vertex of Q_i, where $i \neq 0$, and y is a vertex of Q_{i+1}, where $i + 1 \neq 0$.

Lemma 10 Suppose that x is a vertex of Q_0, y is a vertex of Q_1 and $|P_0| = 2n - 2$. There is a Hamiltonian cycle in Q_n^k containing every edge of P.

Proof As x is incident with at most 1 edge of P_0, let (a, b) be some edge of P_0 that is not incident with x. Applying the induction hypothesis to $(P_0 \setminus \{(a, b)\}, Q_0)$ results in a Hamiltonian cycle C_0 in Q_0 containing every edge of $P_0 \setminus \{(a, b)\}$. Suppose that C_0 also contains (a, b). Let x' and x'' be the neighbours of x on C_0, with y' and y'', respectively, their neighbours in Q_1.

W.l.o.g. $(x, x') \notin P_0$ (as $(x, y) \in P$). Consider the bridge (x, y, y', x'). Apply the induction hypothesis to $(\{(y, y')\}, Q_1)$ to obtain a Hamiltonian cycle C_1 in Q_1 that contains (y, y'). We can join C_0 and C_1 using the bridge (x, y, y', x') to obtain a cycle spanning all vertices of Q_0 and Q_1 and containing all edges of P. The result follows by Lemma 6.

So, suppose that $(a, b) \notin P_0$. Let a' and a'' be the neighbours of a on C_0, and let b' and b'' be the neighbours of b on C_0. Moreover, assume that there is a sub-path of C_0 joining to a' to b' on which x does not lie. Let x' be the neighbour of x on C_0 so that x' does not lie on the sub-path of C_0 from x to a avoiding b, and let x'' be the other neighbour of x on C_0. W.l.o.g. we may assume that $(x, x') \notin P_0$ (as $(x, y) \in P$). Note that $a' \neq b'$ as otherwise there would be a cycle of length 3 in Q_n^k. However, it may be the case that $x' = b''$, $x' = b$, $x'' = a''$ or $x'' = a$.

There are 4 different essential cases to consider depending upon whether or not the edges (a, a') and (b, b') are in P_0. Some of these cases have sub-cases. All cases and their sub-cases are described below and illustrated in Fig. 5.

1. If $(a, a') \notin P_0$ and $(b, b') \notin P_0$ then
   - define ρ_1 to be the sub-path of C_0 from x to a avoiding b, concatenated with (a, b), concatenated with the sub-path of C_0 from b to x' avoiding a
   - define ρ_2 to be the sub-path of C_0 from a' to b' avoiding a.

2.a If $(a, a') \notin P_0$, $(b, b') \in P_0$ and $b'' \neq x' \neq b$ then
   - define ρ_1 to be the sub-path of C_0 from a' to b avoiding a, concatenated with (b, a), concatenated with the sub-path of C_0 from a to x avoiding b
   - define ρ_2 to be the sub-path of C_0 from b'' to x' avoiding x.

   Note that $(b, b'') \notin P_0$.

2.b If $(a, a') \notin P_0$, $(b, b') \in P_0$ and $b'' = x'$ then
   - define ρ_1 to be the sub-path of C_0 from x to a avoiding b, concatenated with (a, b), concatenated with the sub-path of C_0 from b to a' avoiding a.

Note that this path does not include $x' = b''$ and that $(b, x') \notin P_0$.

2.c If $(a, a') \notin P_0$, $(b, b') \in P_0$ and $x' = b$ then

- define ρ_1 to be the sub-path of C_0 from x to a avoiding b, concatenated with (a, b), concatenated with the sub-path of C_0 from b to a' avoiding a.

3.a If $(a, a') \in P_0$, $(b, b') \notin P_0$ and $a'' \neq x$ then

- define ρ_1 to be the sub-path of C_0 from x' to b avoiding a, concatenated with (b, a), concatenated with the sub-path of C_0 from a to b' avoiding b
- define ρ_2 to be the sub-path of C_0 from x to a'' avoiding a.

Note that $(a, a'') \notin P_0$.

3.b If $(a, a') \in P_0$, $(b, b') \notin P_0$ and $a'' = x$ then

- define ρ_1 to be the sub-path of C_0 from b' to a avoiding b, concatenated with (a, b), concatenated with the sub-path of C_0 from b to x avoiding a.

