An integration scheme for electromagnetic scattering using plane wave edge elements

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\textbf{A B S T R A C T}

Finite element techniques for the simulation of electromagnetic wave propagation are, like all conventional element based approaches for wave problems, limited by the ability of the polynomial basis to capture the sinusoidal nature of the solution. The Partition of Unity Method (PUM) has recently been applied successfully, in finite and boundary element algorithms, to wave propagation. In this paper, we apply the PUM approach to the edge finite elements in the solution of Maxwell's equations. The electric field is expanded in a set of plane waves, the amplitudes of which become the unknowns, allowing each element to span a region containing multiple wavelengths. However, it is well known that, with PUM enrichment, the burden of computation shifts from the solver to the evaluation of oscillatory integrals during matrix assembly. A full electromagnetic scattering problem is not simulated or solved in this paper. This paper is an addition to the work of Ledger and concentrates on efficient methods of evaluating the oscillatory integrals that arise. A semi-analytical scheme of the Filon character is presented.

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\section{1. Introduction}

The problem of computing the electromagnetic field scattered by a body when subjected to an incident wave has practical importance in radar cross section prediction. There are several numerical methods that have been proposed for the simulation of these fields, for example the discontinuous Galerkin method \cite{1} is attracting much attention. Node based finite elements impose a continuity of all components of the field across inter-element boundaries which is not a necessary property of the field. Using such node based finite elements may lead to the occurrence of spurious modes in the numerical solutions. Edge elements, first introduced by Nédélec \cite{2–4}, assign degrees of freedom to the edges rather than to the nodes of the elements and have the property that they ensure the continuity of the tangential component of the field across inter-element boundaries while allowing for jumps in the normal component of the field. Edge elements are free of spurious modes.

For the lowest order edge element, the tangential component of the solution is constant on each edge. Consequently the accuracy achieved by this element is low, leading to a requirement for very dense meshes. Compatible, arbitrary order, quadrilateral and triangular edge elements have been developed by Demkowicz and Rachowicz \cite{5} and Ainsworth and Coyle \cite{6}; the latter have been shown to have better conditioning properties and thus will be used here.

The partition of unity method (PUM) developed by Melenk and Babuška \cite{7,8} is a general numerical approximation technique in which the approximation space is enriched by the inclusion of a set of analytical functions known to form a basis for the solution. The approach has been implemented in a finite element scheme under the heading of the Partition of Unity Finite Element Method (PUFEM). The motivation for the use of these new elements is to escape the limitations of conventional finite element procedures, which for wave problems impose an upper bound on nodal spacing of around 10% of the wavelength under consideration. This has led to a rule of thumb being adopted requiring a minimum of around 10 degrees of freedom per wavelength to be used in conventional finite element models. In PUFEM the field is expanded in a discrete series of plane waves, each propagating at a specified angle. These angles can be uniformly distributed or may be carefully chosen. This expansion allows each element to span many wavelengths.

For the simulation of wave phenomena, the PUFEM has been applied to a range of Helmholtz wave diffraction problems in the frequency domain. Laghrouche et al. \cite{9} showed that (when compared with conventional fine finite element meshes) the number of variables could be reduced by up to 96%. Mayer and Mandel \cite{10} presented a similar method with the name Finite Ray Element Method. Farhat et al. \cite{11–13} have proposed the Discontinuous Enrichment Method in which the standard finite element polynomial field is enriched by plane waves. Ortiz and Sanchez \cite{14} have

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developed a three-node wave finite element based on the partition of unity model. Strouboulis et al. [15] apply the approach on Cartesian meshes in the generalised finite element method and analyse the convergence properties.

Plane waves were used in the approximation of integral equations in electromagnetic scattering by de La Bourdonnaye [16] under the name of Microlocal Discretisation. Further experiences in implementing the PUM in boundary element approximations (PUBEM) are reported by Perrey-Debain et al. [17–19]. Trevelyan et al. [20] presented a successful adaptive PUBEM scheme in which wave directions are added in response to an error indicator. A particularly effective strategy for integral equation approximations can be found in the enrichment of the approximation space using plane wave basis comes at the price of some computationally intensive numerical integration over the elements. Since a single element might contain many wavelengths, the integrands to be oscillatory integrals that arise in the matrix entries.

