Colouring vertices of triangle-free graphs without forests

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Abstract

The VERTEX COLOURING problem is known to be NP-complete in the class of triangle-free graphs. Moreover, it is NP-complete in any subclass of triangle-free graphs defined by a finite collection of forbidden induced subgraphs, each of which contains a cycle. In this paper, we study the VERTEX COLOURING problem in subclasses of triangle-free graphs obtained by forbidding graphs without cycles, i.e. forests, and prove polynomial-time solvability of the problem in many classes of this type. In particular, our paper, combined with some previously known results, provides a complete description of the complexity status of the problem in subclasses of triangle-free graphs obtained by forbidding a forest with at most 6 vertices.

Keywords: MSC 05C15, vertex colouring, triangle-free graphs, polynomial-time algorithm, clique-width

1. Introduction

A vertex colouring is an assignment of colours to the vertices of a graph G in such a way that no edge connects two vertices of the same colour. The VERTEX COLOURING problem consists of finding a vertex colouring with the minimum possible number of colours. This number is called the chromatic number of G and is denoted by $\chi(G)$. If G admits a vertex colouring with at most k colours, we say that G is k-colourable. The k-COLOURABILITY problem consists of deciding whether a graph is k-colourable and finding such a colouring, if it exists.

From a computational point of view, VERTEX COLOURING and k-COLOURABILITY ($k \geq 3$) are difficult problems, i.e. both of them are NP-complete. Moreover, the problems remain NP-complete in many restricted graph families. For instance, 3-COLOURABILITY is NP-complete for planar graphs [11], 4-COLOURABILITY is NP-complete for graphs containing no induced path on 8 vertices [6], VERTEX COLOURING is NP-complete for line graphs [16]. On the other hand, for graphs in some special classes, the problems can be solved in polynomial time. For instance, 3-COLOURABILITY is solvable for graphs containing no induced path on 6 vertices [32], k-COLOURABILITY (for any value of k) is solvable for graphs containing no induced path on 5 vertices [15], and VERTEX COLOURING (and therefore also k-COLOURABILITY for any value of k) is solvable for perfect graphs [14].

Recently, much attention has been paid to the complexity of the problems in graph classes defined by forbidden induced subgraphs. Many results of this type were mentioned above, some

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others can be found in [2, 5, 7, 17, 18, 20, 21, 22, 27, 33, 36]. In [21], the authors systematically study VERTEX COLOURING on graph classes defined by a single forbidden induced subgraph, and give a complete characterisation of those for which the problem is polynomial-time solvable and those for which it is NP-complete. In particular, the problem is NP-complete for triangle-free graphs. More generally, from the results in [17] it follows that the problem is NP-complete in any subclass of triangle-free graphs defined by a finite collection of forbidden induced subgraphs, each of which contains a cycle. This motivates us to study the problem in subclasses of triangle-free graphs obtained by forbidding graphs without cycles, i.e. forests. In this paper we prove polynomial-time solvability of the problem in many classes of this type. In particular, our results, combined with some previously known facts, provide a complete description of the complexity status of the problem in subclasses of triangle-free graphs obtained by forbidding a forest with at most 6 vertices.

All preliminary information related to the topic of the paper can be found in Section 2, while open problems are discussed in Section 7.

2. Preliminaries

All graphs in this paper are finite, undirected, without loops or multiple edges. For any graph theoretical terms not defined here, the reader is referred to [12]. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. If v is a vertex of G, then $N(v)$ denotes the neighbourhood of v (i.e. the set of vertices adjacent to v) and $|N(v)|$ is the degree of v. The subgraph of G induced by a set of vertices $U \subseteq V(G)$ is denoted by $G[U]$. For disjoint sets $A, B \subseteq V(G)$, we say that A is *complete* to B if every vertex in A is adjacent to every vertex in B, and that A is *anticomplete* to B if every vertex in A is non-adjacent to every vertex in B.

As usual, P_n is a chordless path, C_n is a chordless cycle, and K_n is a complete graph on n vertices. Also, $K_{n,m}$ denotes a complete bipartite graph with parts of size n and m. $S_{i,j,k}$ denotes a tree with exactly three leaves, which are at distance i, j and k from the only vertex of degree 3. In particular, $S_{1,1,1} = K_{1,3}$ is known as a claw, and $S_{1,2,2}$ is sometimes denoted by E, since this graph can be drawn as the capital letter E. H denotes the graph that can be drawn as the capital letter H, i.e. H has vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{v_1v_2, v_2v_3, v_2v_4, v_4v_5, v_4v_6\}$. The graph obtained from a $K_{1,4}$ by subdividing exactly one edge exactly once is called a *cross*. Given two graphs G and G', we write $G + G'$ to denote the disjoint union of G and G'. In particular, mG is the disjoint union of m copies of G.