Note that $(a, a'') \notin P_0$.

4.a If $(a, a') \in P_0$, $(b, b') \in P_0$, $x'' \neq a$ and $x' \neq b''$ then let $(c, d) \notin P_0$ be an edge on the sub-path of C_0 joining a' and b' avoiding a so that c is closer to a' on this path than d is and

- define ρ_1 to be the sub-path of C_0 from c to a avoiding b, concatenated with (a, b), concatenated with the sub-path of C_0 from b to d avoiding a
- define ρ_2 to be the sub-path of C_0 from x to a'' avoiding a
- if $x' \neq b$ then define ρ_3 to be the sub-path of C_0 from x' to b'' avoiding a.

Note that both (a, a'') and (b, b'') are not in P_0.

4.b If $(a, a') \in P_0$, $(b, b') \in P_0$, $x'' \neq a$ and $x' = b''$ then let $(c, d) \notin P_0$ be an edge on the sub-path of C_0 joining a' and b' avoiding a so that c is closer to a' on this path than d is and

   – define ρ_1 to be the sub-path of C_0 from c to a avoiding b, concatenated with (a, b), concatenated with the sub-path of C_0 from b to d avoiding a

   – define ρ_2 to be the sub-path of C_0 from x to a'' avoiding a.

   Note that these paths do not include $x' = b''$ and that both (a, a'') and (b, b'') are not in P_0.

4.c If $(a, a') \in P_0$, $(b, b') \in P_0$ and $x'' = a$ then let $(c, d) \notin P_0$ be an edge on the sub-path of C_0 joining a' and b' avoiding a so that c is closer to a' on this path than d is and

   – define ρ_1 to be the sub-path of C_0 from c to a avoiding b, concatenated with (a, b), concatenated with the sub-path of C_0 from b to d avoiding a

   – define ρ_2 to be the sub-path of C_0 from x to b'' avoiding a.

   Note that both (a, a'') and (b, b'') are not in P_0, and that $x' \neq b$ as otherwise there would be a cycle of length 3 in Q_n^k.

Consider Case 1. Let ρ_1' and ρ_2' be the isomorphic copies of ρ_1 and ρ_2, respectively, in Q_1. Join ρ_1 and ρ_1' using the edges $(a', n_1(a'))$ and $(b', n_1(b'))$ to form the cycle D_1, and join ρ_2 and ρ_2' using the edges (x, y) and $(x', n_1(x'))$ to form the cycle D_2. Every edge of P lies on one of these cycles. Take any edge f of D_1 lying within Q_1 and any edge g of D_2 lying within Q_1, and let f' and g' be the isomorphic copies of f and g, respectively, in Q_2. By the induction hypothesis applied to $(\{f', g'\}, Q_2)$, there is a Hamiltonian cycle C_2 in Q_2 containing f' and g'. Join D_1 and D_2 to C_2 using the bridges involving f and f' and g and g', respectively. We obtain a cycle spanning all vertices of Q_0, Q_1 and Q_2 and containing all edges of P. We can extend this cycle to a Hamiltonian cycle of Q_n^k containing all edges of P by Lemma 6 (as $k^{n-1} - 2 > 6n - 9$). Analogous constructions apply in Cases 2.a, 2.c, 3.a, 3.b and 4.c.