The reduction in the number of active variables by using such a plane wave basis comes at the price of some computationally intensive numerical integration over the elements. Since a single element might contain many wavelengths, the integrands to be evaluated in order to determine terms in the governing matrices become highly oscillatory. It is well known that polynomial representations of trigonometrical functions are not accurate and are expensive to compute. Therefore, conventional Gauss–Legendre integration requires a very large number of integration points. Computational integration of oscillatory functions is currently an active area of research. Readers are referred to the semi-analytical integration rules of Bettess et al. [26], for example, and other works by Iserles et al. [27], Huybrechs and Vandewalle [28,29], Langdon and Chandler-wilde [30], Honnor and Trevelyan [31].

2. Mathematical model of electromagnetics

Electromagnetic phenomena are governed by Maxwell’s equations, which for harmonically oscillating functions with a single frequency, \( \omega \), lead to the vector wave equation. The vector wave equation can be expressed either in terms of electric field intensity, \( E \), or in terms of magnetic field intensity, \( H \). The vector wave equation for the electric field is [32,33]

\[
\nabla \times (\mu^{-1} \nabla \times E) - \varepsilon_0 \omega^2 E = -k_0 J
\]

where \( \mu \) is the permeability, \( \varepsilon \) is the permittivity, and \( J \) is the electric current density. The vector wave equation for the magnetic field is [32,33]

\[
\nabla \times (\varepsilon \nabla \times H) - \varepsilon_0 \omega^2 H = \nabla \times (\varepsilon^{-1} J)
\]

The vector wave equations (1) or (2), is used in conjunction with the continuity equation [32,33]

\[
\nabla \cdot J = -\mu \rho
\]

where \( \rho \) is the electric charge density. In this work we assume that the conductivity of the medium is negligible and the permittivity, \( \varepsilon \), and permeability, \( \mu \), are unity. This does not lead to any loss in generality since for homogeneous media they are scalar constants. For inhomogeneous media they become functions of position.

3. Geometry representation and basis functions

The edge element basis cannot be used to represent the geometry. Instead, traditional nodal methods must be used. Where the order of approximation of the field variable is increased and the mesh kept constant, the use of a linear sub-parametric discretisation can result in inaccurate solutions. In this work we use the linear blending function approach [34]. There is no requirement for additional nodes on boundary edges of the element and the curve of the boundary edges exactly follows the true curve of the boundary. In this section we describe the hierarchic compatible arbitrary order quadrilateral and triangular edge elements of Ainsworth and Coyle [6].

3.1. Quadrilateral edge element

The master quadrilateral element is shown in Fig. 1. For an element of order \( p \), the variation of the electric field over this element is given in the interpolated form

\[
\nU(\xi_1, \xi_2) = \sum_{i=1}^{4} \sum_{j=0}^{p} u_i^j \phi_i^j + \sum_{k=1}^{\text{p}} \sum_{i=1}^{4} \sum_{j=0}^{p} U^j_k \phi_i^j
\]

where \( U \) denotes either \( E \) for transverse electric polarisation or \( H \) for transverse magnetic polarisation, \( \phi \) are the vector shape functions and \( u \) are the unknowns. The basis functions associated with the element edges, \( \phi_i \), and interior basis functions, \( \phi_i^j \), are piecewise polynomials constructed from shape functions, edge vectors, Legendre polynomials and integrated Legendre polynomials as defined in [6].

3.2. Triangular edge element

The master triangular element is shown in Fig. 2. For an element of order \( p \), the variation of the electric field over this element is
where $\phi$ denotes the vector shape functions and $u$ are the unknowns. The basis functions associated with the element edges, $\phi^i$, pseudo-interior basis functions, $\phi^m$, and genuine interior basis functions, $\phi^k$, are piecewise polynomials constructed from shape functions, Legendre polynomials and Jacobi polynomials as defined in [6].