The clique-width of a graph G is the minimum number of labels needed to construct G using the following four operations:

(i) Creating of a new vertex v with label i (denoted by $i(v)$).

(ii) Taking the disjoint union of two labelled graphs G and H (denoted by $G \oplus H$).

(iii) Joining each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$).

(iv) Renaming label i to j (denoted by $\rho_{i \to j}$).

Every graph can be defined by an algebraic expression using these four operations. For instance, an induced path on five consecutive vertices a, b, c, d, e has clique-width equal to 3 and it can be defined as follows:

$$\eta_{3,2}(3(e) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))))))$$

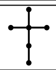
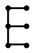
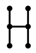
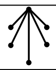
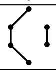
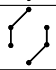
Graph	Graph Name	Complexity	Reference
	Cross	P	[31]
	$S_{1,2,2}$	P	[30]
	H	P	[30] (see also Theorem 7 for a shorter proof)
	$K_{1,5}$	NPC	[27]
	$P_4 + P_2$	P	[6] (see also Theorem 4 for a more general result)
	$2P_3$	P	[7]

Table 1: Forests F for which the complexity of VERTEX COLOURING in the class Free(K_3, F) is known.

If a graph G does not contain induced subgraphs isomorphic to graphs from a set M, we say that G is M-free. The class of all M-free graphs is denoted by $Free(M)$, and M is called the set of forbidden induced subgraphs for this class. Note that such classes $\mathcal{C}$ are *hereditary* in the sense that if $G \in \mathcal{C}$ and $v \in V(G)$ then $G \setminus v \in \mathcal{C}$. Many graph classes that are important from a practical or theoretical point of view can be described in terms of forbidden induced subgraphs. For instance, by definition, forests form the class of graphs without cycles, and due to König's Theorem, bipartite graphs are graphs without odd cycles. Bipartite graphs are precisely the 2-colourable graphs, and recognising 2-colourable graphs is a polynomially solvable task. However, the recognition of k-colourable graphs is an NP-complete problem for any $k \geq 3$.

In the present paper, we study the computational complexity of the VERTEX COLOURING problem in subclasses of triangle-free graphs. The family of these classes contains both NP-complete and polynomially solvable cases of the problem. For classes defined by a single additional forbidden induced subgraph, a summary of known results is presented in the following theorem (see also Table 1), where we also prove one more result that can easily be derived from known results.

Theorem 1. *Let F be a graph. If F contains a cycle or $F = K_{1,5}$, then the* VERTEX COLOURING *problem is NP-complete in the class $Free(K_3, F)$. If F is isomorphic to $S_{1,2,2}, H, cross, P_4 + P_2, 2P_3$ or P_6, then the problem is polynomial-time solvable in the class $Free(K_3, F)$.*

Proof. If F contains a cycle, then the NP-completeness of the problem follows from the fact that it is NP-complete for graphs of girth at least $k+1$, i.e. in the class $Free(C_3, C_4, \ldots, C_k)$, for any fixed value of k (see e.g. [17, 21]). The NP-completeness of the problem in the class of $(K_3, K_{1,5})$-free graphs was shown in [27].

In [29, 30, 31] Randerath et al. showed that every graph in the following three classes is 3-colourable and that a 3-colouring can be found in polynomial time: $Free(K_3, H)$, $Free(K_3, S_{1,2,2})$, $Free(K_3, cross)$. Therefore VERTEX COLOURING is polynomial-time solvable in these three classes.

The polynomial-time solvability of the problem in the class $Free(K_3, P_4 + P_2)$ was shown in [6] and for the class $Free(K_3, 2P_3)$, it was proved in [7].

The conclusion that the problem is solvable for (K_3, P_6)-free graphs can be derived from two facts. First, the clique-width of graphs in this class is bounded by a constant [4]. Second, the

VERTEX COLOURING problem is solvable in polynomial time on graphs of bounded clique-width [34]. □

A particular corollary of this theorem is that the VERTEX COLOURING problem is solvable in any subclass of triangle-free graphs defined by forbidding a forest with at most 5 vertices.