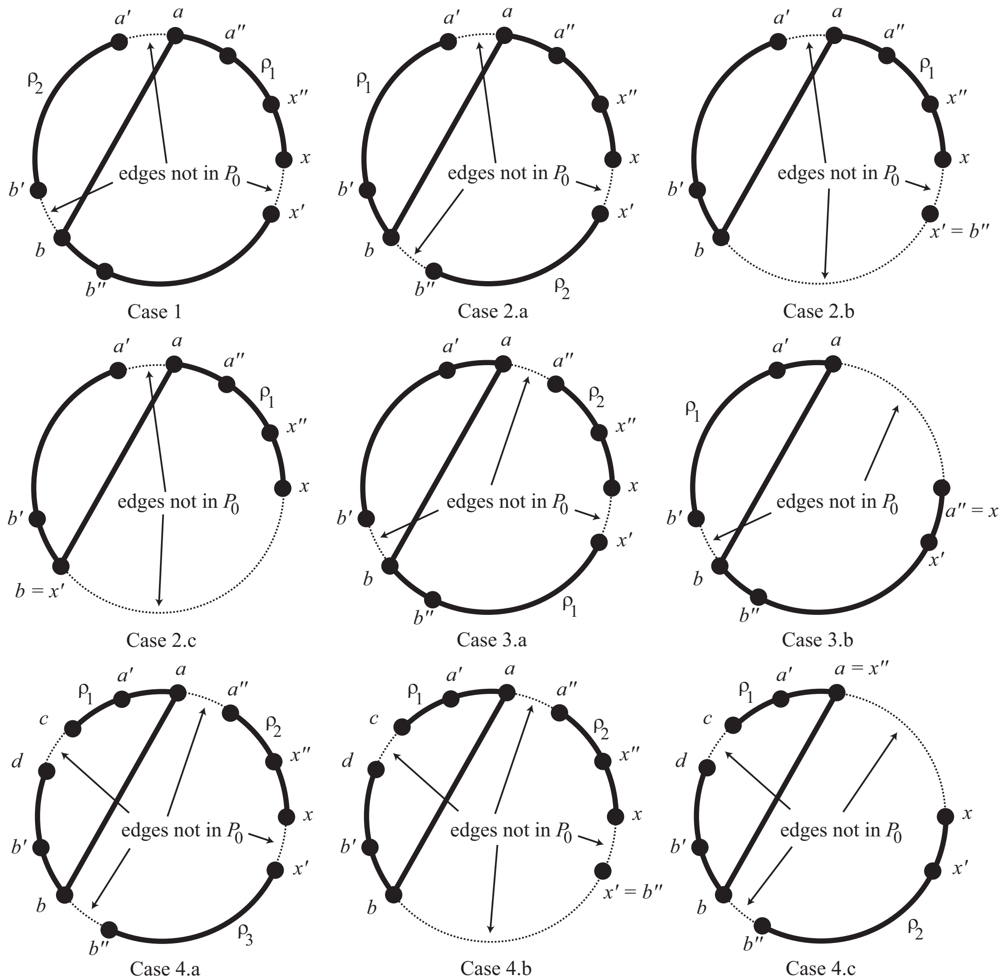

Figure 5: The different cases for the edge (a, b).

We are left with Cases 2.b, 4.a and 4.b. Consider Case 2.b. Let ρ_1' be the isomorphic copy of ρ_1 in Q_1. Join ρ_1 and ρ_1' using the edges (x, y) and $(a', n_1(a'))$ to form the cycle D.

Suppose that k is even. For every $j \in \{2, 3, \ldots, k-1\}$, apply the induction hypothesis to $(\{(n_j(x'), n_j(x)), (n_j(x), n_j(x''))\}, Q_j)$ to obtain a Hamiltonian cycle C_j in Q_j upon which both edges $(n_j(x'), n_j(x))$ and $(n_j(x), n_j(x''))$ lie. For $j \in \{2, 3, \ldots, k-1\}$, let π_j by the sub-path of C_j from $n_j(x')$ to $n_j(x)$ of length $k^{n-1} - 1$. Form the cycle D' by starting from the path $(x', n_1(x'), n_2(x'))$, concatenating π_2, concatenating the edge $(n_2(x), n_3(x))$, concatenating the path π_3, concatenating the edge $(n_3(x'), n_4(x'))$, concatenating π_4, ..., concatenating the edge $(n_{k-2}(x), n_{k-1}(x))$, concatenating the path π_{k-1} and finally concatenating the edge $(n_{k-1}(x'), x')$. The cycles D and D' span all vertices of Q_n^k. Join D and D' using the bridge $(y, n_2(x), n_2(x''), n_1(x''))$ to obtain a Hamiltonian cycle of Q_n^k containing all edges of P (note that neither $(y, n_1(x''))$ nor $(n_2(x), n_2(x''))$ lies in P). The construction can be visualised as in Fig. 6. There is an analogous construction for Case 4.b except that instead of one cycle D we have two cycles D_1 and D_2, formed by composing the paths ρ_1 and ρ_1' and the paths ρ_2 and ρ_2', respectively. We build the cycle D' as we did before except that when building D' we ensure that all Hamiltonian cycles C_j, for $j \in \{2, 3, \ldots, k-1\}$, contain the edges of $\{(n_j(x'), n_j(x)), (n_j(x), n_j(x'')), (n_j(a), n_j(b))\}$. We join D_1 and D using the bridge $(n_1(a), n_2(a), n_2(b), n_1(b))$ and the resulting cycle to D_2 using the bridge $(y, n_2(x), n_2(x''), n_1(x''))$.

