Colouring vertices of triangle-free graphs*

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Abstract

The VERTEX COLOURING problem is known to be NP-complete in the class of triangle-free graphs. Moreover, it remains NP-complete even if we additionally exclude a graph F which is not a forest. We study the computational complexity of the problem in (K_3, F)-free graphs with F being a forest. From known results it follows that for any forest F on 5 vertices, the VERTEX COLOURING problem is polynomial-time solvable in the class of (K_3, F)-free graphs. In the present paper, we study the problem for (K_3, F)-free graphs with F being a forest on 6 vertices. It is know that in the case when F is the star $K_{1,5}$, the problem is NP-complete. We show that in nearly all other cases the problem is polynomial-time solvable. The only exception is the class of $(K_3, 2P_3)$-free graphs for which the complexity status of the problem remains an open question.

Keywords: Vertex colouring; Triangle-free graphs; Polynomial-time algorithm; Clique-width

1 Introduction

A vertex colouring is an assignment of colours to the vertices of a graph G in such a way that no edge connects two vertices of the same colour. The VERTEX COLOURING problem consists in finding a vertex colouring with a minimum number of colours. This number is called the chromatic number of G and is denoted by $\chi(G)$. If G admits a vertex colouring with at most k colours, we say that G is k-colourable. The k-COLOURABILITY problem consists in deciding whether a graph is k-colourable.

From a computational point of view, VERTEX COLOURING and k-COLOURABILITY ($k \geq 3$) are difficult problems, i.e. both of them are NP-complete. Moreover, the problems remain NP-complete in many restricted graph families. For instance, 3-COLOURABILITY is NP-complete for planar graphs [10], 4-COLOURABILITY is NP-complete for graphs containing no induced path on 12 vertices [34], VERTEX COLOURING is NP-complete for line graphs [14]. On the other hand, for graphs in some special classes, the problems can be solved in polynomial time. For instance, 3-COLOURABILITY is solvable for graphs containing no induced path on 6 vertices [28], k-COLOURABILITY (for any value of k) is solvable for graphs containing no induced path on 5

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vertices [13], and VERTEX COLOURING (and therefore also k-COLOURABILITY for any value of k) is solvable for perfect graphs.

Recently, much attention has been paid to the complexity of the problems in graph classes defined by forbidden induced subgraphs. Many results of this type were mentioned above, some others can be found in [2, 5, 15, 16, 18, 19, 20, 25, 31]. In [19], the authors systematically study VERTEX COLOURING on graph classes defined by a single forbidden induced subgraph, and give a complete characterisation of those for which the problem is polynomial-time solvable and those for which the problem is NP-complete. In particular, the problem is NP-complete for K_3-free graphs, i.e. for triangle-free graphs. Moreover, the problem is NP-complete for (K_3, F)-free graphs for any graph F which is not a forest [15, 19]. Here we study the computational complexity of the problem in (K_3, F)-free graphs with F being a forest. From known results it follows that for any forest F on 5 vertices the VERTEX COLOURING problem is polynomial-time solvable in the class of (K_3, F)-free graphs. In the present paper, we show that the problem is also polynomial-time solvable in many classes of (K_3, F)-free graphs with F being a forest on 6 vertices.

2 Preliminaries

All graphs in this paper are finite, undirected, without loops or multiple edges. For any graph theoretical terms not defined here, the reader is referred to [11]. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. If v is a vertex of G, then $N(v)$ denotes the neighbourhood of v (i.e. the set of vertices adjacent to v) and $|N(v)|$ is the degree of v. The subgraph of G induced by a set of vertices $U \subseteq V(G)$ is denoted by $G[U]$. For disjoint sets $A, B \subseteq V(G)$, we say that A is complete to B if every vertex in A is adjacent to every vertex in B, and that A is anticomplete to B if every vertex in A is non-adjacent to every vertex in B.

As usual, P_n is a chordless path, C_n is a chordless cycle, and K_n is a complete graph on n vertices. Also, $K_{n,m}$ denotes a complete bipartite graph with parts of size n and m. By $S_{i,j,k}$ we denote a tree with exactly three leaves, which are of distance i, j and k from the only vertex of degree 3. In particular, $S_{1,1,1} = K_{1,3}$ is known as a claw, and $S_{1,2,2}$ is sometimes denoted by E, since this graph can be drawn as the capital letter E. H denotes the graph that can be drawn as the capital letter H, i.e. H has vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{v_1v_2, v_2v_3, v_2v_4, v_4v_5, v_4v_6\}$. The graph obtained from a $K_{1,4}$ by subdividing exactly one edge exactly once is called a *cross*. Given two graphs G and G', we write $G + G'$ to denote the disjoint union of G and G'. In particular, mG is the disjoint union of m copies of G.

The clique-width of a graph G is the minimum number of labels needed to construct G using the following four operations:

(i) Creation of a new vertex v with label i (denoted by $i(v)$).

(ii) Disjoint union of two labelled graphs G and H (denoted by $G \oplus H$).

(iii) Joining each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$).

(iv) Renaming label i to j (denoted by $\rho_{i\to j}$).

Every graph can be defined by an algebraic expression using these four operations. For instance, an induced path on five consecutive vertices a, b, c, d, e has clique-width equal to 3 and it can be defined as follows:

$$\eta_{3,2}(3(e) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))))))$$

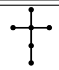
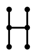
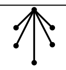
Graph	Graph Name	Complexity	Reference
	Cross	P	[27, 29]
	$S_{1,2,2}$	P	[27, 29]
	H	P	[27, 29] (see also Theorem 7)
	$K_{1,5}$	NPC	[25]

Table 1: Forests F for which the complexity of VERTEX COLOURING in the class Free(K_3, F) is known.

If a graph G does not contain induced subgraphs isomorphic to graphs from a set M, we say that G is M-free. The class of all M-free graphs is denoted by $Free(M)$, and M is called the set of forbidden induced subgraphs for this class. Many graph classes that are important from a practical or theoretical point of view can be described in terms of forbidden induced subgraphs. For instance, by definition, forests form the class of graphs without cycles, and due to König's Theorem, bipartite graphs are graphs without odd cycles. Bipartite graphs are precisely the 2-colourable graphs, and recognising 2-colourable graphs is a polynomially solvable task. However, the recognition of k-colourable graphs is an NP-complete problem for any $k \geq 3$.

In the present paper, we study the computational complexity of the VERTEX COLOURING problem in graph classes defined by two forbidden induced subgraphs one of which is a triangle, i.e. a K_3. The following theorem summarises known results of this type (see also Table 1) and proves one more that can easily be derived from known results.

Theorem 1. *Let F be a graph. If F contains a cycle or $F = K_{1,5}$, then the* VERTEX COLOURING *problem is NP-complete in the class $Free(K_3, F)$. If F is isomorphic to $S_{1,2,2}, H, P_6$ or a cross, then the problem is polynomial-time solvable in the class $Free(K_3, F)$.*

Proof. If F contains a cycle, then the NP-completeness of the problem follows from the fact that it is NP-complete for graphs of girth at least $k + 1$, i.e. in the class $Free(C_3, C_4, \ldots, C_k)$, for any fixed value of k (see e.g. [15, 19]). The NP-completeness of the problem in the class of $(K_3, K_{1,5})$-free graphs was shown in [25].

In [27, 29] Randerath et al. showed that every graph in the following three classes is 3-colourable and that a 3-colouring can be found in polynomial time: $Free(K_3, H)$, $Free(K_3, S_{1,2,2})$, $Free(K_3, cross)$. Therefore VERTEX COLOURING is polynomial-time solvable in these three classes.