4. Plane wave edge elements

The electric field over the element, $U$, i.e. Eq. (4) for quadrilateral elements and Eq. (5) for triangular elements, may be expanded in a discrete series of planar waves, each propagating at a specified angle [24,25]. These angles can be uniformly distributed, as in this work, or may be carefully chosen. This leads to an expansion of the basis functions, thus

$$\phi_n = \phi^q e^{ik(x\cos\theta_n + y\sin\theta_n)}, \quad q = 1, 2, \ldots, m$$

(6)

where $\phi^q$ represents all the basis functions, $\phi^i_1, \phi^i_2, \phi^m_1, \phi^m_2, \phi^k_1, \phi^k_2$, $k$ is the wavenumber, $k = \alpha / \sqrt{\mu}$, $m$ is the number of plane wave directions and $(x,y)$ are the global coordinates. In this work the directions are evenly spaced such that

$$\theta_n = \frac{q \pi}{m}$$

(7)

The finite element discretisation for the scattering problem gives a matrix expression

$$[K - \omega^2 M] u = F$$

(8)

in which the unknowns $u_n$ forming the vector $u$ are now the amplitudes of the scattered plane waves. The terms of the stiffness matrix, $K$, the mass matrix, $M$, and the right hand side vector, $F$, are given by

$$K_{jk} = \int_\Omega \nabla \times \phi_j \cdot \nabla \times \phi_k \, d\Omega$$

(9)

$$M_{jk} = \int_\Omega \phi_j \cdot \phi_k \, d\Omega$$

(10)

$$F_j = -\int_\Gamma \phi_j \cdot (n \times U) \, d\Gamma$$

(11)

where the $F_j$ term appears only for those elements that have an edge that lies on the boundary $\Gamma$.

5. Numerical integration

The integrals in Eqs. (9)–(11) are highly oscillatory because of the presence of the exponential terms of imaginary argument, which are trigonometrical functions. Since the wavenumber, $k$, may be large, the exponential function may contain many wavelengths in a single element. So the reduction in the number of active variables through using plane waves comes at the price of some computationally intensive numerical integration. Previous research in PUFEM and PUBEM for the simulation of wave phenomena has resorted to very high order Gauss–Legendre integration. Table 1 shows the difference in the stiffness and mass norms for a single element for various numbers of Gauss–Legendre (G–L) integration points when compared to a reference (Ref) solution obtained by using 4096 x 4096 points for Gauss–Legendre numerical integration. The element is a unit square for the quadrilateral element, and half a unit square for the triangular element. The wavelength is 0.1, giving 10 wavelengths in the single element, using element order, $p = 0$, and there are 8 plane wave directions for each edge. The difference norm is defined by

$$\text{Difference norm} = \frac{\sum \left| \text{term}_{\text{cal}} - \text{term}_{\text{ref}} \right|}{\sum \left| \text{term}_{\text{ref}} \right|}$$

(12)

where term represents either the stiffness or mass term of the 32 by 32, quadrilateral element, or 24 by 24, triangular element, matrix.

This set of results suggests the optimum number of integration points for element order, $p = 0$, for Gauss–Legendre numerical integration to be about 6 per wavelength. It should be noted that the quadrilateral element exhibits much better convergence than the triangular element.

5.1. Semi-analytical integration

The integrals in Eqs. (9) and (10) are of the form

$$\text{Term} = e^{i\beta \xi_1 + \gamma \xi_2} \int_{\Omega} f(\xi_1, \xi_2) \exp (i\beta \xi_1 + i\gamma \xi_2) \, d\xi_1 \, d\xi_2$$

(13)

where $f$ is the Jacobian of the geometric mapping, $\alpha, \beta$ and $\gamma$ are real constants and $f(\xi_1, \xi_2)$ represents a polynomial in $\xi_1$ and $\xi_2$ that is assumed to be slowly varying over the element.