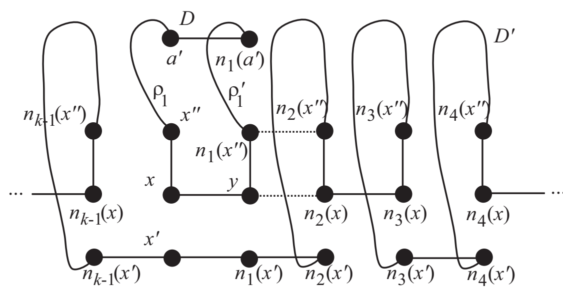

Figure 6: Case 2.b when k is even.

Suppose that k is odd. Let ρ_1'' be the isomorphic copy of ρ_1 in Q_2. For every $j \in \{3, 4, \ldots, k-2\}$, we use Lemma 1 to obtain a Hamiltonian path π_j in Q_j from $n_j(a')$ to $n_j(x)$. We use Lemma 1 to obtain a Hamiltonian path π_{k-1} in Q_{k-1} from $n_{k-1}(x)$ to $n_{k-1}(x')$. We build the cycle D' by starting from the path $(x', n_1(x'), n_2(x'))$, concatenating the edge $(n_2(x'), n_2(x))$, concatenating the path ρ_1'', concatenating the edge $(n_2(a'), n_3(a'))$, concatenating the path π_3, concatenating the edge $(n_3(x), n_4(x))$, concatenating the path π_4, concatenating the edge $(n_4(a'), n_5(a'))$, concatenating the path π_5, ..., concatenating the edge $(n_{k-2}(x), n_{k-1}(x))$, concatenating the path π_{k-1} and concatenating the edge $(n_{k-1}(x'), x')$. The cycles D and D' span all vertices of Q_n^k. Join D and D' using the bridge $(y, n_2(x), n_2(x''), n_1(x''))$ to obtain a Hamiltonian cycle of Q_n^k containing all edges of P (note that neither $(y, n_1(x''))$ nor $(n_2(x), n_2(x''))$ lies in P). There is an analogous construction for Case 4.b except that instead of one cycle D we have two cycles D_1 and

D_2, formed by composing the paths ρ_1 and ρ_1' and the paths ρ_2 and ρ_2', respectively. We build the cycle D' as we did before except that when building D' we ensure that: the Hamiltonian path π_2 in Q_2 is the sub-path of length $k^{n-1} - 1$ of the isomorphic copy of C_0 in Q_2 from $n_2(x')$ to $n_2(b)$; the Hamiltonian paths π_j in Q_j are from $n_j(b)$ to $n_j(x)$, for $j \in \{3, 4, \ldots, k-2\}$; and the Hamiltonian path π_{k-1} in Q_{k-1} is from $n_{k-1}(x)$ to $n_{k-1}(x')$. We join D_1 and D' using the bridge $(n_1(a), n_2(a), n_2(a'), n_1(a'))$, and we join the resulting cycle with D_2 using the bridge $(y, n_2(x), n_2(x''), n_1(x''))$ to obtain a Hamiltonian cycle of Q_n^k containing all edges of P. The construction can be visualised as in Fig. 7.