The conclusion that the problem is solvable for (K_3, P_6)-free graphs can be derived from three facts. First, the clique-width of graphs in this class is bounded by a constant [4]. Second, the chromatic number of graphs in this class is bounded by a constant (see e.g. [33]). Third, for each fixed k, the k-colourability problem on graphs of bounded clique-width is solvable in polynomial time [8]. □

Corollary 1. *For each forest F on 5 vertices, the* VERTEX COLOURING *problem in the class $Free(K_3, F)$ is solvable in polynomial time.*

Proof. If F contains no edge, then the problem is trivial in the class of $Free(K_3, F)$, since the size of graphs in this class is bounded by a constant (by Ramsey's theorem). If F contains at least one edge, then it is an induced subgraph of at least one of the following graphs: H, $S_{1,2,2}$, $cross$, P_6. Therefore $Free(K_3, F)$ is a subclass of one the classes $Free(K_3, H)$, $Free(K_3, S_{1,2,2})$, $Free(K_3, cross)$, $Free(K_3, P_6)$, and thus the result follows from Theorem 1. □

3 (K_3, F)-free graphs with F containing an isolated vertex

In this section, we study graph classes $Free(K_3, F)$ with F being a forest on 6 vertices, at least one of which is isolated. Without loss of generality we may assume that F contains at least one edge, since otherwise there are only finitely many graphs in the class $Free(K_3, F)$ (by Ramsey's theorem). Throughout the section, an isolated vertex in F is denoted by v and the rest of the graph is denoted by F_0, i.e. $F_0 = F - v$.

Lemma 1. *Let F be a forest on 6 vertices with at least one edge and at least one isolated vertex. Then the chromatic number of any graph G in the class $Free(K_3, F)$ is at most 4 and a 4-colouring can be found in polynomial time.*

Proof. Suppose that $F_0 \neq P_3 + P_2$. Then it is not difficult to verify that F_0 is an induced subgraph of H, $S_{1,2,2}$ or $cross$. Therefore the chromatic number of (K_3, F_0)-free graphs is at most 3 (see [27, 29]). As a result, the chromatic number of any (K_3, F)-free graph is at most 4. To see this, observe that for any vertex x, the graph $G \setminus N(x)$ is 3-colourable, while $N(x)$ is an independent set.

Now let $F_0 = P_3 + P_2$. Let ab be an edge in a (K_3, F)-free graph G. (If G has no edges, the chromatic number is 1 and we are done.) We will show that $G_0 := G - (N(a) \cup N(b))$ is a bipartite graph. Notice that since G is K_3-free, both $N(a)$ and $N(b)$ induce an independent set. We may assume that at least one of $N(a)\setminus\{b\}, N(b)\setminus\{a\}$ is non-empty (otherwise each connected component of G has at most two vertices and thus G is trivially 4-colourable). Obviously G_0 is C_k-free for any odd $k \geq 7$, since otherwise G contains a $P_3 + P_2$. Therefore we may assume that G_0 contains a C_5 (otherwise G_0 is bipartite). Let $c \in N(b) \setminus \{a\}$. Since G is triangle-free, c can have at most two neighbours in the C_5, and if it has two, they must be non-consecutive vertices of the C_5. Thus c is non-adjacent to at least three vertices in C_5, say d, e, f, such that $G[d, e, f]$ is isomorphic to $P_2 + K_1$. But now $G[a, b, c, d, e, f]$ is isomorphic to $P_3 + P_2 + K_1$, which is a forbidden graph for G. This contradiction shows that G_0 has no odd cycles, i.e. G_0 is a bipartite graph. If V_0^1, V_0^2 are two colour classes of G_0, then $N(b) \cup \{a\}$, $N(a) \cup \{b\}$, V_0^1, V_0^2 are four colour classes of G. □

In view of Lemma 1 and the polynomial-time solvability of 2-COLOURABILITY, all we have to do to solve the problem in the classes under consideration is to develop a tool for deciding 3-colourability in polynomial time. For this, we use a result from [31]. A set $D \subseteq V(G)$ is dominating in G if every vertex $x \in V(G) \setminus D$ has at least one neighbour in D.

Lemma 2. *([31]) For a graph $G = (V, E)$ with a dominating set D, we can decide 3-colourability and determine a 3-colouring in time $O(3^{|D|}|E|)$.*

If a graph $G \in Free(K_3, F)$ is F_0-free, then the problem is solvable for G by Corollary 1. If G has an induced F_0, then the vertices of F_0 form a dominating set in G. Summarising the above discussion, we obtain the following result.

Theorem 2. *Let F be a forest on 6 vertices with at least one isolated vertex. Then the* VERTEX COLOURING *problem is polynomial-time solvable in the class $Free(K_3, F)$.*

All forests satisfying the conditions of Theorem 2 are listed in Table 2.

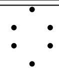
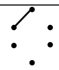
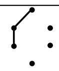
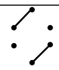
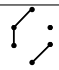
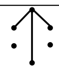
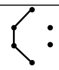
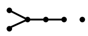
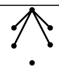
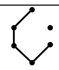
Graph	Graph Name
	Empty
	$P_2 + 4K_1$
	$P_3 + 3K_1$
	$2P_2 + 2K1$
	$P_3 + P_2 + K_1$
	$K_{1,3} + 2K_1$
	$P_4 + 2K_1$
	$S_{1,1,2} + K_1$
	$K_{1,4} + K_1$
	$P_5 + K_1$

Table 2: Forests F for which polynomial time solvability of VERTEX COLOURING in the class Free(K_3, F) follows from Theorem 2.

4 $(K_3, P_2 + P_4)$-free graphs

In this section we solve the VERTEX COLOURING problem in the class of $(K_3, P_2 + P_4)$-free graphs. First, we derive an easy upper bound on the chromatic number of graphs in this class.

Lemma 3. *The chromatic number of any $(K_3, P_2 + P_4)$-free graph G is at most 4 and a 4-colouring of G can found in polynomial time.*

Proof. If G has no edges, its chromatic number is 1, so we may assume that G has an edge ab. Since G is K_3-free, the vertices a, b together with their neighbours induce a bipartite graph. And since G is $(P_2 + P_4)$-free, the set of vertices adjacent to neither a nor b induces a (K_3, P_4)-free graph, i.e. a bipartite graph too. Therefore, the chromatic number of G is at most 4 and a 4-colouring of G can found in polynomial time. □

Lemma 3 reduces the VERTEX COLOURING problem in the class of $(K_3, P_2 + P_4)$-free graphs to 3-COLOURABILITY. To solve the latter problem we prove the following two lemmas:

Lemma 4. *Let G be a connected $(K_3, P_2 + P_4)$-free graph containing an induced cycle C of length seven. Then the vertices of C form a dominating set.*

Proof. Let $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_1$ be an induced cycle C of length seven in G. Suppose that there exists a vertex $v \in V(G)$ that is anticomplete to $V(C)$. Since G is connected, we may assume that v is adjacent to some vertex u which has a neighbour in $V(C)$, say v_1. Then u must be nonadjacent to v_2, v_7, since G is K_3-free. First, we claim that u has a neighbour in $\{v_4, v_5\}$. Indeed, if u is anticomplete to $\{v_4, v_5\}$, then $G[v, u, v_7, v_1, v_4, v_5]$ is isomorphic to $P_2 + P_4$, a contradiction. By symmetry, we may assume that u is adjacent to v_4 (and thus nonadjacent to v_3, v_5, since G is K_3-free). It follows that u must be adjacent to v_6, since otherwise $G[v, u, v_1, v_2, v_5, v_6]$ would be isomorphic to $P_2 + P_4$. But now the vertices v, u, v_6, v_7, v_2, v_3 induce a $P_2 + P_4$. This contradiction shows that every vertex of G in $V(G) \setminus V(C)$ has a neighbour in $V(C)$, and hence $V(C)$ is a dominating set. □

Lemma 5. *The chromatic number of any $(K_3, P_2 + P_4, C_7)$-free graph G containing a P_6 is at most 3 and a 3-colouring of G can found in polynomial time.*

Proof. Let Q denote the graph obtained from a C_6 cycle by adding a vertex which has exactly one neighbour on the cycle. Clearly, if a graph contains Q as an induced subgraph, it also contains a P_6. We split the proof into two cases.

