On the fixed-parameter tractability of parameterized model-checking problems

Iain A. Stewart

Department of Computer Science, University of Durham,
Science Labs, South Road, Durham DH1 3LE, U.K.

Abstract

In this note, we show, through the use of examples, how generic results for proving fixed-parameter tractability which apply to restricted classes of structures can sometimes be more widely applied.

Key words: computational complexity, analysis of algorithms, fixed-parameter tractability
1991 MSC: 68Q15

1 Introduction

Fixed-parameter tractability has emerged in recent years as an alternative to the traditional notion of tractability, namely that of solvability in polynomial-time. It is a more general notion of tractability, in that even some **NP**-complete problems are fixed-parameter tractable, yet the notion remains a realistic one in an empirical sense. A structural parameterized complexity theory now exists, and there is a considerable ongoing effort to both classify the (parameterized) complexity of problems, with much research as to whether specific problems are fixed-parameter tractable or not, and also to develop increasingly better algorithms for specific problems (especially those that are fixed-parameter tractable). The reader is referred to [3,5,8] for a thorough overview of the current status of parameterized complexity theory and fixed-parameter algorithms.

Email address: `i.a.stewart@durham.ac.uk` (Iain A. Stewart).
URL: `http://www.durham.ac.uk/i.a.stewart` (Iain A. Stewart).

In this note we are concerned with generic techniques for proving problems to be fixed-parameter tractable. We have in mind results such as the following.

Theorem 1 ([6]) *Let C be a polynomial-time decidable class of structures of effectively bounded local tree-width. There is a computable function f and an algorithm α such that, given $\mathcal{A} \in C$ and a first-order sentence $\varphi \in FO$, α decides whether $\mathcal{A} \models \varphi$ in time $O(||\mathcal{A}||) + f(|\varphi|) \cdot |\mathcal{A}|^2$.*

(Here, $||\mathcal{A}||$ is essentially the length of the encoding of $\mathcal{A}$ as a string of symbols [5, p. 74].)

Theorem 2 ([4]) *Let C be a non-trivial class of graphs that is minor closed. There is a computable function f, a polynomial p and an algorithm α such that, given $\mathcal{A} \in C$ and a first-order sentence $\varphi \in FO$, α decides whether $\mathcal{A} \models \varphi$ in time $f(|\varphi|) \cdot p(|\mathcal{A}|)$.*

Theorem 3 ([1]) *Let C be a class of graphs locally excluding a minor. There is a computable function f, a polynomial p and an algorithm α such that, given $\mathcal{A} \in C$ and a first-order sentence $\varphi \in FO$, α decides whether $\mathcal{A} \models \varphi$ in time $f(|\varphi|) \cdot p(|\mathcal{A}|)$.*

Such results are only applicable to restricted classes of structures yet are still very powerful. For example, the class of structures of degree at most d, for some fixed d, has effectively bounded local tree-width, as does the class of planar graphs (see [5, p. 314]); consequently, any first-order definable parameterized problem on these classes of structures is necessarily fixed-parameter tractable (when parameterized by the length of the input formula). We have two remarks. First, there are other generic results, such as Courcelle's Theorem [2] which states that there is a computable function f such that given a structure $\mathcal{A}$ and a sentence φ of monadic second-order logic, we can decide whether $\mathcal{A} \models \varphi$ in time $f(|\varphi|, \mathrm{tw}(\mathcal{A})) \cdot |\mathcal{A}| + O(||\mathcal{A}||)$, where $\mathrm{tw}(\mathcal{A})$ is the tree-width of $\mathcal{A}$. However, (results similar to) Courcelle's Theorem are too limiting (in comparison to Theorems 1, 2 and 3); for example, there are planar grids of arbitrary-size tree-width. Second, there are generic results in traditional complexity theory that automatically yield tractability bounds on classes of problems, one of which is Seese's result that any first-order definable problem on a class of structures of bounded degree is solvable in linear time [9].

However, it turns out that (a theorem derived from the proof of) Theorem 1 can sometimes be applied to show that a specific parameterized problem whose inputs come from a class of *arbitrary* structures is fixed-parameter tractable (where the applicability of the theorem depends upon the problem in hand). This note simply remarks upon this observation and is intended to act as a stimulus to further research into generalizing the application of results such as Theorems 1, 2 and 3.

2 An example

Henceforth, we use [5] as our standard reference text for (proofs of) existing results of interest to us in this paper. We motivate and explain our ideas with an example. Consider deciding whether the following problem *k-Almost s-Regular Graph*, ARG(s), where $s \geq 0$, is fixed-parameter tractable.

