Some very weak height 1 identities

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Bulatov-Zhuk

Theorem (Bulatov-Zhuk, [Bulatov, 17], [Zhuk, 17], [Siggers, 10], et al.**)**
Let $\mathbb{B}$ be a finite structure. Exactly one of the following holds:

1. *There exists a minor-preserving map* $\mathrm{Pol}(\mathbb{B}) \to \mathscr{P}$*, and* $\mathrm{CSP}(\mathbb{B})$ *is NP-complete,*
2. $\mathrm{Pol}(\mathbb{B})$ *contains a function s satisfying the identity*

$$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y),$$

   and $\mathrm{CSP}(\mathbb{B})$ *is in P.*

In particular, $\mathrm{CSP}(\mathbb{B})$ *is in P or NP-complete.*

($\mathscr{P}$ is the clone of projections on a two-element set.)

The question

Conjecture (Bodirsky-Pinsker; [Barto, __, Pinsker, 18])

Let $\mathbb{B}$ be a reduct of a finitely bounded homogeneous structure. Exactly one of the following holds:

1. *There exists a uniformly continuous minor-preserving map from* $\mathrm{Pol}(\mathbb{B})$ *to* $\mathscr{P}$*, and* $\mathrm{CSP}(\mathbb{B})$ *is NP-complete,*
2. $\mathrm{Pol}(\mathbb{B})$ *does not have a uniformly continuous minor-preserving map to* $\mathscr{P}$*, and* $\mathrm{CSP}(\mathbb{B})$ *is in P.*

Can the non-existence of a minor-preserving map to $\mathscr{P}$ be replaced by a statement positing that some fixed set of identities holds in $\mathrm{Pol}(\mathbb{B})$?

Overview

1. Some very weak height 1 identities
2. There is no weakest height 1 condition (Manuel Bodirsky)
3. Topology is relevant (Antoine Mottet)

Height 1 identities

Height 1 identity is an identity of the form

$$f(x_{\pi(1)}, \dots, x_{\pi(n)}) \approx g(x_{\sigma(1)}, \dots, x_{\sigma(m)}).$$

Height 1 condition is a finite set of height 1 identities over a finite algebraic language.

The function symbols are considered to be variables.

Generaly, a finite system of identities in some algebraic language is called a strong Maltsev condition.

Some very weak identities

[Tay88] Walter Taylor. *Some very weak identities.* Algebra Universalis, 25(1):27–35, Dec 1988.

Is there a weakest strong Maltsev condition?

- No! [García, Taylor, 84], [Taylor, 88]

There is a weakest idempotent str. Maltsev condition. [Olšák, 2017]

Is there a weakest height 1 condition?

- Yes, for finite algebras. [Siggers, 10]

$$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y)$$

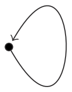
$\mathbb{G}$-conditions

Start with an undirected graph $\mathbb{G} = (V, E)$, construct a height 1 condition $\Sigma_{\mathbb{G}}$ in the following way:

1. for each $v \in V$, introduce a ternary symbol f_v, and
2. for each edge $(u, v) \in E$, introduce a 6-ary symbol $g_{(u,v)}$ and add to $\Sigma_{\mathbb{G}}$ the identities

$$f_u(x, y, z) \approx g_{(u,v)}(x, y, x, z, y, z)$$
$$f_v(x, y, z) \approx g_{(u,v)}(y, x, z, x, z, y).$$

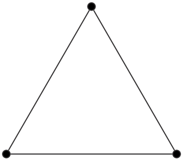
Examples



Siggers!

$$f(x, y, z) \approx g(x, y, x, z, y, z)$$
$$f(x, y, z) \approx g(y, x, z, x, z, y).$$

Examples



Trivial! Take $f_i(x_1, x_2, x_3) = x_i$, and extend to $g_{(i,j)}$'s.

$$y \approx g_{(2,3)}(x, y, x, z, y, z)$$
$$z \approx g_{(2,3)}(y, x, z, x, z, y).$$

Observations

$\Sigma_{\mathbb{G}}$ essentially means

'If a compatible graph contains $\mathbb{K}_3$ then it contains a homomorphic image of $\mathbb{G}$.'

Lemma

Let $\mathbb{G}$ and $\mathbb{H}$ be two finite graphs.

- *If $\mathbb{G}$ maps homomorphically into $\mathbb{H}$ then $\Sigma_{\mathbb{H}}$ implies $\Sigma_{\mathbb{G}}$.*
- *$\Sigma_{\mathbb{G}}$ is trivial if and only if $\mathbb{G}$ is 3-colorable.*

Note

$\mathbb{G}$-conditions can be generalized to $\mathbb{H}, \mathbb{G}$-conditions. Meaning:

$$\mathbb{H} \Rightarrow \mathbb{G}$$

- $\Sigma_{\mathbb{G}}$ is the $\mathbb{K}_3, \mathbb{G}$-condition.
- If $\mathbb{G}$ is a loop, we get loop conditions.
- These conditions are inherent in the algebraic CSP reductions. [Bulín, Krokhin, __, 19]

As good as it gets

Lemma

For any non-trivial height 1 condition Σ, there exists a graph $\mathbb{G}$ that is not 3-colorable and such that $\Sigma_{\mathbb{G}}$ is weaker than (or equivalent to) Σ.

Proof sketch.

- Observe that Σ is not satisfied in $\mathrm{Pol}(\mathbb{K}_3)$.
- This is witnessed by the fact that the indicator graph $\mathbb{G}$ of Σ in $\mathbb{K}_3$ does not map to $\mathbb{K}_3$.

$$V_{\mathbb{G}} = \{f(a_1, \dots, a_k) : a_i \in [3], f \in \mathrm{Lang}(\Sigma)\}/\Sigma$$
$$E_{\mathbb{G}} = \{(f(a_1, \dots, a_k), f(b_1, \dots, b_k)) : a_i \neq b_i\}$$

- Notice that Σ implies $\Sigma_{\mathbb{G}}$. ∎

Next in this session...

You will learn that none of the conditions $\Sigma_{\mathbb{G}}$ is the weakest.

■

Lemma

For any non-trivial height 1 condition Σ, there exists a graph $\mathbb{G}$ that is not 3-colorable and such that $\Sigma_{\mathbb{G}}$ is weaker than (or equivalent to) Σ.

Lemma

Let $\mathbb{G}$ be a finite graph. Then $\mathbb{G}$ maps homomorphically to any graph $\mathbb{H}$ that contains $\mathbb{K}_3$, and whose polymorphisms satisfy $\Sigma_{\mathbb{G}}$.