Case 1: G contains an induced subgraph isomorphic to Q. Say Q is induced by vertices $a, b, c, d, e, f, g \in V(G)$, where $a - b - c - d - e - f - a$ is a chordless cycle and the only neighbour of g on the cycle is e. The vertices of G outside the set $\{a, b, c\}$ can be partitioned into at most 5 non-empty subsets in the following way:

- V_a is the set of vertices adjacent to a and nonadjacent to b and c,
- V_b and V_c are defined by analogy with V_a,
- V_{ac} is the set of vertices adjacent to a and c and nonadjacent to b,
- W is the set of vertices anticomplete to $\{a, b, c\}$.

Note that V_a, V_b, V_c and V_{ac} are independent sets, since G is K_3-free. We will split W into independent sets. Then we will investigate the possible edges between all these independent sets and finally, we will show how to obtain a 3-colouring of G.

(i) *Every connected component of* $G[W]$ *is a complete bipartite graph, and exactly one of the components contains an edge.* First, every connected component in $G[W]$ is (K_3, P_4)-free, which follows readily from the fact that G is $(K_3, P_2 + P_4)$-free. Therefore, every connected component of $G[W]$ is a complete bipartite graph. Next, assume there exist two connected components in $G[W]$ each of which contains an edge, say a component C^1 contains an edge v_1v_2 and a component C^2 contains an edge v_3v_4. Clearly v_1 and v_2 cannot both be adjacent to f, since G is K_3-free. Also, they cannot both be nonadjacent to f, since otherwise $G[v_1, v_2, f, a, b, c]$ is isomorphic to $P_2 + P_4$. Therefore f has exactly one neighbour in $\{v_1, v_2\}$, say v_1, and analogously, f has exactly one neighbour in $\{v_3, v_4\}$, say v_3. But then $G[b, c, v_2, v_1, f, v_3]$ is isomorphic to $P_2 + P_4$. This contradiction shows that there exists at most one connected component in $G[W]$ which contains an edge. Since eg is an edge in $G[W]$, it follows that there exists exactly one such component.

Let C^1 be the unique connected component of $G[W]$ containing an edge. Let W_1, W_2 be the vertex sets of C^1 defining the bipartition with $e \in W_1$, and let $W_0 = W \setminus (W_1 \cup W_2)$. Thus W_0 is a set of isolated vertices.

(ii) W_1 *is complete to* $V_a \cup V_c$ *and* W_2 *is anticomplete to* $V_a \cup V_c$. Let $u \in V_a$. From the proof of (i) we know that for each edge xy in C^1, exactly one of x, y is adjacent to u. Suppose that vertex $x \in W_2$ is adjacent to u (notice that $u \neq f$ since otherwise $G[e, u, x]$ is isomorphic to K_3, a contradiction). But then $G[b, c, u, x, e, f]$ is isomorphic to $P_2 + P_4$, a contradiction. Thus W_2 is anticomplete to V_a and hence W_1 is complete to V_a. By symmetry, W_2 is anticomplete to V_c and W_1 is complete to V_c.

(iii) V_a *is anticomplete to* V_c, which follows from (ii) and the fact that G is K_3-free.

(iv) W_0 *is anticomplete to* $V_a \cup V_c$. If $w \in W_0$ and $u \in V_a$ were adjacent, then these two vertices, together with e, g, b, c would induce a $P_2 + P_4$. So W_0 is anticomplete to V_a. By symmetry, W_0 is also anticomplete to V_c.

(v) W_2 *and* W_0 *have no common neighbours in* V_{ac}. Indeed, if a vertex $v \in V_{ac}$ is adjacent to a vertex $w \in W_2$ and a vertex $u \in W_0$, then v is not adjacent to e (since G is K_3-free) and therefore $G[e, f, u, v, c, b]$ is isomorphic to $P_2 + P_4$, a contradiction (notice that v is nonadjacent to f, since G is K_3-free).

Let X denote the subset of vertices of V_{ac} that have neighbours in W_2 and let Y denote the remaining vertices of V_{ac}. Notice that X is anticomplete to W_1, since G is K_3-free. From the above discussion and the fact that G is K_3-free, we now conclude that each of the following three sets is independent: $V_a \cup V_c \cup W_2 \cup Y$, $W_1 \cup W_0 \cup X \cup \{b\}$, $V_b \cup \{a, c\}$. Therefore G is 3-colourable and such a colouring can be found in polynomial time.

Case 2: G contains no induced subgraph isomorphic to Q. We assume that P_6 is induced in G, say by the vertices a, b, c, d, e, f with the edges $\{ab, bc, cd, de, ef\}$. The vertices outside the set $\{b, c, d, e\}$ can be partition into at most 8 non-empty sets as follows:

V_b is the set of vertices adjacent to b and nonadjacent to c, d, e,

V_c, V_d and V_e are defined by analogy with V_b,

V_{bd} is the set of vertices adjacent to b and d and nonadjacent to c and e,

V_{ce} and V_{be} are defined by analogy with V_{bd},

W is the set of vertices anticomplete to $\{b, c, d, e\}$.

(i) $G[W]$ *contains no edge*, since any two adjacent vertices in W together with b, c, d, e would induce a $P_2 + P_4$.

Note that $V_b, V_c, V_d, V_e, V_{bd}, V_{ce}$ and V_{be} are independent sets (since G is K_3-free), as is W. We will investigate the possible edges between these sets and then show how to obtain a 3-colouring of G.

(ii) V_b *is anticomplete to* V_e. Note that $a \in V_b$ and $f \in V_e$. We know $af \notin G$. Suppose a had a neighbour $u \in V_e \setminus \{f\}$. Then $G[a, b, c, d, e, u, f]$ is isomorphic to Q, a contradiction. Therefore a is anticomplete to V_e and by symmetry f is anticomplete to V_b. Now suppose that there exist two adjacent vertices $u \in V_b \setminus \{a\}, v \in V_e \setminus \{f\}$. Then $G[b, c, d, e, v, u, f]$ would be isomorphic to Q. This contradiction shows that V_b is anticomplete to V_e.

(iii) W *is anticomplete to* $V_b \cup V_e$. For suppose there exists a vertex $w \in W$ adjacent to a vertex $u \in V_b$. Then w is adjacent to f, since otherwise $G[e, f, w, u, b, c]$ would be isomorphic to $P_2 + P_4$. But now w, u, b, c, d, e, f induce a cycle of length seven, a contradiction. Thus W is anticomplete to V_b, and by symmetry, W is anticomplete to V_e.