5.2. Quadrilaterals

For a quadrilateral element, the global Cartesian coordinates $x = (x,y)$ are related to the local element coordinates $(\xi_1,\xi_2)$ by

$$x(\xi_1, \xi_2) = a + b \xi_1 + c \xi_2 + d \xi_1 \xi_2$$

(14)

where the coefficients $a$, $b$, $c$, $d$ are vectors of dimension 2. The $\xi_1 \xi_2$ term complicates the evaluation of the integrals considerably and will not be considered. Here we are only concerned with parallelogram elements so that $d = (0,0)$. This implies that the Jacobian of

<table>
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<th>Number of points</th>
<th>Quadrilateral element</th>
<th>Triangular element</th>
</tr>
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<td>Mass</td>
<td>Stiffness</td>
</tr>
<tr>
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<td>0.2460</td>
<td>0.2650</td>
</tr>
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<tr>
<td>128</td>
<td>5.9951E-13</td>
<td>2.0683E-13</td>
</tr>
</tbody>
</table>

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the geometric mapping is constant. This essential simplifying assumption does not restrict the applicability of the current work excessively since one can control a mesh generator to produce predominantly parallelogram elements. The integral in Eq. (13) may be approximated using

$$\text{Term} = e^{i\theta} \sum_{n=0}^{N} \sum_{q=0}^{P} \int_{-1}^{1} \int_{-1}^{1} \exp(i\phi \xi_1) \times \exp(i\psi \xi_2) L_{pq}(\xi_1, \xi_2) d\xi_1 d\xi_2$$

(15)

where

$$f(\xi_1, \xi_2) = \sum_{n=0}^{N} \sum_{q=0}^{P} f(\xi_1, \xi_2) L_{pq}(\xi_1, \xi_2)$$

(16)

$L_{pq}(\xi_1, \xi_2)$ is an interpolating polynomial, which takes unit value at an integration point, and is zero at all the others. For parallelogram elements it can be constructed from the product of Lagrange polynomials, $L_{pq}(\xi_1, \xi_2) = L_p^L(\xi_1) L_q^R(\xi_2)$. In Eqs. (15) and (16) $n$ is the number of integration points in each direction, which depends on the order of $Q_{1,2}$, which in turn depends on the element order $p$. Given that $n$ points are required to integrate polynomials of order $2n - 1$ exactly, this leads to $n = 2(p + 1) + 1$ points in each direction; this is independent of the wavenumber. Thus quadrilateral elements of order $p = 0, 1, 2, 3$ require a total of 9, 25, 49, 81 Gauss points respectively to integrate exactly using this semi-analytical integration scheme regardless of how large the wavenumber may be. This leads to an integration rule of the following form:

$$\text{Term} = \|f\| e^{i\theta} \sum_{p=0}^{N} \sum_{q=0}^{P} f(\xi_1, \xi_2) w_{pq}(\beta, \gamma)$$

(17)

where

$$w_{pq}(\beta, \gamma) = \int_{-1}^{1} \int_{-1}^{1} \exp(i\phi \xi_1) \exp(i\psi \xi_2) L_{pq}(\xi_1, \xi_2) d\xi_1 d\xi_2$$

(18)

For the parallelogram, the evaluation of the integration weights is simplified because the integral of products becomes a product of integrals

$$w_{pq}(\beta, \gamma) = \int_{-1}^{1} \exp(i\phi \xi_1) L_{p}^L(\xi_1) d\xi_1 \int_{-1}^{1} \exp(i\psi \xi_2) L_{q}^R(\xi_2) d\xi_2$$

(19)

Both integrals can be evaluated analytically by repeated application of integration by parts. However, practical implementation problems are found when evaluating Eq. (19) when either $\beta$ or $\gamma$ are zero or are very small. Consider the integral

$$\int e^{i\phi \xi_1} d\xi_1 = -\frac{e^{i\phi \xi_1}}{\beta} + C$$

(20)

A difficulty arises when $\beta$ is zero or very small. In these cases it is necessary to adopt a series form for the exponential

$$e^{i\phi \xi_1} \approx \sum_{n=0}^{N} \left( \frac{\xi_1^n}{\beta^n} \right) i^n$$

(21)

where $N$ denotes the number of terms retained in the approximation, which is adjusted so that the truncation error is of the order of machine accuracy. This problem will be addressed again when we turn to integrations over a triangle. However, for parallelograms an elegant transformation both simplifies the integral and removes the singular behaviour when either $\beta$ or $\gamma$ are zero or are very small.