Corollary 1. *For each forest F on 5 vertices, the* VERTEX COLOURING *problem in the class $Free(K_3, F)$ is solvable in polynomial time.*

Proof. If F contains no edge, then the problem is trivial in the class of $Free(K_3, F)$, since the size of graphs in this class is bounded by a constant (by Ramsey's Theorem). If F contains at least one edge, then it is an induced subgraph of at least one of the following graphs: H, $S_{1,2,2}$, $cross$, P_6. Therefore $Free(K_3, F)$ is a subclass of one the classes $Free(K_3, H)$, $Free(K_3, S_{1,2,2})$, $Free(K_3, cross)$, $Free(K_3, P_6)$, and thus the result follows from Theorem 1. □

In the subsequent sections we study subclasses of triangle-free graphs defined by forbidding forests with more than 5 vertices and prove polynomial-time solvability of the problem in many classes of this type.

3. (K_3, F)-free graphs with F containing an isolated vertex

In this section we study graph classes $Free(K_3, F)$ with F being a forest on 6 vertices, at least one of which is isolated. Without loss of generality we may assume that F contains at least one edge, since otherwise there are only finitely many graphs in the class $Free(K_3, F)$ (by Ramsey's Theorem). Throughout the section, an isolated vertex in F is denoted by v and the rest of the graph is denoted by F_0, i.e. $F_0 = F - v$.

Lemma 1. *Let F be a forest on 6 vertices with at least one edge and at least one isolated vertex. Then the chromatic number of any graph G in the class $Free(K_3, F)$ is at most 4 and a 4-colouring can be found in polynomial time.*

Proof. Suppose that $F_0 \neq P_3 + P_2$. Then it is not difficult to verify that F_0 is an induced subgraph of H, $S_{1,2,2}$ or $cross$. Therefore the chromatic number of (K_3, F_0)-free graphs is at most 3 (see [30, 31]). As a result, the chromatic number of any (K_3, F)-free graph is at most 4. To see this, observe that for any vertex x, the graph $G \setminus N(x)$ is 3-colourable, while $N(x)$ is an independent set.

Now let $F_0 = P_3 + P_2$. Let ab be an edge in a (K_3, F)-free graph G. (If G has no edges, the chromatic number is 1 and we are done.) We will show that $G_0 := G - (N(a) \cup N(b))$ is a bipartite graph. Notice that since G is K_3-free, both $N(a)$ and $N(b)$ induce an independent set. We may assume that at least one of $N(a)\setminus\{b\}, N(b)\setminus\{a\}$ is non-empty (otherwise each connected component of G has at most two vertices and thus G is trivially 4-colourable). Obviously G_0 is C_k-free for any odd $k \geq 7$, since otherwise G contains a $P_3 + P_2$. Therefore we may assume that G_0 contains a C_5 (otherwise G_0 is bipartite). Let $c \in N(b) \setminus \{a\}$. Since G is triangle-free, c can have at most two neighbours in the C_5, and if it has two, they must be non-consecutive vertices of the C_5. Thus c is non-adjacent to at least three vertices in C_5, say d, e, f, such that $G[d, e, f]$ is isomorphic to $P_2 + K_1$. But now $G[a, b, c, d, e, f]$ is isomorphic to $P_3 + P_2 + K_1$, which is a forbidden graph for G. This contradiction shows that G_0 has no odd cycles, i.e. G_0 is a bipartite graph. If V_0^1, V_0^2 are two colour classes of G_0, then $N(a)$, $N(b)$, V_0^1, V_0^2 are four colour classes of G. □

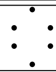
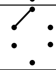
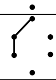
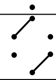
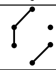
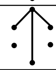
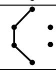
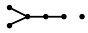
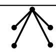
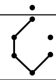
Graph	Graph Name
	Empty
	$P_2 + 4K_1$
	$P_3 + 3K_1$
	$2P_2 + 2K1$
	$P_3 + P_2 + K_1$
	$K_{1,3} + 2K_1$
	$P_4 + 2K_1$
	$S_{1,1,2} + K_1$
	$K_{1,4} + K_1$
	$P_5 + K_1$

Table 2: Forests F for which polynomial-time solvability of VERTEX COLOURING in the class Free(K_3, F) follows from Theorem 2.

In view of Lemma 1 and the polynomial-time solvability of 2-COLOURABILITY, all we have to do to solve the problem in the classes under consideration is to develop a tool for deciding 3-colourability in polynomial time. For this, we use a result from [33]. A set $D \subseteq V(G)$ is dominating in G if every vertex $x \in V(G) \setminus D$ has at least one neighbour in D.