(iv) W *is anticomplete to* $V_c \cup V_d$. For suppose there exists a vertex $w \in W$ adjacent to a vertex $u \in V_c$. Then u is adjacent to f, since otherwise, the graph $G[e, f, b, c, u, w]$ would be isomorphic to $P_2 + P_4$, a contradiction. Furthermore, u is adjacent to a, since otherwise, the graph $G[a, b, e, f, u, w]$ would be isomorphic to $P_2 + P_4$, a contradiction. But now $G[w, u, a, b, d, e]$ is isomorphic to $P_2 + P_4$, a contradiction. Thus W is anticomplete to V_c, and by symmetry W is also anticomplete to V_d.

(v) *If* W *is anticomplete to* V_{be}*, then* G *is* 3*-colourable.* We obtain a feasible 3-colouring by assigning colour 1 to the vertices of $V_b \cup V_e \cup V_{be} \cup W \cup \{d\}$, colour 2 to the vertices of $\{b\} \cup V_c \cup V_{ce}$, and colour 3 to the vertices of $\{c, e\} \cup V_d \cup V_{bd}$.

It follows from (v) that we may assume that W is not anticomplete to V_{be}, i.e. there exists a vertex $w \in W$ which has a neighbour $u \in V_{be}$. We claim that u is complete to $V_c \cup V_d$. Indeed, suppose that u is nonadjacent to a vertex $v \in V_c$. Then v is adjacent to f, since otherwise, the graph $G[c, v, w, u, e, f]$ is isomorphic to $P_2 + P_4$, a contradiction. But then $G[v, f, e, u, b, c, w]$ is isomorphic to Q, a contradiction. Thus u is complete to V_c, and by symmetry, u is complete to V_d. Since G is K_3-free, we conclude that V_c is anticomplete to V_d, and therefore, each of the following three sets is independent: $V_c \cup V_d \cup W \cup \{b, e\}$, $V_b \cup V_{bd} \cup \{c\}$, $V_e \cup V_{ce} \cup V_{be} \cup \{d\}$. Therefore, G is 3-colourable. □

Now we combine Lemmas 3, 4 and 5 to derive the main result of this section.

Theorem 3. *The* VERTEX COLOURING *problem can be solved for* $(K_3, P_2 + P_4)$*-free graphs in polynomial time.*

Proof. By Lemmas 2 and 4, the 3-COLOURABILITY problem can be solved in polynomial time for any connected $(K_3, P_2 + P_4)$-free graph G containing a C_7. If G is C_7-free and contains a P_6, 3-COLOURABILITY is solvable in polynomial time for G by Lemma 5. If G is P_6-free, 3-COLOURABILITY is polynomial-time solvable for G according to [28] (or using Theorem 1 and the fact that G is K_3-free). Finally, if G is not 3-colourable, then by Lemma 3 it is 4-colourable and a 4-colouring can be found in polynomial time.

□

5 Graphs of bounded clique-width

In Section 2, we mentioned that the polynomial-time solvability of the VERTEX COLOURING problem in the class of (K_3, P_6)-free graphs follows from the facts that both the clique-width and the chromatic number of graphs in this class are bounded by a constant. In the present section we use that same idea to solve the problem in the following two classes: $Free(K_3, S_{1,1,3})$ and $Free(K_3, K_{1,3} + K_2)$.

It is known that if G is an F-free graph, where F is a subdivision of a star $K_{1,n}$, then the chromatic number of G is bounded by a function of its clique number (see e.g. [33]). Therefore the chromatic number of $(K_3, S_{1,1,3})$-free graphs and $(K_3, K_{1,3} + K_2)$-free graphs is bounded by a constant. This means that in order to prove polynomial-time solvability of the VERTEX COLOURING problem in the classes $Free(K_3, S_{1,1,3})$ and $Free(K_3, K_{1,3} + K_2)$, all we have to do is to show that the clique-width of graphs in these classes is bounded. In our proofs we use the following helpful facts:

Fact 1: The clique-width of graphs with vertex degree at most 2 is bounded by 4 (see e.g. [9]).

Fact 2: The clique-width of $S_{1,1,3}$-free bipartite graphs [22] and $(K_{1,3}+K_2)$-free bipartite graphs [24] is bounded by a constant.

Fact 3: For a constant k and a class of graphs X, let $X_{[k]}$ denote the class of graphs obtained from graphs in X by deleting at most k vertices. Then the clique-width of graphs in X is bounded if and only if the clique-width of graphs in $X_{[k]}$ is bounded [23].

Fact 4: For a graph G, the subgraph complementation is the operation that consists of complementing the edges in an induced subgraph of G. Also, given two disjoint subsets of vertices in G, the bipartite subgraph complementation is the operation which consists of complementing the edges between the subsets. For a constant k and a class of graphs X, let $X^{(k)}$ be the class of graphs obtained from graphs in X by applying at most k subgraph complementations or bipartite subgraph complementations. Then the clique-width of graphs in $X^{(k)}$ is bounded if and only if the clique-width of graphs in X is bounded [17].

Fact 5: The clique-width of graphs in a hereditary class X is bounded if and only if it is bounded for connected graphs in X (see e.g. [9]).

Facts 2 and 5 allow us to reduce the problem to connected non-bipartite graphs in the classes $Free(K_3, S_{1,1,3})$ and $Free(K_3, K_{1,2}+K_2)$, i.e. to connected graphs in these classes that contain an odd induced cycle of length at least five.

Lemma 6. *Let G be a connected $(K_3, S_{1,1,3})$-free graph containing an odd induced cycle C of length at least 7. Then $G = C$.*

Proof. Let $C = v_1 - v_2 - \ldots - v_{2k} - v_{2k+1} - v_1$ be an induced cycle in G, of length $2k+1$, $k \geq 3$. Suppose that there exists a vertex $v \in V(G)\setminus V(C)$, which is adjacent to a vertex of C. Without loss of generality, we may assume that v is adjacent to v_1. We claim that v is non-adjacent to v_4. Otherwise, since G is K_3-free, it follows that v is non-adjacent to v_{2k+1}, v_2, v_3, v_5. But now $G[v_4, v_3, v_5, v, v_1, v_{2k+1}]$ is isomorphic to $S_{1,1,3}$, a contradiction. Thus v is non-adjacent to v_4. This implies that v is adjacent to v_3, since otherwise $G[v, v_{2k+1}, v_1, v_2, v_3, v_4]$ would be isomorphic to $S_{1,1,3}$. Now repeating the same argument with v_3 playing the role of v_1, we conclude that v is adjacent to v_5. But now $G[v_1, v_2, v_{2k+1}, v, v_5, v_4]$ is isomorphic to $S_{1,1,3}$. This contradiction shows that $G = C$. □

Lemma 7. *Let G be a connected $(K_3, K_{1,3}+K_2)$-free graph containing an odd induced cycle C_{2k+1}, $k \geq 3$. If $k \geq 4$ then $G = C_{2k+1}$ and if $k = 3$ then $|V(G)| \leq 28$.*