Input: Positive integer k, graph $G = (V, E)$.
Parameter: k.
Question: Does there exist a set $S \subseteq V$ of size at most k such that the subgraph of G induced by the vertices of $V \setminus S$ is regular of degree s?

This problem has recently been shown to be fixed-parameter tractable [7] using combinatorial techniques (the algorithm in [7] runs in time $O(n(k + s) + ks^2(k+s)^2(s+2)^k)$), but ideally we would like to use results such as Theorems 1, 2 and 3.

Note that the problem ARG(s) can be defined in first-order logic, as the following sentence χ_k^s witnesses:

$$\begin{aligned}
&\exists x_1 \exists x_2 \dots \exists x_k \forall y (\bigwedge_{1 \leq i \leq k} (x_i \neq y) \Rightarrow \\
&\ \exists z_1 \exists z_2 \dots \exists z_s (\bigwedge_{1 \leq i < j \leq s} (z_i \neq z_j) \wedge \bigwedge_{1 \leq i \leq k} \bigwedge_{1 \leq j \leq s} (x_i \neq z_j) \\
&\ \wedge \bigwedge_{1 \leq j \leq s} (E(y, z_j) \vee E(z_j, y)) \\
&\ \wedge \forall w ((E(y, w) \vee E(w, y)) \Rightarrow \bigvee_{1 \leq j \leq s} (w = z_j)))).
\end{aligned}$$

Given any input (G, k) to ARG(s), we can construct the pair (G, χ_k^s) in $O(n^2 + ks)$ time, where n is the number of vertices of G (note that $|\chi_k^s| = O(ks(\log(k) + \log(s)))$. However, we cannot apply any of Theorems 1, 2 or 3 to show that ARG(s) is fixed-parameter tractable as the inputs of ARG(s) potentially involve graphs of unbounded local treewidth, come from a class that is not minor-closed or come from a class that does not locally exclude a minor.

The problem ARG(s) can be reduced (as we explain below) to a similar problem ARG$'$(s) where the degrees of the vertices of any input graph are bounded in terms of (a function of) the parameter. The problem ARG$'$(s) is defined as follows.

Input: Positive integer k, graph $G = (V, E)$ for which every vertex of V

has degree no less than k and no more than $k + s$.

Parameter: k.

Question: Does there exist a set $S \subseteq V$ of size at most k such that the subgraph of G induced by the vertices of $V \setminus S$ is regular of degree s?

The following algorithm reduces ARG(s) to ARG$'$(s).

```
while k ≥ 0 and there exists a vertex v ∈ V of degree in G
> k + s or < k do
  remove v from V;
  set G to be the subgraph of (the old) G induced by V;
  k := k − 1;
od
if k ≥ 0 then return (G, k)
else return (K_s, 1) fi
```

Clearly, the above algorithm is a parameterized reduction from ARG(s) to ARG$'$(s) (running in time $O(kn^2)$; note that any vertex of G of degree more than $k + s$ or less than s must necessarily be in the chosen subset S of V). Thus, ARG(s) is fixed-parameter tractable if, and only if, ARG$'$(s) is fixed-parameter tractable, and we may as well work with the problem ARG$'$(s).

Nevertheless, we still cannot apply Theorem 1 as in order to do so, we require that the local tree-width of any input graph is explicitly bounded and this is not the case for the input graphs of ARG$'$(s). However, a simple observation opens up our problem to application. Let FIN be the class of finite structures and let FO be first-order logic. Furthermore, let $\mathrm{tw}(\mathcal{A})$ be the treewidth of any structure and let $N_r^{\mathcal{A}}(a)$ be the r-neighbourhood of the element $a \in |\mathcal{A}|$ in the Gaifman graph of $\mathcal{A}$ (again, see [5, p. 314]).

Definition 4 *Let $C \subseteq FIN \times FO$. We say that C has* effective p-bounded local tree-width *if there is a computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every $r \in \mathbb{N}$, $ltw(\mathcal{A}, r) \leq f(r, |\varphi|)$, for every $(\mathcal{A}, \varphi) \in C$, where $ltw(\mathcal{A}, r) = \max\{tw(N_r^{\mathcal{A}}(a) : a \in |\mathcal{A}|\}$.*

The definition of effective local bounded tree-width in [5] is as in Definition 4 except that the computable function f has domain $\mathbb{N}$ and for each $r \in \mathbb{N}$, $\mathrm{ltw}(\mathcal{A}, r) \leq f(r)$.

The proof of the following result is identical to the proof of Theorem 12.22 of [5] (and so there is no need to include the proof here).