Lemma 2. *([33]) For a graph $G = (V, E)$ with a dominating set D, we can decide 3-colourability and determine a 3-colouring in time $O(3^{|D|}|E|)$.*

If a graph $G \in Free(K_3, F)$ is F_0-free, then by Corollary 1, the problem is solvable for G in polynomial time. If G has an induced F_0, then the vertices of F_0 form a dominating set in G. Summarising the above discussion, we obtain the following result.

Theorem 2. *Let F be a forest on 6 vertices with at least one isolated vertex. Then the* VERTEX COLOURING *problem is polynomial-time solvable in the class $Free(K_3, F)$.*

All forests satisfying the conditions of Theorem 2 are listed in Table 2.

4. Graphs of bounded clique-width

In Section 2, we mentioned that the polynomial-time solvability of the VERTEX COLOURING problem in the class of (K_3, P_6)-free graphs follows from the fact that the clique-width of graphs in this class is bounded by a constant. In the present section we use that same idea to solve the problem in the following two classes: $Free(K_3, S_{1,1,3})$ and $Free(K_3, K_{1,3} + K_2)$.

This means that in order to prove polynomial-time solvability of the VERTEX COLOURING problem in the classes $Free(K_3, S_{1,1,3})$ and $Free(K_3, K_{1,3} + K_2)$, all we have to do is to show that the clique-width of graphs in these classes is bounded. In our proofs we use the following helpful facts:

Fact 1: The clique-width of graphs with vertex degree at most 2 is bounded by 4 (see e.g. [10]).

Fact 2: The clique-width of $S_{1,1,3}$-free bipartite graphs [24] and $(K_{1,3}+K_2)$-free bipartite graphs [26] is bounded by a constant.

Fact 3: For a constant k and a class of graphs X, let $X_{[k]}$ denote the class of graphs obtained from graphs in X by deleting at most k vertices. Then the clique-width of graphs in X is bounded if and only if the clique-width of graphs in $X_{[k]}$ is bounded [25].

Fact 4: For a graph G, the subgraph complementation is the operation that consists of complementing the edges in an induced subgraph of G. Also, given two disjoint subsets of vertices in G, the bipartite subgraph complementation is the operation which consists of complementing the edges between the subsets. For a constant k and a class of graphs X, let $X^{(k)}$ be the class of graphs obtained from graphs in X by applying at most k subgraph complementations or bipartite subgraph complementations. Then the clique-width of graphs in $X^{(k)}$ is bounded if and only if the clique-width of graphs in X is bounded [19].

Fact 5: The clique-width of graphs in a hereditary class X is bounded if and only if it is bounded for connected graphs in X (see e.g. [10]).

Facts 2 and 5 allow us to reduce the problem to connected non-bipartite graphs in the classes $Free(K_3, S_{1,1,3})$ and $Free(K_3, K_{1,3}+K_2)$, i.e. to connected graphs in these classes that contain an odd induced cycle of length at least five.

Lemma 3. *Let G be a connected $(K_3, S_{1,1,3})$-free graph containing an odd induced cycle C of length at least 7. Then $G = C$.*

Proof. Let $C = v_1 - v_2 - \cdots - v_{2k} - v_{2k+1} - v_1$ be an induced cycle in G, of length $2k+1$, $k \geq 3$. Suppose that there exists a vertex $v \in V(G) \setminus V(C)$, which is adjacent to a vertex of C. Without loss of generality, we may assume that v is adjacent to v_1. We claim that v is non-adjacent to v_4. Otherwise, since G is K_3-free, it follows that v is non-adjacent to v_{2k+1}, v_2, v_3, v_5. But now $G[v_4, v_3, v_5, v, v_1, v_{2k+1}]$ is isomorphic to $S_{1,1,3}$, a contradiction. Thus v is non-adjacent to v_4. This implies that v is adjacent to v_3, since otherwise $G[v_1, v, v_{2k+1}, v_2, v_3, v_4]$ would be isomorphic to $S_{1,1,3}$. Now repeating the same argument with v_3 playing the role of v_1, we conclude that v is adjacent to v_5. But now $G[v_1, v_2, v_{2k+1}, v, v_5, v_4]$ is isomorphic to $S_{1,1,3}$. This contradiction shows that $G = C$. □

Lemma 4. *Let G be a connected $(K_3, K_{1,3}+K_2)$-free graph containing an odd induced cycle C_{2k+1}, $k \geq 3$. If $k \geq 4$ then $G = C_{2k+1}$ and if $k = 3$ then $|V(G)| \leq 28$.*