Proof. Let $C = v_1 - v_2 - \ldots - v_{2k} - v_{2k+1} - v_1$ be an induced cycle of length $2k+1$ in G. First consider the case where $k \geq 4$. Suppose that there exists a vertex $v \in V(G) \setminus V(C)$ which is adjacent to some vertex of C, say v_1. Since G is K_3-free, it follows that v is non-adjacent to v_{2k+1}, v_2. We claim that for every pair of vertices $\{v_i, v_{i+1}\}$, with $i = 4, 5, \ldots, 2k-2$, vertex v is adjacent to exactly one of v_i, v_{i+1}. Clearly, since G is K_3-free, v has a non-neighbour in $\{v_i, v_{i+1}\}$. If v has no neighbours in $\{v_i, v_{i+1}\}$, then $G[v_1, v_2, v, v_{2k+1}, v_i, v_{i+1}]$ is isomorphic to $K_{1,3}+K_2$, a contradiction. Now suppose that v is adjacent to v_4. Then it follows that v is complete to $\{v_4, v_6, \ldots, v_{2k-2}\}$ and anticomplete to $\{v_5, v_7, \ldots, v_{2k-1}\}$. But then $G[v_{2k-2}, v, v_{2k-3}, v_{2k-1}, v_2, v_3]$ is isomorphic to $K_{1,3}+K_2$, a contradiction. Thus we may assume that v is adjacent to v_5. This implies that v is complete to $\{v_5, v_7, \ldots, v_{2k-1}\}$ and anticomplete to $\{v_4, v_6, \ldots, v_{2k-2}\}$. It follows that v is non-adjacent to v_{2k}, since G is K_3-free. But now $G[v_5, v_v, v_6, v, v_{2k}, v_{2k+1}]$ is isomorphic to $K_{1,3}+K_2$. This contradiction shows that $G = C$.

Now consider the case where $k = 3$ and let $v \in V(G) \setminus V(C)$ be adjacent to v_1. As before, v has exactly one neighbour in $\{v_4, v_5\}$. By symmetry, we may assume that v is adjacent to v_4. Hence v has no neighbours in $\{v_2, v_3, v_5, v_7\}$. Finally, observe that v is non-adjacent to v_6, since otherwise $G[v_6, v_5, v_7, v, v_2, v_3]$ would be isomorphic to $K_{1,3}+K_2$. Therefore we conclude that each vertex $v \in V(G) \setminus V(C)$ that is adjacent to some vertex $v_i \in V(C)$, is either complete to $\{v_i, v_{i+3}\}$ and anticomplete to $V(C) \setminus \{v_i, v_{i+3}\}$, or complete to $\{v_i, v_{i+4}\}$ and anticomplete to $V(C) \setminus \{v_i, v_{i+4}\}$ (here subscripts are taken modulo 7).

Let U_j denote the set of vertices at distance j from the cycle. We claim that:

- $|U_1| \leq 7$. Indeed, if $|U_1| > 7$, then there exist two vertices $z, z' \in U_1$ that are complete to $\{v_i, v_{i+3}\}$ for some value of i (and thus anticomplete to $V(C) \setminus \{v_i, v_{i+3}\}$). Since G is K_3-free, z, z' must be non-adjacent. But then $G[v_i, z, z', v_{i+1}, v_{i+4}, v_{i+5}]$ is isomorphic to $K_{1,3}+K_2$, a contradiction.
- *each vertex of U_1 has at most one neighbour in U_2*. Indeed, suppose a vertex $x \in U_1$ has two neighbours $y, z \in U_2$, and without loss of generality let x be complete to $\{v_i, v_{i+3}\}$ (and thus anticomplete to $V(C) \setminus \{v_i, v_{i+3}\}$). Since G is K_3-free, it follows that y, z are non-adjacent. But then $G[x, y, z, v_i, v_{i+4}, v_{i+5}]$ is isomorphic to $K_{1,3}+K_2$, a contradiction.
- *each vertex of U_2 has at most one neighbour in U_3*, which can be proved by analogy with the previous claim.
- *for each $i \geq 4$, U_i is empty.* Indeed, assume without loss of generality that $U_4 \neq \emptyset$ and let u_4, u_3, u_2, u_1 be a path from U_4 to C with $u_j \in U_j$ and u_1 being adjacent to v_i. Then $G[v_i, v_{i-1}, v_{i+1}, u_1, u_3, u_4]$ is isomorphic to $K_{1,3}+K_2$, a contradiction.

From the above claims we conclude that $V(G) = V(C) \cup U_1 \cup U_2 \cup U_3$, $|U_3| \leq |U_2| \leq |U_1| \leq 7 = |V(C)|$, and therefore $|V(G)| \leq 28$. □

Thus Lemmas 6 and 7 and Fact 2 further reduce the problem to graphs containing a C_5.

Lemma 8. *If G is a connected $(K_3, S_{1,1,3})$-free graph containing a C_5, then the clique-width of G is bounded by a constant.*

Proof. Let G be a connected $(K_3, S_{1,1,3})$-free graph and let $C = v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ be an induced cycle of length five in G. If $G = C$ then the clique-width of G is at most 4 (Fact 1). Therefore we may assume that there exists at least one vertex $v \in V(G) \setminus V(C)$. Since G is

K_3-free, v can be adjacent to at most two vertices of C, and if v has two neighbours in C, they must be non-consecutive vertices of the cycle. We denote the set of vertices in $V(G) \setminus V(C)$ that have exactly i neighbours in C by N_i, $i \in \{0, 1, 2\}$. Also, for $i = 1, \ldots, 5$, we let V_i denote the set of vertices in N_2 adjacent to $v_{i-1}, v_{i+1} \in V(C)$ (throughout the proof subscripts i are taken modulo 5). We call two different sets V_i and V_j *consecutive* if v_i and v_j are consecutive vertices of C, and *opposite* otherwise. Finally, we call V_i *large* if $|V_i| \geq 2$, and *small* otherwise. The proof of the lemma will be given through a series of claims.

(1) *Each V_i is an independent set.* This immediately follows from the fact that G is K_3-free.

(2) *N_0 is an independent set.* Indeed, assume xy is an edge connecting two vertices $x, y \in N_0$, and let, without loss of generality, y be adjacent to a vertex $z \in N_1 \cup N_2$. Assume z is adjacent to $v_i \in V(C)$. Since G is K_3-free, z is non-adjacent to x, v_{i-1}, v_{i+1}. But then $G[v_i, v_{i+1}, v_{i-1}, z, y, x]$ is isomorphic to $S_{1,1,3}$, a contradiction.

(3) *Any vertex $x \in N_1 \cup N_2$ has at most one neighbour in N_0.* Suppose $x \in N_1 \cup N_2$ is adjacent to $z, z' \in N_0$, and let $v_i \in V(C)$ be a neighbour of x. Since G is K_3-free, it follows that x is non-adjacent to v_{i-1}, v_{i+1}. Furthermore, x is adjacent to at most one of v_{i-2}, v_{i+2}. By symmetry we may assume that x is non-adjacent to v_{i-2}. But now $G[x, z, z', v_i, v_{i-1}, v_{i-2}]$ is isomorphic to $S_{1,1,3}$, a contradiction.

(4) $|N_1| \leq 5$. Indeed, if there are two vertices $x, x' \in N_1$ which are adjacent to the same vertex $v_i \in V(C)$, then $G[x, x', v_i, v_{i+1}, v_{i+2}, v_{i+3}]$ is isomorphic to $S_{1,1,3}$, a contradiction.

(5) *If V_i and V_j are opposite sets, then no vertex of V_i is adjacent to a vertex of V_j.* This immediately follows from the fact that G is K_3-free.

(6) *If V_i and V_j are consecutive, then every vertex $x \in V_i$ has at most one non-neighbour in V_j.* Suppose $x \in V_i$ has two non-neighbours $y, y' \in V_j$. By symmetry, we may assume that $j = i+1$. But now $G[v_{i-3}, y, y', v_{i-2}, v_{i-1}, x]$ is isomorphic to $S_{1,1,3}$, a contradiction.