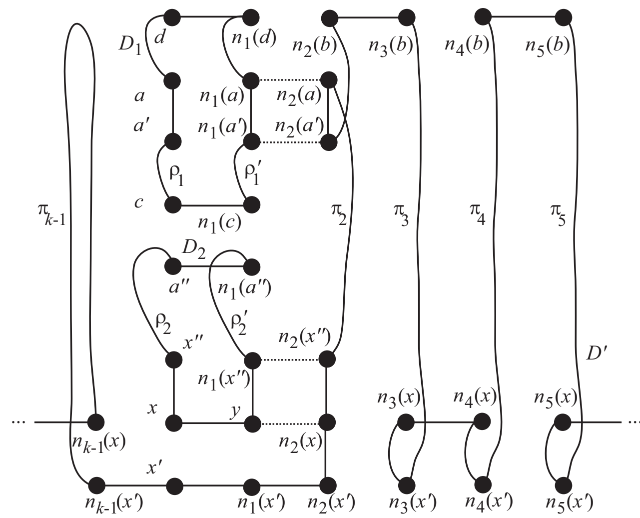

Figure 7: Case 4.b when k is odd.

Finally, consider Case 4.a. Let ρ_1', ρ_2' and ρ_3' be the isomorphic copies of ρ_1, ρ_2 and ρ_3, respectively, in Q_1. Join corresponding pairs (as we have done earlier) to form three cycles D_1, D_2 and D_3, respectively, which span all vertices of Q_0 and Q_1. Choose an edge f_i of D_i that lies in Q_1 and let f_i' be the isomorphic copy in Q_2, for $i = 1, 2, 3$. Apply the induction hypothesis to $(\{f_1', f_2', f_3'\}, Q_2)$ to obtain a Hamiltonian cycle D in Q_2 containing f_1', f_2' and f_3'. Join D to D_1, D_2 and D_3 using the corresponding bridge to obtain a cycle D' spanning all vertices of Q_1, Q_2 and Q_3 and containing all edges of P. The result follows by Lemma 6. □

Lemma 11 Suppose that x is a vertex of Q_i and y is a vertex of Q_{i+1} where $i \neq 0 \neq i+1$. Suppose that $|P_0| = 2n - 2$. There is a Hamiltonian cycle in Q_n^k containing every edge of P.

Proof Let (p, q) be some edge of P_0. By the induction hypothesis applied to $(P \setminus \{(p, q)\}, Q_0)$, there is a Hamiltonian cycle C_0 in Q_0 containing every edge of $P_0 \setminus \{(p, q)\}$. If (p, q) lies in C_0 then let D be the cycle C_0.

Suppose that (p, q) does not lie on C_0. There are two possibilities: we have a Hamiltonian path ρ_1 in Q_0 (a sub-path of C_0) containing all edges of P_0; or we have two vertex-disjoint (non-trivial) paths ρ_1 and ρ_2 in Q_0 (sub-paths of C_0) which span all vertices of Q_0 and contain all edges of P_0 (see the diagrams in Case 1 and Case 2.a in Fig. 5). In the first case, let ρ'_1 be the isomorphic copy of ρ_1 in Q_1 and let D be the cycle spanning all vertices of Q_0 and Q_1 obtained by joining ρ_1 and ρ'_1. In the second case, let ρ'_1 and ρ'_2 be the isomorphic copies of ρ_1 and ρ_2, respectively, in Q_1, and let D_1 and D_2 be the cycles obtained by joining ρ_1 and ρ'_1 and by joining ρ_2 and ρ'_2, respectively, so that the cycles D_1 and D_2 span the vertices of Q_0 and Q_1. Now choose some edge f_1 of D_1 that lies in Q_1 and some edge f_2 of D_2 that lies in Q_1, ensuring that f_1 is incident with x if x lies in Q_1. Let f'_1 and f'_2 be the isomorphic copies of f_1 and f_2, respectively, in Q_2. By the induction hypothesis applied to $(\{f'_1, f'_2\}, Q_2)$, there is a Hamiltonian cycle C_2 in Q_2 containing f'_1 and f'_2. Join D_1 and D_2 to C_2 using the bridges involving f_1 and f'_1 and f_2 and f'_2, respectively, to obtain a cycle D.

Whatever the situation, we have a cycle D that contains every edge of P_0 (and possibly the remaining edge of P). We iteratively work through the remaining k-ary $(n-1)$-cubes not yet spanned by the cycle D and, using the induction hypothesis, extend D so that we ensure that the edge (x, y) appears in the extension of D (we do this as we did in the last paragraph by always choosing the bridge by which we extend so that it contains (x, y)). The result follows. □

Suppose that $|P_0| = 2n-1$. By Lemmas 10 and 11, there is a Hamiltonian cycle in Q^k_n containing all edges of P.