Proof. Let $C = v_1 - v_2 - \cdots - v_{2k} - v_{2k+1} - v_1$ be an induced cycle of length $2k+1$ in G. First consider the case when $k \geq 4$. Suppose that there exists a vertex $v \in V(G) \setminus V(C)$ which is adjacent to some vertex of C, say v_1. Since G is K_3-free, it follows that v is non-adjacent to v_{2k+1}, v_2. We claim that for every pair of vertices $\{v_i, v_{i+1}\}$, with $i = 4, 5, \ldots, 2k-2$, vertex v is adjacent to exactly one of v_i, v_{i+1}. Clearly, since G is K_3-free, v has a non-neighbour in $\{v_i, v_{i+1}\}$. If v has no neighbours in $\{v_i, v_{i+1}\}$, then $G[v_1, v_2, v, v_{2k+1}, v_i, v_{i+1}]$ is isomorphic to $K_{1,3}+K_2$, a contradiction. Now suppose that v is adjacent to v_4. Then it follows that v is complete to $\{v_4, v_6, \ldots, v_{2k-2}\}$ and anticomplete to $\{v_5, v_7, \ldots, v_{2k-1}\}$. But then $G[v_{2k-2}, v, v_{2k-3}, v_{2k-1}, v_2, v_3]$ is isomorphic to $K_{1,3}+K_2$, a contradiction. Thus we may assume that v is adjacent to v_5. This implies that v is complete to $\{v_5, v_7, \ldots, v_{2k-1}\}$ and anticomplete to $\{v_4, v_6, \ldots, v_{2k-2}\}$. It follows that v is non-adjacent to v_{2k}, since G is K_3-free. But now $G[v_5, v_4, v_6, v, v_{2k}, v_{2k+1}]$ is isomorphic to $K_{1,3} + K_2$. This contradiction shows that $G = C$.

Now consider the case where $k = 3$ and let $v \in V(G) \setminus V(C)$ be adjacent to v_1. As before, v has exactly one neighbour in $\{v_4, v_5\}$. By symmetry, we may assume that v is adjacent to v_4. Hence v has no neighbours in $\{v_2, v_3, v_5, v_7\}$. Finally, observe that v is non-adjacent to v_6, since otherwise $G[v_6, v_5, v_7, v, v_2, v_3]$ would be isomorphic to $K_{1,3} + K_2$. Therefore we conclude that each vertex $v \in V(G) \setminus V(C)$ that is adjacent to some vertex $v_i \in V(C)$, is either complete to $\{v_i, v_{i+3}\}$ and anticomplete to $V(C) \setminus \{v_i, v_{i+3}\}$, or complete to $\{v_i, v_{i+4}\}$ and anticomplete to $V(C) \setminus \{v_i, v_{i+4}\}$ (here subscripts are taken modulo 7).

Let U_j denote the set of vertices at distance j from the cycle. We claim that:

- $|U_1| \leq 7$. Indeed, if $|U_1| > 7$, then there exist two vertices $z, z' \in U_1$ that are complete to $\{v_i, v_{i+3}\}$ (and thus anticomplete to $V(C) \setminus \{v_i, v_{i+3}\}$) for some value of i. Since G is K_3-free, z, z' must be non-adjacent. But then $G[v_i, z, z', v_{i+1}, v_{i+4}, v_{i+5}]$ is isomorphic to $K_{1,3} + K_2$, a contradiction.

- *Each vertex of U_1 has at most one neighbour in U_2.* Indeed, suppose a vertex $x \in U_1$ has two neighbours $y, z \in U_2$, and without loss of generality let x be complete to $\{v_i, v_{i+3}\}$ (and thus anticomplete to $V(C) \setminus \{v_i, v_{i+3}\}$). Since G is K_3-free, it follows that y, z are non-adjacent. But then $G[x, y, z, v_i, v_{i+4}, v_{i+5}]$ is isomorphic to $K_{1,3}+K_2$, a contradiction.

- *Each vertex of U_2 has at most one neighbour in U_3*, which can be proved by analogy with the previous claim.

- *For each $i \geq 4$, U_i is empty.* Indeed, assume without loss of generality that $U_4 \neq \emptyset$ and let u_4, u_3, u_2, u_1 be a path from U_4 to C with $u_j \in U_j$ and u_1 being adjacent to v_i. Then $G[v_i, v_{i-1}, v_{i+1}, u_1, u_3, u_4]$ is isomorphic to $K_{1,3} + K_2$, a contradiction.