(7) *If V_i and V_j are two opposite large sets, then no vertex in N_0 has a neighbour in $V_i \cup V_j$.* Without loss of generality assume that $i = 1$ and $j = 4$, and suppose for a contradiction that a vertex $x \in N_0$ has a neighbour $y \in V_1$. If x is non-adjacent to some vertex $z \in V_4$, then $G[v_3, v_4, z, v_2, y, x]$ is isomorphic to $S_{1,1,3}$, a contradiction. Therefore x is complete to V_4. But now $G[x, z, z', y, v_2, v_1]$ with $z, z' \in V_4$ is isomorphic to $S_{1,1,3}$, a contradiction.

Since G is connected and N_0 is an independent set, every vertex of N_0 has a neighbour in $N_1 \cup N_2$. Let V_0 be the set of vertices in N_0, all of whose neighbours belong to the large sets V_i. Let G_0 be the subgraph of G induced by V_0 and the large sets. From Claims (2),(3) and (4), it follows that at most 25 vertices of G do not belong to G_0. Therefore, by Fact 3, the clique-width of G is bounded if and only if it is bounded for G_0. We may assume that G has at least one large set, since otherwise G_0 is empty. We will show that G_0 has bounded clique-width by examining all possible combinations of large sets.

Case 1: Assume that for every large set V_i there is an opposite large set V_j. Then it follows from Claim (7) that $V_0 = \emptyset$. In order to see that G_0 has bounded clique-width, we complement the edges between every pair of consecutive large sets. By Claim (6), the resulting graph has maximum degree at most 2. From Fact 1 it follows that this graph is of bounded clique-width, and therefore, applying Fact 4, G_0 has bounded clique-width.

Case 1 allows us to assume that G contains a large set such that the opposite sets are small. Without loss of generality we let V_1 be large, and V_3 and V_4 be small. The rest of the proof is based on the analysis of the size of the sets V_2 and V_5.

Case 2: V_2 and V_5 are large. Then, by Claims (1), (2), (5), and (7), G_0 is a bipartite graph with bipartition $(V_1, V_2 \cup V_5 \cup V_0)$. Therefore by Fact 2, G_0 has bounded clique-width.

Case 3: V_2 and V_5 are small. Then G_0 is a bipartite graph with bipartition (V_1, V_0), and therefore, by Fact 2, G_0 has bounded clique-width.

Case 4: V_2 is large and V_5 is small, i.e. G_0 is induced by $V_0 \cup V_1 \cup V_2$. Consider a vertex $x \in V_0$ that has a neighbour $y \in V_1$ and a neighbour $z \in V_2$. Then y and z are non-adjacent (since G is K_3-free) and therefore, by Claim (6), y is complete to $V_2 \setminus \{z\}$ and z is complete to $V_1 \setminus \{y\}$. From the K_3-freeness of G it follows that x is anticomplete to $(V_1 \cup V_2) \setminus \{y, z\}$.

Let V_0' denote the vertices of V_0 that have neighbours both in V_1 and V_2, and let V_i' $(i = 1, 2)$ denote the vertices of V_i that have neighbours in V_0'. Also, let $V_i'' = V_i - V_i'$ for $i = 0, 1, 2$, and $G_0' = G_0[V_0' \cup V_1' \cup V_2']$, $G_0'' = G_0[V_0'' \cup V_1'' \cup V_2'']$.

By Claim (3), V_0'' is anticomplete to $V_1' \cup V_2'$. Also, it follows from the above discussion that V_0' is anticomplete to $V_1'' \cup V_2''$, that V_1' is complete to V_2'', and that V_2' is complete to V_1''. Therefore by complementing the edges between V_1' and V_2'', and between V_2' and V_1'', we disconnect G_0' from G_0''. The graph G_0'' is a bipartite graph, since every vertex of V_0'' has neighbours either in V_1'' or in V_2'' but not in both. Thus it follows from Fact 2 that G_0'' has bounded clique-width. To see that G_0' has bounded clique-width, we complement the edges between V_1' and V_2'. This operation transforms G_0' into a collection of disjoint triangles. Therefore the clique-width of G_0' is bounded. Now it follows from Fact 4 that G_0 has bounded clique-width. □

Similarly to Lemma 8, one can prove the following result.

Lemma 9. *If G is a connected $(K_3, K_{1,3}+K_2)$-free graph containing a C_5, then the clique-width of G is bounded by a constant.*

Proof. The proof is similar to the proof of Lemma 8. Let G be a connected $(K_3, K_{1,3}+K_2)$-free graph and let $C = v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ be an induced cycle of length five in G. If $G = C$ then the clique-width of G is at most 4 (Fact 1). Therefore we may assume that there exists at least one vertex $v \in V(G) \setminus V(C)$. Since G is K_3-free, v can be adjacent to at most two vertices in C, and if v has two neighbours in C, they must be non-consecutive vertices of C. We denote the set of vertices in $V(G) \setminus V(C)$ that have exactly i neighbours in C by N_i, $i \in \{0, 1, 2\}$. Also, for $i = 1, \ldots, 5$, we let V_i denote the set of vertices in N_2 adjacent to $v_{i-1}, v_{i+1} \in V(C)$ (throughout the proof subscripts i are taken modulo 5). We call two different sets V_i and V_j *consecutive* if v_i and v_j are consecutive vertices of C, and *opposite* otherwise. Finally, we call V_i *large* if $|V_i| \geq 7$, and *small* otherwise. The proof of the lemma will be given through a series of claims .

(1) *Each V_i is an independent set.* This immediately follows from the fact that G is K_3-free.

(2) $|N_1| \leq 10$. Indeed, if there are three vertices $x, x', x'' \in N_1$ which are adjacent to a same vertex $v_i \in V(C)$, then $G[v_i, x, x', x'', v_{i+2}, v_{i+3}]$ is isomorphic to $K_{1,3} + K_2$, a contradiction (notice that x, x', x'' are pairwise non-adjacent, since G is K_3-free).

(3) *If V_i and V_j are opposite sets, then no vertex of V_i is adjacent to a vertex of V_j.* This immediately follows from the fact that G is K_3-free.

(4) *If V_i and V_j are consecutive, then every vertex of V_i has at most two non-neighbours in V_j.* By symmetry, we may assume $j = i + 1$. Suppose $x \in V_i$ has three non-neighbours $y, y', y'' \in V_j$. Then $G[v_{i+2}, y, y', y'', v_{i-1}, x]$ is isomorphic to $K_{1,3} + K_2$, a contradiction.

(5) *Each vertex* $w \in N_0$ *is adjacent to at most two vertices in a set* V_i. Indeed, if $w \in N_0$ were adjacent to three vertices $z, z', z'' \in V_i$, then $G[w, z, z', z'', v_{i+2}, v_{i+3}]$ would be isomorphic to $K_{1,3} + K_2$, a contradiction.

(6) N_0 *induces a graph of vertex degree at most two. Moreover, if there exists at least one large set, then* N_0 *is an independent set.* If a vertex $w \in N_0$ has three neighbours $z, z', z'' \in N_0$, then $G[w, z, z', z'', v_1, v_2]$ is isomorphic to $K_{1,3} + K_2$. This contradiction proves the first part of the claim. To prove the second part, assume V_i is a large set and suppose that two vertices $w, w' \in N_0$ are adjacent. Since V_i is large, it follows from Claim (5) that there exist at least three vertices $z, z', z'' \in V_i$ which are anticomplete to $\{w, w'\}$. But now $G[v_{i-1}, z, z', z'', w, w']$ is isomorphic to $K_{1,3} + K_2$, a contradiction.