Case (b): $P = \cup_{i=0}^{k-1} P_i$.

Suppose that $|P_0| \leq 2n-3$. By the induction hypothesis applied to (P_0, Q_0), there is a Hamiltonian cycle C_0 in Q_0 containing every edge of P_0. The result follows by Lemma 6.

Suppose that $|P_0| = 2n-2$. Let (p, q) be some edge of P_0. By the induction hypothesis applied to $(P_0 \setminus \{(p, q)\}, Q_0)$, there is a Hamiltonian cycle C_0 of Q_0 containing every edge of $P_0 \setminus \{(p, q)\}$. If (p, q) lies on C_0 then set $D = C_0$.

Suppose that (p, q) does not lie on C_0. There are two possibilities: we have a Hamiltonian path ρ_1 in Q_0 (a sub-path of C_0) containing all edges of P_0; or we have two vertex-disjoint (non-trivial) paths ρ_1 and ρ_2 in Q_0 (sub-paths of C_0) which span all vertices of Q_0 and contain all edges of P_0. In the first case, w.l.o.g. we may assume that $P_1 = \emptyset$ (otherwise work in Q_{k-1}). Let ρ_1' be the isomorphic copy of ρ_1 in Q_1. Let D be the cycle spanning all vertices of Q_0 and Q_1 obtained by joining ρ_1 and ρ_1'. In the second case, w.l.o.g. we may assume that $P_1 = \emptyset$. Let ρ_1' and ρ_2' be the isomorphic copies of ρ_1 and ρ_2, respectively, in Q_1. Let D_1 and D_2 be the cycles obtained by joining ρ_1 and ρ_1' and ρ_2 and ρ_2', respectively. Again, w.l.o.g. we may assume that $P_2 = \emptyset$. Choose edges f_1 and f_2 in Q_1 that lie in D_1 and D_2, respectively, and let f_1' and f_2' be the isomorphic copies of f_1 and f_2 in Q_2. By the induction hypothesis applied to $(\{f_1', f_2'\}, Q_2)$, there is a Hamiltonian cycle C_2 in Q_2 containing f_1' and f_2'. Join D_1, D_2 and C_2 using the bridges involving f_1 and f_1' and f_2 and $f2'$ to obtain the cycle D.

Whatever the situation, we obtain the result using Lemma 6.

Suppose that $|P_0| = 2n - 1$. Let e and f be two edges of P_0. Applying the induction hypothesis to $(P_0 \setminus \{e, f\}, Q_0)$ yields a Hamlitonian cycle C_0 of Q_0 containing every edge of $P_0 \setminus \{e, f\}$. Suppose that C_0 contains at least one of e and f also. Now we proceed exactly as we did in the case above when $|P_0| = 2n - 2$ and the edge (p, q) does not lie on (the previous cycle) C_0. Doing so, and then applying Lemma 6, yields the result. Hence, we may assume that both e and f do not appear in C_0. Consider e. Suppose that there is an edge of P_0 lying on C_0 and incident with e. Let e' and e'' be the edges of the maximal path ρ' of $\langle P_0 \rangle$ containing e that are incident with the terminal vertices of ρ'. Reapply the induction hypothesis to $(P_0 \setminus \{e', e''\}, Q_0)$ to obtain a Hamiltonian cycle C_0' of Q_0 containing every edge of $P_0 \setminus \{e', e''\}$. As above, we may assume that neither e' nor e'' lies on C_0'. Let ρ be the sub-path of C_0' joining the terminal vertices of ρ' and which contains no other vertex of ρ'. Let the terminal vertex of ρ' incident with e' (resp. e'') be c' (resp. c''), and let d' (resp. d'') be the other vertex incident with e' (resp. e''). W.l.o.g. we may assume that there is a sub-path of C_0' from c' to d' on which neither c'' nor d'' appears (in the alternative situation we proceed almost identically). There are 3 cases: $|\rho| = 1$; $|\rho| = 2$; and $|\rho| > 2$. These sub-cases can be visualised as in Fig. 8 (note that the sub-path $\rho' \setminus \{e', e''\}$ might consist of one vertex only). Note that none of the edges of C_0' incident with c' or c'' are in P_0 and that all edges of ρ', apart from e' and e'', lie on C_0'. Let a (resp. b) be the vertex of the sub-path of C_0' from c' to d' (resp. c'' to