From the above claims we conclude that $V(G) = V(C) \cup U_1 \cup U_2 \cup U_3$, $|U_3| \leq |U_2| \leq |U_1| \leq 7 = |V(C)|$, and therefore $|V(G)| \leq 28$. □

Thus Lemmas 3 and 4 and Fact 2 further reduce the problem to graphs containing a C_5.

Lemma 5. *If G is a connected $(K_3, S_{1,1,3})$-free graph containing a C_5, then the clique-width of G is bounded by a constant.*

Proof. Let G be a connected $(K_3, S_{1,1,3})$-free graph and let $C = v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ be an induced cycle of length five in G. If $G = C$ then the clique-width of G is at most 4 (Fact 1). Therefore we may assume that there exists at least one vertex $v \in V(G) \setminus V(C)$. Since G is K_3-free, v can be adjacent to at most two vertices of C, and if v has two neighbours in C, they must be non-consecutive vertices of the cycle. We denote the set of vertices in $V(G) \setminus V(C)$ that have exactly i neighbours in C by N_i, $i \in \{0, 1, 2\}$. Also, for $i = 1, \ldots, 5$, we let V_i denote the set of vertices in N_2 adjacent to $v_{i-1}, v_{i+1} \in V(C)$ (throughout the proof subscripts i are taken modulo 5). We call two different sets V_i and V_j *consecutive* if v_i and v_j are consecutive vertices of C, and *opposite* otherwise. Finally, we call V_i *large* if $|V_i| \geq 2$, and *small* otherwise. The proof of the lemma will be given through a series of claims.

(1) *Each V_i is an independent set.* This immediately follows from the fact that G is K_3-free.

(2) *N_0 is an independent set.* Indeed, suppose xy is an edge connecting two vertices $x, y \in N_0$, and, without loss of generality, let y be adjacent to a vertex $z \in N_1 \cup N_2$. Let $v_i \in V(C)$ be a neighbour of z. Since G is K_3-free, z is non-adjacent to x, v_{i-1}, v_{i+1}. But then $G[v_i, v_{i-1}, v_{i+1}, z, y, x]$ is isomorphic to $S_{1,1,3}$, a contradiction.

(3) *Any vertex $x \in N_1 \cup N_2$ has at most one neighbour in N_0.* Suppose $x \in N_1 \cup N_2$ is adjacent to $z, z' \in N_0$, and let $v_i \in V(C)$ be a neighbour of x. Since G is K_3-free, it follows that x is non-adjacent to v_{i-1}, v_{i+1}. Furthermore, x is adjacent to at most one of v_{i-2}, v_{i+2}. By symmetry we may assume that x is non-adjacent to v_{i-2}. But now $G[x, z, z', v_i, v_{i-1}, v_{i-2}]$ is isomorphic to $S_{1,1,3}$, a contradiction.

(4) $|N_1| \leq 5$. Indeed, if there are two vertices $x, x' \in N_1$ which are adjacent to the same vertex $v_i \in V(C)$, then $G[v_i, x, x', v_{i+1}, v_{i+2}, v_{i+3}]$ is isomorphic to $S_{1,1,3}$, a contradiction.

(5) *If V_i and V_j are opposite sets, then no vertex of V_i is adjacent to a vertex of V_j.* This immediately follows from the fact that G is K_3-free.

(6) *If V_i and V_j are consecutive, then every vertex $x \in V_i$ has at most one non-neighbour in V_j.* Suppose $x \in V_i$ has two non-neighbours $y, y' \in V_j$. By symmetry, we may assume that $j = i + 1$. But now, by Claim (1), $G[v_{i-3}, y, y', v_{i-2}, v_{i-1}, x]$ is isomorphic to $S_{1,1,3}$, a contradiction.

(7) *If V_i and V_j are two opposite large sets, then no vertex in N_0 has a neighbour in $V_i \cup V_j$.* Without loss of generality assume that $i = 1$ and $j = 4$, and suppose for a contradiction that a vertex $x \in N_0$ has a neighbour $y \in V_1$. If x is non-adjacent to some vertex $z \in V_4$, then $G[v_3, v_4, z, v_2, y, x]$ is isomorphic to $S_{1,1,3}$, a contradiction. Therefore x is complete to V_4. But now, by Claim (1), $G[x, z, z', y, v_2, v_1]$ with $z, z' \in V_4$ is isomorphic to $S_{1,1,3}$, a contradiction.