(7) *If* V_i *and* V_j *are two opposite large sets, then no vertex in* N_0 *has a neighbour in* $V_i \cup V_j$. Without loss of generality, assume that $i = 1$ and $j = 4$, and suppose for contradiction, that a vertex $w \in N_0$ has a neighbour $y \in V_1$. Since V_4 is large and since w is adjacent to at most two vertices in V_4 (Claim (5)), it follows that w has two non-neighbours $z, z' \in V_4$. But now $G[v_3, v_4, z, z', w, y]$ is isomorphic to $K_{1,3} + K_2$, a contradiction.

(8) *Any vertex* $x \in N_1 \cup N_2$ *has at most two neighbours in* N_0. Indeed, for any vertex $x \in N_1 \cup N_2$ there exist at least two consecutive vertices of C non-adjacent to x. These two vertices together with x and any three neighbours of x in N_0 would induce a $K_{1,3}+K_2$.

From Claim (6) and Fact 1 we know that the clique-width of $G[N_0]$ is at most 4. Therefore, if all sets V_i are small, then G has bounded clique-width, which follows from Claim (2) and Fact 3.

From now on, we assume that there exists at least one large set V_i. This implies that N_0 is an independent set (Claim (6)). Since G is connected, every vertex of N_0 has a neighbour in $N_1 \cup N_2$. Let V_0 be the set of vertices in N_0, all of whose neighbours belong to the large sets V_i. Let G_0 be the subgraph of G induced by V_0 and the large sets. From Claims (2) and (8), it follows that only finitely many vertices of G do not belong to G_0. Therefore, by Fact 3, the clique-width of G is bounded if and only if it is bounded for G_0. We will show that G_0 has bounded clique-width by examining all possible combinations of large sets.

Case 1: Assume that for every large set V_i there is an opposite large set V_j. Then it follows from Claim (7) that $V_0 = \emptyset$. Let V_{i-1} and V_{i+1} be large sets. We claim that every vertex $x \in V_i$ is complete to $V_{i-1} \cup V_{i+1}$. For suppose not: let $y \in V_{i+1}$ be a non-neighbour of x. Since V_{i-1} is large, it follows from Claim (4) that x has at least two neighbours $z, z' \in V_{i-1}$. But now $G[x, z, z', v_{i-1}, v_{i+2}, y]$ is isomorphic to $K_{1,3} + K_2$, a contradiction. In order to see that G_0 is of bounded clique-width, we complement the edges between every pair of consecutive large sets. From Claim (4) and the discussion above, it follows that the resulting graph is of vertex degree at most 2. From Fact 1 it follows that this graph has bounded clique-width, and therefore applying Fact 4, G_0 has bounded clique-width.

Case 1 allows us to assume that G contains a large set such that the opposite sets are small. Without loss of generality we let V_1 be large, and V_3 and V_4 be small. The rest of the proof is based on the analysis of the size of the sets V_2 and V_5.

Case 2: V_2 and V_5 are large. Then by Claims (1),(3),(6) and (7), G_0 is a bipartite graph with bipartition $(V_1, V_2 \cup V_5 \cup V_0)$. Therefore by Fact 2, G_0 has bounded clique-width.

Case 3: V_2 and V_5 are small. Then G_0 is a bipartite graph with bipartition (V_1, V_0), and therefore, by Fact 2, G_0 has bounded clique-width.

Case 4: V_2 is large and V_5 is small, i.e. G_0 is induced by $V_0 \cup V_1 \cup V_2$. Consider a vertex $w \in V_0$ that is adjacent to some vertex $x \in V_1$ (resp. $y \in V_2$). We claim that

(9) *w is complete to all the non-neighbours of x in V_2 (resp. of y in V_1).* By symmetry we let x belong to V_1 and for contradiction, suppose that w is non-adjacent to a non-neighbour $z \in V_2$ of x. Since V_1 is large, it follows from Claims (4) and (5) that V_1 contains three vertices x_1, x_2, x_3 adjacent to z and non-adjacent to w. But now $G_0[z, x_1, x_2, x_3, x, w]$ is isomorphic to $K_{1,3} + K_2$, a contradiction.

In order to see that G_0 has bounded clique-width, we complement the edges between V_1 and V_2. Let us denote the resulting graph by G'_0. From Facts 4 and 5, it follows that it is enough to show that each connected component of G'_0 has bounded clique-width. Let C^* be a component of G'_0. If C^* has maximum vertex degree at most two, then C^* has bounded clique-width by Fact 1. So we may assume that C^* contains a vertex x of degree at least three.

First suppose that $x \in V_1 \cup V_2$. By symmetry, we may assume $x \in V_1$. We know that in the graph G'_0 vertex x has at most two neighbours in V_0 (Claim (8)) and at most two neighbours in V_2 (Claim (4)). Therefore, x is adjacent to some vertex $y \in V_2$ and to some vertex $w \in V_0$ in the graph G'_0. Since in the graph G_0 vertex y is a non-neighbour of x, it follows from Claim (9) that y, w are adjacent. Repeating this argument, we conclude that w is complete to $V(C^*) \cap (V_1 \cup V_2)$. By Claim (5), we obtain that $|V(C^*) \cap (V_1 \cup V_2)| \leq 4$. Since each vertex in $V_1 \cup V_2$ has at most two neighbours in V_0 (Claim (8)), we finally conclude that $|V(C^*)| \leq 12$ and therefore the clique-width of C^* is at most 12.

Now assume that $x \in V_0$ and all vertices of C^* in $V_1 \cup V_2$ have degree at most 2. Since V_0 is an independent set, all neighbours of x are in $V_1 \cup V_2$. Let z, z', z'' denote three neighbours of x. Without loss of generality we may assume that $z, z' \in V_1$ and $z'' \in V_2$ (Claim (5)). Since G is K_3-free, it follows that in C^*, vertex z'' is adjacent to both z, z'. But now $z'' \in V_2$ has degree at least three, contradicting our assumption. □

From Lemmas 6, 7, 8, and 9, we derive the main result of this section.

Theorem 4. *The clique-width of $(K_3, S_{1,1,3})$-free graphs and $(K_3, K_{1,3} + K_2)$-free graphs is bounded by a constant and therefore the* VERTEX COLOURING *problem is polynomial-time solvable in these classes of graphs.*

6 Further results

In this section, we prove a few additional results. The first two results deal with graph classes $Free(K_3, F)$ where F is a "big" forest of simple structure.

Theorem 5. *For every fixed m, the* VERTEX COLOURING *problem is polynomial-time solvable in the class $Free(K_3, mK_2)$.*

Proof. Obviously, if a graph G is k-colourable, then it admits a k-colouring in which one of the colour classes is a maximal independent set.

It is known that for every fixed m the number of maximal independent sets in the class $Free(mK_2)$ is bounded by a polynomial [1] and all of them can be found in polynomial time [35]. Therefore given a mK_2-free graph G, we can solve the 3-COLOURABILITY problem for G by generating all maximal independent sets and solving 2-COLOURABILITY for the remaining vertices of the graph. Then by induction on k, we conclude that for any fixed k the k-COLOURABILITY problem can be solved in the class $Free(mK_2)$ in polynomial time. Since the chromatic number of (K_3, mK_2)-free graphs is bounded by $2m - 2$ (see e.g. [3]), the VERTEX COLOURING problem is polynomial-time solvable in the class $Free(K_3, mK_2)$ for any fixed m. □

Theorem 6. *For every fixed m, the* VERTEX COLOURING *problem is polynomial-time solvable in the class $Free(K_3, P_3 + mK_1)$.*

Proof. To prove the theorem, we will show that for any fixed m, graphs in the class $Free(K_3, P_3+mK_1)$ are either bounded in size, or they are 3-colourable and a 3-colouring can be found in polynomial time.