5.3. Shirron transformation for parallelogram elements

Without any loss of generality the integral (19) can be expressed in terms of Legendre polynomials of order $m$, $P_m(x)$, so that

$$w_{pq}(\beta, \gamma) = \int_{-1}^{1} \exp(i\phi \xi_1) P_{m}(\xi_1) d\xi_1 \int_{-1}^{1} \exp(i\psi \xi_2) P_{q}(\xi_2) d\xi_2$$

(22)

The property of spherical Bessel functions, $j_m$, is now used

$$\int_{-1}^{1} \exp(izx) P_m(x) dx = \sum_{n=0}^{\infty} (2n + 1) j_m(x) \int_{-1}^{1} P_m(x) P_n(x) dx$$

(23)

The above property follows from [35, p. 440, 10.1.47] and the fact that $\exp(1/2n!x^2)$, The Legendre polynomials are orthogonal over the interval $(-1, 1)$, i.e.

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{nm}$$

(24)

Thus Eq. (23) simplifies to

$$\int_{-1}^{1} \exp(izx) P_m(x) dx = 2 i^m j_m(x)$$

(25)

The final result for the weights for the rectangular domain is thus

$$w_{pq}(\beta, \gamma) = 4 i^{p+q} j_p(\beta) j_q(\gamma)$$

(26)

This is an extremely valuable result. The spherical Bessel functions can be evaluated as rapidly as any other trigonometrical functions and they exhibit none of the instabilities for small arguments which are encountered in dealing with the integrals of exponential functions. The expression (26) for the weights is used in the integration of the parallelogram element matrices. A necessary step, which is not described here, is to convert the results for the integration weights in terms of Legendre polynomials into those for the Lagrange polynomials, $L_p^L(\xi_1)$ and $L_q^R(\xi_2)$ (Eq. (19)) while interpolating through the sampling points. Since the Lagrange polynomials for any given set of integration abscissae can be expressed in terms of the Legendre polynomials this is a straightforward exercise.

5.4. Triangles

In carrying out the integrations over the triangular elements it is simpler to transform the integral over the triangle domain into integrals along the edges of the triangle and then into terms at the ends of each edge. This is done by using Green’s theorem. The starting point is the integral of Eq. (13), which is written in the form

$$\int_{\partial \Omega} \int \left[ \frac{p(\xi)}{w(\xi)} \exp(iw \cdot \xi) \right] d\Omega$$

(27)

where $\xi = (\xi_1, \xi_2)$ is the local coordinate system and $w$ is the local wavenumber vector ($w = s \hat{w}_1 + t \hat{w}_2 = s \hat{\beta} + t \gamma$, $s$ and $t$ being unit vectors in the $\xi_1$ and $\xi_2$ directions), and $p(\xi)$ is a polynomial function which corresponds to the product of the shape functions or their derivatives as defined in Eqs. (9), (10). $p(\xi)$ thus contains the term $\exp(iz\xi)$, $f(\xi_1, \xi_2)$ and $||z||$, of Eq. (13).1 Without any loss of generality $p(\xi)$ can be considered to be composed of terms of the form $\xi_1^m \xi_2^n$, where $l$ and $m$ are the powers of $\xi_1$ and $\xi_2$. The next step is to transform the integral in expression (27) into a closed form expression using the method of Gordon [36]. Gordon’s method was originally used to evaluate the double integrals of wave functions over arbitrary flat polygons in three dimensional space. In the current application it can be simplified. Gordon needed to set up a local coordinate system in the plane of the polygon in his theory, but that is not required here.

5.5. Gordon’s method

For convenience the method described by Gordon for finding integrals over a flat polygonal surface, $S$, is summarised here, with

1 It should be noted that $w$ is not the wavenumber of the overall problem, but is arrived at as a result of combining waves by taking their product in the element formulation.
some simplifications. The problem is to evaluate a surface integral of the form

$$\int \int \int \exp(iw \cdot \zeta) \, d\Omega$$  \hspace{1cm} (28)