d'') and avoiding c'' (resp. c') that is adjacent to d'. In all cases, let ρ_1 be the path starting with the sub-path of C_0' from a to c' avoiding b, concatenated with the path ρ', and concatenated with the sub-path of C_0' from c'' to b avoiding a. If $|\rho| = 2$ then let x be the solitary internal vertex of ρ, and if $|\rho| > 2$ then let the vertex of ρ adjacent to c' (resp. c'') be x (resp. y). We now proceed essentially as we did in Case 2.c, Case 2.b and Case 1 of Lemma 10, as appropriate, to obtain the result.

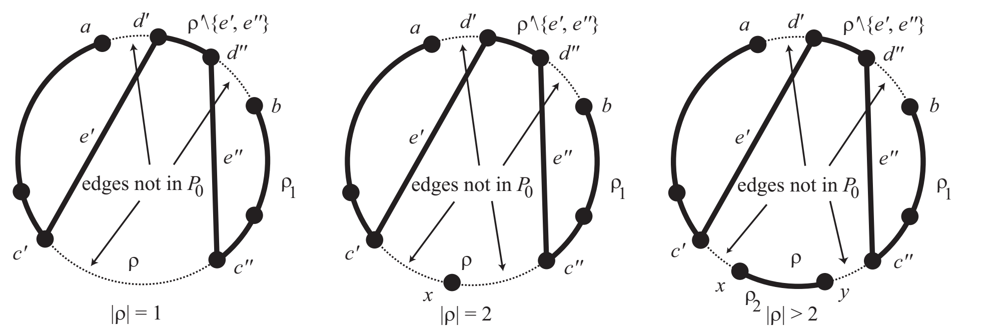

Figure 8: The case when $|P_0| = 2n - 1$.

Hence, we may assume that we have a Hamiltonian cycle C_0 of Q_0 containing all edges of $P_0 \setminus \{e, f\}$ so that the edges e and f do not lie on C_0 and are such that neither e nor f is incident with an edge of C_0 lying on P_0. We proceed as above depending upon which of the situations as in Fig. 8 occurs. The result follows. □

5 Conclusions

In this paper, we have shown that we can build a Hamiltonian cycle in a k-ary n-cube so that up to $2n-1$ prescribed edges can be guaranteed to be on the cycle if, and only if, these edges induce a subgraph consisting of a vertex-disjoint collection of paths. A simple induction shows that we can select a set P of $4n-2$ edges in Q_n^k, where $n \geq 2$ and $k \geq 3$, so that $\langle P \rangle$ consists of a set of vertex-disjoint paths and there exists a vertex x of Q_n^k so that all but 1 of x's neighbours in Q_n^k are incident with exactly 2 edges of P. Thus, the maximal size of a set P of edges of Q_n^k for which a version of Theorem 3 or Theorem 4 holds is at most $4n-3$. It would be interesting to establish exactly where between $2n-1$ and $4n-3$ this threshold lies. We expect that given the more complex structure of the k-ary n-cube, the exact threshold will be much more difficult to obtain than it was for the hypercube.

As we mentioned earlier, there has been a significant amount of research undertaken as regards the necessity of the existence of Hamiltonian cycles in hypercubes and k-ary n-cubes either avoiding or containing prescribed sets of edges of a given size. The general question of given a set of edges of a hypercube or a k-ary n-cube (with no bound on the size of the set), does there exist a Hamiltonian cycle containing these edges, has yet to be considered as regards its computational complexity. It could well be that the proof of related complexity-theoretic results from [1, 4] will provide an entry point into such an investigation.

Finally, hypercubes and k-ary n-cubes are not the only Hamiltonian graphs used as interconnection networks. There are many other such Hamiltonian graphs including, for example, star graphs, (n, k)-star graphs, pancake graphs, crossed cubes, twisted cubes and Möbius cubes (see, for example, [6]). It would be interesting to consider some of these graphs as to the existence of Hamiltonian cycles through prescribed edges as we have done here.

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