Since G is connected and N_0 is an independent set, every vertex of N_0 has a neighbour in $N_1 \cup N_2$. Let V_0 be the set of vertices in N_0, all of whose neighbours belong to the large sets V_i. Let G_0 be the subgraph of G induced by V_0 and the large sets. From Claims (2),(3) and (4), it follows that at most 25 vertices of G do not belong to G_0. Therefore, by Fact 3, the clique-width of G is bounded if and only if it is bounded for G_0. We may assume that G has at least one large set, since otherwise G_0 is empty. We will show that G_0 has bounded clique-width by examining all possible combinations of large sets.

Case 1: Suppose that for every large set V_i there is an opposite large set V_j. Then it follows from Claim (7) that $V_0 = \emptyset$. In order to see that G_0 has bounded clique-width, we complement the edges between every pair of consecutive large sets. By Claims (5) and (6), the resulting graph has maximum degree at most 2. From Fact 1 it follows that this graph is of bounded clique-width, and therefore, applying Fact 4, G_0 has bounded clique-width.

Case 1 allows us to assume that G contains a large set such that the opposite sets are small. Without loss of generality we let V_1 be large, and V_3 and V_4 be small. The rest of the proof is based on the analysis of the size of the sets V_2 and V_5.

Case 2: V_2 and V_5 are large. Then, by Claims (1), (2), (5), and (7), G_0 is a bipartite graph with bipartition $(V_1, V_2 \cup V_5 \cup V_0)$. Therefore by Fact 2, G_0 has bounded clique-width.

Case 3: V_2 and V_5 are small. Then by Claims (1) and (2), G_0 is a bipartite graph with bipartition (V_1, V_0), and therefore, by Fact 2, G_0 has bounded clique-width.

Case 4: V_2 is large and V_5 is small, i.e. G_0 is induced by $V_0 \cup V_1 \cup V_2$. Consider a vertex $x \in V_0$ that has a neighbour $y \in V_1$ and a neighbour $z \in V_2$. Then y and z are non-adjacent (since G is K_3-free) and therefore, by Claim (6), y is complete to $V_2 \setminus \{z\}$ and z is complete to $V_1 \setminus \{y\}$. From the K_3-freeness of G it follows that x is anticomplete to $(V_1 \cup V_2) \setminus \{y, z\}$.

Let V_0' denote the vertices of V_0 that have neighbours both in V_1 and V_2, and let V_i' $(i = 1, 2)$ denote the vertices of V_i that have neighbours in V_0'. Also, let $V_i'' = V_i - V_i'$ for $i = 0, 1, 2$, and $G_0' = G_0[V_0' \cup V_1' \cup V_2']$, $G_0'' = G_0[V_0'' \cup V_1'' \cup V_2'']$.

By Claim (3), V_0'' is anticomplete to $V_1' \cup V_2'$. Also, it follows from the above discussion that V_0' is anticomplete to $V_1'' \cup V_2''$, that V_1' is complete to V_2'', and that V_2' is complete to V_1''. Therefore by complementing the edges between V_1' and V_2'', and between V_2' and V_1'', we disconnect G_0' from G_0''. The graph G_0'' is a bipartite graph, since every vertex of V_0'' has neighbours either in V_1'' or in V_2'' but not in both. Thus it follows from Fact 2 that G_0'' has bounded clique-width. To see that G_0' has bounded clique-width, we complement the edges between V_1' and V_2'. This operation transforms G_0' into a collection of disjoint triangles. Therefore the clique-width of G_0' is bounded. Now it follows from Fact 4 that G_0 has bounded clique-width. □

Similarly to Lemma 5, one can prove the following result.

Lemma 6. *If G is a connected $(K_3, K_{1,3}+K_2)$-free graph containing a C_5, then the clique-width of G is bounded by a constant.*

Proof. The proof is similar to the proof of Lemma 5. Let G be a connected $(K_3, K_{1,3}+K_2)$-free graph and let $C = v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ be an induced cycle of length five in G. If $G = C$ then the clique-width of G is at most 4 (Fact 1). Therefore we may assume that there exists at least one vertex $v \in V(G) \setminus V(C)$. Since G is K_3-free, v can be adjacent to at most two vertices in C, and if v has two neighbours in C, they must be non-consecutive vertices of C. We denote the set of vertices in $V(G) \setminus V(C)$ that have exactly i neighbours in C by N_i, $i \in \{0, 1, 2\}$. Also, for $i = 1, \ldots, 5$, we let V_i denote the set of vertices in N_2 adjacent to $v_{i-1}, v_{i+1} \in V(C)$ (throughout the proof subscripts i are taken modulo 5). We call two different sets V_i and V_j *consecutive* if v_i and v_j are consecutive vertices of C, and *opposite* otherwise. Finally, we call V_i *large* if $|V_i| \geq 7$, and *small* otherwise. The proof of the lemma will be given through a series of claims.