Let G be a $(K_3, P_3 + mK_1)$-free graph. We start by finding a maximum independent set in G. For each fixed m, this problem is solvable in polynomial time, which can easily be seen by induction on m. Let S be a maximum independent set in G. Let R denote the remaining vertices of G, i.e. $R = V(G) - S$. We may assume that R contains an induced odd cycle $C = v_1 - v_2 - \ldots - v_p - v_1$ with $p \geq 5$. Since S is a maximum independent set, each vertex of C has at least one neighbour in S. Let us call a vertex $v_i \in V(C)$ strong if it has at least 2 neighbours in S and weak otherwise. Since C is an odd cycle, it has either two consecutive weak vertices or two consecutive strong vertices.

If C has two consecutive weak vertices, say v_1, v_2, then jointly they are adjacent to two vertices of S, say v_1 is adjacent to s_1, and v_2 is adjacent to s_2, and therefore, they have $|S| - 2$ common non-neighbours in S. If $|S| - 2 \geq m$, then s_1, v_1, v_2 together with m vertices in $S \setminus \{s_1, s_2\}$ induce a subgraph isomorphic to $P_3 + mK_1$, a contradiction. Therefore $|S| < m+2$. But then the number of vertices in G is bounded by the Ramsey number $R(3, m+2)$, since G is K_3-free and contains no independent set of size $m+2$.

Now assume C has two consecutive strong vertices, say v_1, v_2. Since the graph is (P_3+mK_1)-free, every strong vertex has at most $m-1$ non-neighbours in S, and since the graph is K_3-free, consecutive vertices of C cannot have common neighbours. Therefore each of v_1 and v_2 has at most $m-1$ neighbours in S. But then $|S| < 2m-1$ and hence the number of vertices of G is bounded by the Ramsey number $R(3, 2m-1)$ by the same argument as before.

Thus, if R has an odd cycle, then the number of vertices in G is bounded by a constant. If R has no odd cycles, then $G[R]$ is bipartite, and hence G is 3-colourable. Finding a maximum independent set in a $(P_3 + mK_1)$-free graph can be done in polynomial time, so any $(K_3, P_3 + mK_1)$-free graph is either bounded in size, or can be 3-coloured in this way in polynomial time. Thus VERTEX COLOURING of (K_3, P_3+mK_1)-free graphs can be solved in polynomial time. □

We conclude the paper with an alternative proof of the fact that every (K_3, H)-free graph is 3-colourable. Observe that the original proof given in [27] is about 6 pages long. We give a much shorter proof.

Theorem 7. *Every (K_3, H)-free graph is 3-colourable and a 3-colouring can be found in polynomial time.*

Proof. Let G be a (K_3, H)-free graph and S any maximal (with respect to set inclusion) independent set in G. We assume that S admits no augmenting $K_{1,2}$ (i.e. a triple x, y, z such that x and y are nonadjacent vertices outside S with $N(x) \cap S = N(y) \cap S = \{z\}$), since finding an augmenting $K_{1,2}$ can be done in polynomial time.

Assume that the graph $G[V \setminus S]$ is not bipartite, and let vertices $x_1, \ldots, x_k$ induce in $G[V \setminus S]$ a cycle C of odd length $k \geq 5$. By maximality of S, every vertex outside S has a neighbour in S.

Suppose that each vertex of C has exactly one neighbour in S, and let $y_2 \in S$ and $y_3 \in S$ be the neighbours of x_2 and x_3, respectively. Then $x_1, x_2, x_3, x_4, y_2, y_3$ induce a copy of the graph H (by lack of triangles and augmenting $K_{1,2}$s). Thus, C must contain vertices with at least two neighbours in S. Assume without loss of generality that x_2 is of this type. If C has two consecutive vertices each of which has at least two neighbours in S, then an induced H can be easily found. Therefore, each of x_1 and x_3 has exactly one neighbour in S. If $y_2 \in S$ is a neighbour of x_2 and $y_3 \in S$ is a neighbour of x_3, then x_4 is adjacent to y_2, since otherwise $x_1, x_2, y_2, x_3, y_3, x_4$ induce a copy of H. Therefore, $N(x_2) \cap S \subseteq N(x_4) \cap S$, and by

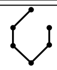
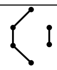
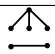
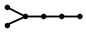
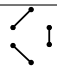
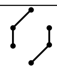
Graph	Graph Name	Complexity	Reference
	P_6	P	Theorem 1
	$P_4 + P_2$	P	Theorem 3
	$K_{1,3} + P_2$	P	Theorem 4
	$S_{1,1,3}$	P	Theorem 4
	$3P_2$	P	Theorem 5
	$2P_3$	Open	

Table 3: Forests F on six vertices for which the complexity of VERTEX COLOURING in the class Free(K_3, F) is either contributed in this paper or remains open.

symmetry, $N(x_4) \cap S \subseteq N(x_2) \cap S$, i.e. x_2 and x_4 have the same neighbourhood in S. This in turn implies that x_5 has exactly one neighbour in S. Continuing inductively, we conclude that the even-indexed vertices of C have the same neighbourhood in S consisting of at least two vertices, and each of the odd-indexed vertices of C has exactly one neighbour in S. But then $x_1, x_2, x_k, x_{k-1}, y_1, y_k$ induce a copy of the graph H, where $y_1 \in S$ and $y_k \in S$ are the neighbours of x_1 and x_k, respectively. □

7 Concluding remarks and open problems

In this paper we studied the complexity of the VERTEX COLOURING problem in classes of (K_3, F)-free graphs with F being a forest on 6 vertices. It is know that this problem is NP-complete if F is a star $K_{1,5}$. We showed that in all other cases, with possibly one exception, the problem is solvable in polynomial time (see Tables 2 and 3 for a summary of the results obtained in this paper). The only exception is the class $Free(K_3, 2P_3)$, where the complexity of the problem remains an open question. We conjecture that this case is NP-hard.

One more natural direction of research is investigation of the problem in extensions of triangle-free graphs. Let us observe that all results on triangle-free graphs can be extended, with no extra work, to so-called paw-free graphs, where a paw is the graph obtained from a triangle by adding a pendant edge. This follows from two facts: first, the problem can obviously be reduced to connected graphs, and second, according to [26], a connected paw-free graph is either complete multipartite (i.e. $\overline{P}_3$-free), in which case the problem is trivial, or triangle-free.

Further extensions make the problem much harder. For instance, by adding a pendant edge to each vertex of a triangle, we obtain a graph known in the literature as a net, and according to [32] the problem is NP-hard even for $(net, 2K_2)$-free graphs and $(net, 4K_1)$-free graphs. An interesting intermediate class between paw-free and net-free graphs is the class of bull-free graphs, where a bull is the graph obtained by adding a pendant edge to two vertices of a triangle. Recently, the class of bull-free graphs received much attention in the literature (see e.g. [7, 6, 12, 21]). In particular, paper [6] provides a structural characterisation of bull-free graphs which may be helpful in designing algorithms for various graph problems, including the vertex colouring problem.

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