Theorem 5 *Let $C \subseteq FIN \times FO$ have effective p-bounded local tree-width. Consider the parameterized problem Ω defined as follows.*

Input: $(\mathcal{A}, \varphi) \in C$.
Parameter: $|\varphi|$.
Question: Is it the case that $\mathcal{A} \models \varphi$*?*

There is a computable function f, a polynomial p and an algorithm α such that, given $(\mathcal{A}, \varphi) \in C$, α decides whether $\mathcal{A} \models \varphi$ in time $O(||\mathcal{A}||) + f(|\varphi|) \cdot |\mathcal{A}|^2$.

We can now apply Theorem 5.

Corollary 6 *For every $s \geq 0$, the decision problem ARG(s) is fixed-parameter tractable.*

PROOF. As there is a parameterized reduction from ARG(s) to ARG$'(s)$, it suffices to show that ARG$'(s)$ is fixed-parameter tractable. Let (G, k) be an input to ARG$'(s)$; in particular, every vertex of G has degree at most $k + s$. Define

$$C = \{(G, \chi_k^s) : (G, k) \text{ is an input to ARG}'(s)\}.$$

As remarked earlier, this reduction can be completed in $O(n^2 + ks)$ time, when the graph G has n vertices. For every $(G, \chi_k^s) \in C$ and for every $r \in \mathbb{N}$, $\mathrm{ltw}(G, r) \leq (k + s)^r \leq (|\chi_k^s|)^r$, and so C has effectively p-bounded local tree-width. By Theorem 5, ARG$'(s)$ is fixed-parameter tractable and the result follows. $\square$

Note that complexity of the resulting algorithm for ARG(s) is $O(f(k) \cdot n^2)$, which (as is to be expected) is not as good as the algorithm in [7], in terms of neither n nor the function corresponding to k (note that no bound is given for this function in Corollary 6). Nevertheless, if all one is interested in is whether a problem is fixed-parameter tractable or not then applying our generic technique is much easier than the approach employed in [7].

The proof in [7] that ARG(s) is fixed-parameter tractable actually also shows that the following problem, *k-Almost Regular Graph*, ARG, is fixed-parameter tractable, though this is not stated explicitly in [7].

Input: Positive integers k and s, graph $G = (V, E)$.
Parameter: $k + s$.
Question: Does there exist a set $S \subseteq V$ of size at most k such that the subgraph of G induced by the vertices of $V \setminus S$ is regular of degree s?

An identical argument to that above yields the following result.

Corollary 7 *The decision problem ARG is fixed-parameter tractable.*

We end with a remark regarding Courcelle's Theorem, stated earlier. As can be seen from Section 11.6 of [5], there exists a technique called *tree-width reduction* where, essentially, one reduces an instance to an instance of bounded tree-width and then applies Courcelle's Theorem. This technique has similarities with ours, though our technique is more powerful in that it enables us to show fixed-parameter tractability by reducing to classes of structures which do not have bounded tree-width.

3 Conclusion

In this note we have demonstrated, with a simple example, the wider applicability of generic results such as Theorems 1, 2 and 3. We hope that this note acts as an impetus for further research into the applicability of results such as these. For example, it would be interesting to derive properties of defining formulae (such as χ_s^k) which necessarily make a problem amenable to attacks such as that demonstrated here.

Acknowledgement. We are indebted to Stefan Szeider for useful comments on an earlier version of this paper, and to two referees for their very useful comments.

References

[1] A. Dawar, M. Grohe and S. Kreutzer, Locally excluding a minor, *Proc. of 22nd IEEE Ann. Symp. on Logic in Computer Science (LICS)*, IEEE Press (2007) 270–279.

[2] B. Courcelle, Graph rewriting: an algebraic and logic approach, *Handbook of Theoretical Computer Science, Volume B* (ed. J. van Leeuwen), Elsevier (1990) 194–242.

[3] R.G. Downey and M.R. Fellows, *Parameterized Complexity*, Springer-Verlag (1999).

[4] J. Flum and M. Grohe, Fixed-parameter tractbility, definability, and model checking, *SIAM Journal on Computing* **31** (2005) 113–145.

[5] J. Flum and M. Grohe, *Parameterized Complexity Theory*, Springer-Verlag (2006).

[6] M. Frick and M. Grohe, Deciding first-order properties of locally tree-decomposable structures, *Journal of the ACM* **48** (2001) 1184–1206.

[7] H. Moser and D.M. Thilikos, *Proc. of 2nd Algorithms and Complexity in Durham Workshop (ACiD)*, Texts in Algorithmics Vol. 7, King's College (2006) 107–118.

[8] R. Niedermeier, *Invitation to Fixed-Parameter Algorithms*, Oxford University Press (2006).

[9] D. Seese, Linear-time computable problems and first-order descriptions, *Mathematical Structures in Computer Science* **6** (1996) 505–526.