Green’s theorem or the divergence theorem can now be applied to transform this double integral into a line integral evaluated around the boundary, \(\partial \Omega\), of the polygon \(\Omega\):

$$\int \int \exp(iw \cdot \zeta) \, d\zeta_1 \, d\zeta_2 = \frac{i}{|w|^2} \int \int \left\{ -w_1 \frac{\partial}{\partial \zeta_1} \left[ \exp(iw \cdot \zeta_1) - \exp(iw \cdot \zeta_2) \right] \right\} \, d\zeta_1 \, d\zeta_2$$  \hspace{1cm} (29)

$$= \frac{i}{|w|^2} \int \int e^{-i/2} (w_2 \cdot \zeta_2 - w_1 \cdot \zeta_1) = \sum_{n=1}^{N_{edges}} \frac{i}{|w|^2} T_n, \quad |w|^2 \neq 0$$  \hspace{1cm} (30)

where \(T_n\) is defined below. Now suppose that \(\partial \Omega\) is a polygon with \(N\) vertices whose coordinates are given by \(a_1, a_2, \ldots, a_N\) and let \(\zeta_n(t)\) be the parametric representation of the \(n\)th side of \(\partial \Omega\):

$$\zeta_n(t) = (1-t) a_n + t a_{n+1}, \quad a_{N+1} = a_1$$  \hspace{1cm} (31)

Thus the line integral in Eq. (30) is equivalent to

$$I_n = \int_{0}^{1} \exp[iw \cdot \zeta_n(t)] \, |w| \cdot \zeta_n(t) \, dt$$  \hspace{1cm} (32)

with \(w = (w_2 - w_1)\) obtained by rotating \(w\) through an angle of \(90^\circ\) clockwise and where \(\zeta_n(t)\) means the derivative of \(\zeta_n(t)\) with respect to \(t\). Then if \(\Delta a = a_{n+1} - a_n\) \((n = 1, N)\), the contribution of the \(n\)th side of \(\partial \Omega\) to the preceding interval is

$$I_n = \int_{0}^{1} \exp[iw \cdot \Delta a_n] \exp[(1-t) a_n + t a_{n+1}] \, dt$$  \hspace{1cm} (33)

$$= (w \cdot \Delta a_n) \int_{0}^{1} \exp[it (w \cdot \Delta a_n)] \, dt$$  \hspace{1cm} (34)

$$= (w \cdot \Delta a_n) \frac{-\exp[iw \cdot \Delta a_n]}{iw \cdot \Delta a_n} \left[ \exp(iw \cdot \Delta a_n) - 1 \right]$$  \hspace{1cm} (35)

$$= 2(w \cdot \Delta a_n) \sin \left( \frac{1}{2} w \cdot \Delta a_n \right)$$  \hspace{1cm} (36)

$$= \frac{2}{w \cdot \Delta a_n} \sin \left( \frac{1}{2} w \cdot \Delta a_n \right) \exp \left[ \frac{1}{2} w \cdot (a_n + a_{n+1}) \right]$$  \hspace{1cm} (37)

$$= T_n$$  \hspace{1cm} (38)

Hence

$$\int \int \exp(iw \cdot \zeta) \, d\zeta_1 \, d\zeta_2 = \sum_{n=1}^{N_{edges}} \frac{i}{|w|^2} T_n, \quad |w|^2 \neq 0$$  \hspace{1cm} (39)

As noted above, if \(|w|^2 = 0\) then the expression (28) cannot be evaluated, because of the division by \(|w|^2\). Physically, this corresponds to the case where the two waves involved in a term in the element matrix are equal and opposite and simply cancel each other out, leaving a polynomial expression. The other case which causes problems is when \(w \cdot \Delta a_n \approx 0\), because then the term \(\sin(\xi w \cdot \Delta a_n) / (\xi w \cdot \Delta a_n)\) in expression (37) tends to 0/0. In this case the two waves involved in the element term have combined to form a new wave which is approximately normal to the current polygon edge. The variation of the wave in the edge direction is then proportional rather than trigonometric. In order to deal with this case, Eq. (37) above must be replaced with one in which the sine term is replaced by a series i.e. writing \(w \cdot \Delta a_n / 2 = z, \sin(z) \approx z - z^3/3 + z^5/5 - z^7/7 + \cdots\) and \(\sin(z) \approx 1 - z^2/2 - z^4/4 + z^6/6 + \cdots\). This is well behaved for \(z \approx 0\).

The scheme described above, in conjunction with the Gordon method will only integrate the plane wave itself. Higher order polynomial terms can be integrated using the method below.

### 5.6. Linear order terms

Consider that we have already evaluated the integral

$$W_{00} = \int \int \exp(iw \cdot \zeta) \, d\Omega$$  \hspace{1cm} (40)

If this is carried out analytically the result for \(W_{00}\) can be regarded as a function of \(w_1\) and \(w_2\), the local wave number components. Now if \(W_{00}\) is differentiated with respect to \(w_1\) the integral \(W_{10}\) can be found, where the first subscript stands for the power of \(\zeta_1\) and the second subscript stands for the power of \(\zeta_2\). That is

$$\frac{\partial}{\partial w_1}(W_{00}) = \int \int \zeta_1 \exp(iw \cdot \zeta) \, d\Omega$$  \hspace{1cm} (41)

Thus

$$W_{10} = -i \frac{\partial}{\partial w_1} (W_{00}) = \int \int \zeta_1 \exp(iw \cdot \zeta) \, d\Omega$$  \hspace{1cm} (42)

### 5.7. Generalisation for terms of any order

The above concept can be generalised to any power of \(\zeta_1\) and \(\zeta_2\). The general expression is

$$W_{pq} = (-1)^{p+q} \cdot i^{p+q} \frac{\partial^p}{\partial w_1^p}(W_{00})$$  \hspace{1cm} (43)

Clearly, it is essential that the integral, \(W_{00}\), be found in symbolic form for this procedure to work.

### 5.8. Programming the numerical integration scheme

The integration formula given in Eq. (39) and the corresponding version using the series form of the sine function, could readily be programmed. However, this would not enable the evaluation of the high order terms, because the result could not be differentiated as shown in Eq. (43). The programming was therefore carried out as follows:

- The two versions of expression (37) were programmed in Maple. The first version used the sine function and the second a series approximation of the sine function. (Other computer algebra programs including Mathematica would also be suitable, provided they have the facility for generating Fortran, or other numerical code.)

- The expressions were then differentiated, the appropriate number of times (depending upon the order of finite element required), as given by expression (43), and the resulting expressions were stored, for whatever powers of \(\zeta_1\) and \(\zeta_2\) are required (this depends upon the order of the element).

- Fortran code was then generated for all the integration expressions using the code generation options in Maple. Switches had to be included in the code so that the series form of the expressions for the integration weights is used where necessary.

- The necessary expressions needed to link the integrals for the powers of \(\zeta_1\) and \(\zeta_2\) with those for the Lagrange polynomials for the integration points were generated in the form of a matrix, consisting entirely of numbers. This can be carried out using Maple, or by other means. By this means all the required integrals of the form of Eq. (27) can be found (see Appendix).

- The entire integration system was then tested. The Maple code was given a range of different values for the wave number vector \(w\), including pathological cases of zero waves in the element, very small waves, and waves normal to the element edges, or almost normal to them. The Maple code can, of course, evaluate expressions, if given the test data, and to any chosen precision. The Fortran code which had been generated was also used to...
integrate the element terms. The Fortran and Maple results were compared automatically and if the differences were of the order of machine precision, then the integration code could be regarded as reliable.

The above code was used in the creation of the element matrices, using semi-analytical integration as outlined above.

6. Results

It is not the intention of this paper to evaluate the performance of electromagnetic scattering using plane wave edge elements as this can be found in references [24,25]. Instead we compare the semi-analytical integration technique presented to the conventional Gauss–Legendre integration.

A test of the semi-analytical integration scheme is performed on a 2.4 GHz P4 win32 PC. Integration is performed over a single element, using 10 plane wave directions on each edge of a unit square, using a wavelength of 0.01, i.e. 100 wavelengths span the element in each coordinate direction. We define the error norm in a Gauss–Legendre (G–L) integration as

\[
\text{Error norm} = \frac{\sum |(\text{term}_i - \text{term}_{\text{analytical}}_i)|}{\sum |\text{term}_{\text{analytical}}_i|}
\]

where term represents either the stiffness or mass term. The results of the test are presented in Table 2 where \( p \) is the element order and the CPU times are given as the ratio of CPU_{G–L}/CPU_{analytical}. The table shows that the results using conventional Gauss–Legendre integration converge to those of the semi-analytical results at about 6 integration points per wavelength for all element orders, \( p \), tested. It is noted, for the square, Gauss–Legendre integration becomes more expensive compared to the semi-analytical integration scheme as the element order, \( p \), increases.

A set of numerical tests for the semi-analytical integration scheme has been run for a triangular element occupying a half a unit square, the remaining parameters for the test being iden-
tical to those used for the quadrilateral integration test. The results of the test are presented in Table 3. Again the table shows that the results using conventional Gauss–Legendre integration converge to those of the semi-analytical results at about 6 integration points per wavelength for all element order, p, tested. It is noted, for the triangle, Gauss–Legendre integration becomes less expensive compared to the semi-analytical integration scheme as the element order, p, increases. As was found in Table 1, the Gauss–Legendre integration for the rectangular element shows much better convergence than for the triangular element.

7. Conclusions

We have reviewed the plane wave edge element formulation for electromagnetic scattering problems using a family of compatible arbitrary order quadrilateral and triangular edge elements. Using the plane wave basis, the objective is to employ elements that may span many wavelengths, so that the determination of the entries in the element matrices requires the integration of highly oscillatory functions. This can be accurately accomplished using Gauss–Legendre quadrature of a very high order, which implies that any reduction in the number of unknowns by using the plane wave basis comes at the cost of having to perform computationally intensive numerical integration. In order to reduce the cost of the integration of highly oscillatory functions we have presented a semi-analytical integration scheme which gave considerable cost savings as shown in Table 2 for the rectangular edge element and Table 3 for the triangular edge element. The number of integration points used by the semi-analytical integration scheme is independent of the wavenumber so the cost savings will increase as the wavenumber increases. It is noted that for the rectangular elements Gauss–Legendre integration becomes more expensive compared to the semi-analytical integration scheme as the element order increases, but for triangular elements the opposite is true. We do not have any explanation why this is so. The rectangular element shows much better convergence than the triangular element. This is also shown in Table 1 where we compare the results using various numbers of Gauss–Legendre integration points to those of a reference solution obtained using a extremely large number of points. It should be noted that in common with most authors using the PUBEM and PUFEM we encountered ill-conditioning which required the use of the Singular Value Decomposition solver [37] for most cases, with the singular values truncated at a threshold of $10^{-12} \times W_{\text{max}}$, where $W_{\text{max}}$ is the largest of the singular values.

Appendix. Transforming integrals to integration weights

A linear triangular element is considered. The method can easily be generalised to higher order elements. The products in Eqs. (9) and (10) will give rise to a complete set of quadratic polynomial terms in $\zeta_1$ and $\zeta_2$, six in total. In order to integrate the above element matrices exactly it will be necessary to choose six integration points. These can be chosen arbitrarily, but one convenient choice is the set of triangle vertices and mid-side points. These are the nodal locations for the classical quadratic triangle finite element. The shape functions for these points are available in many finite element texts. The points are tabulated in Table 4, together with their shape functions. The nodes have been ordered as the three vertices followed by the three mid-side points.

Hence the matrix relating the integration weights for the powers of $\zeta_1$ and $\zeta_2$ to those for the integration points is simply:

$$
\begin{bmatrix}
W_1 \\
W_2 \\
W_3 \\
W_4 \\
W_5 \\
W_6
\end{bmatrix} = 
\begin{bmatrix}
1 & -3 & -3 & 2 & 4 & 2 \\
0 & -1 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 2 \\
0 & 4 & 0 & -4 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 4 & 0 & -4 & 0
\end{bmatrix}
\begin{bmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3 \\
\zeta_4 \\
\zeta_5 \\
\zeta_6
\end{bmatrix}
$$

where, for example, $W_1$ denotes the integration weight for point 2 and $W_{\zeta_1}$ denotes the integral from Eq. (27), where the polynomial term, $p_{\zeta_1}^2$, $p_{\zeta_2}^2$ are used. Similar relations can easily be generated for larger numbers of integration points. In reality triangular elements of order $p = 0,1,2,3$ require a total of 6, 15, 28, 45 integration points respectively to integrate exactly using this semi-analytical integration scheme regardless of how large the wavenumber may be.

References