(1) *Each V_i is an independent set.* This immediately follows from the fact that G is K_3-free.

(2) $|N_1| \leq 10$. Indeed, if there are three vertices $x, x', x'' \in N_1$ which are adjacent to the same vertex $v_i \in V(C)$, then $G[v_i, x, x', x'', v_{i+2}, v_{i+3}]$ is isomorphic to $K_{1,3} + K_2$, a contradiction (notice that x, x', x'' are pairwise non-adjacent since G is K_3-free).

(3) *If V_i and V_j are opposite sets, then no vertex of V_i is adjacent to a vertex of V_j.* This immediately follows from the fact that G is K_3-free.

(4) *If V_i and V_j are consecutive, then every vertex of V_i has at most two non-neighbours in V_j.* By symmetry, we may assume $j = i + 1$. Suppose $x \in V_i$ has three non-neighbours $y, y', y'' \in V_j$. Then by Claim (1), $G[v_{i+2}, y, y', y'', v_{i-1}, x]$ is isomorphic to $K_{1,3} + K_2$, a contradiction.

(5) *Each vertex $w \in N_0$ is adjacent to at most two vertices in a set V_i.* Indeed, if $w \in N_0$ were adjacent to three vertices $z, z', z'' \in V_i$, then by Claim (1), $G[w, z, z', z'', v_{i+2}, v_{i+3}]$ would be isomorphic to $K_{1,3} + K_2$, a contradiction.

(6) *N_0 induces a graph of vertex degree at most two. Moreover, if there exists at least one large set, then N_0 is an independent set.* If a vertex $w \in N_0$ has three neighbours $z, z', z'' \in N_0$,

then $G[w, z, z', z'', v_1, v_2]$ is isomorphic to $K_{1,3}+K_2$, since G is K_3-free. This contradiction proves the first part of the claim. To prove the second part, assume V_i is a large set and suppose that two vertices $w, w' \in N_0$ are adjacent. Since V_i is large, it follows from Claim (5) that there exist at least three vertices $z, z', z'' \in V_i$ which are anticomplete to $\{w, w'\}$. But now, by Claim (1), $G[v_{i-1}, z, z', z'', w, w']$ is isomorphic to $K_{1,3}+K_2$, a contradiction.

(7) *If V_i and V_j are two opposite large sets, then no vertex in N_0 has a neighbour in $V_i \cup V_j$.* Without loss of generality, assume that $i = 1$ and $j = 4$, and suppose for contradiction, that a vertex $w \in N_0$ has a neighbour $y \in V_1$. Since V_4 is large and since w is adjacent to at most two vertices in V_4 (Claim (5)), it follows that w has two non-neighbours $z, z' \in V_4$. But now, by Claim (1), $G[v_3, v_4, z, z', w, y]$ is isomorphic to $K_{1,3} + K_2$, a contradiction.

(8) *Any vertex $x \in N_1 \cup N_2$ has at most two neighbours in N_0.* Indeed, for any vertex $x \in N_1 \cup N_2$ there exist at least two consecutive vertices of C non-adjacent to x. These two vertices together with x and any three neighbours of x in N_0 would induce a $K_{1,3}+K_2$.

From Claim (6) and Fact 1 we know that the clique-width of $G[N_0]$ is at most 4. Therefore, if all sets V_i are small, then G has bounded clique-width, which follows from Claim (2) and Fact 3.

From now on, we assume that there exists at least one large set V_i. This implies that N_0 is an independent set (Claim (6)). Since G is connected, every vertex of N_0 has a neighbour in $N_1 \cup N_2$. Let V_0 be the set of vertices in N_0, all of whose neighbours belong to the large sets V_i. Let G_0 be the subgraph of G induced by V_0 and the large sets. From Claims (2) and (8), it follows that the size of $V(G) \setminus V(G_0)$ is bounded. Therefore, by Fact 3, the clique-width of G is bounded if and only if it is bounded for G_0. We will show that G_0 has bounded clique-width by examining all possible combinations of large sets.

Case 1: Suppose that for every large set V_i there is an opposite large set V_j. Then it follows from Claim (7) that $V_0 = \emptyset$. Let V_{i-1} and V_{i+1} be large sets. We claim that every vertex $x \in V_i$ is complete to $V_{i-1} \cup V_{i+1}$. For suppose not: let $y \in V_{i+1}$ be a non-neighbour of x. Since V_{i-1} is large, it follows from Claim (4) that x has at least two neighbours $z, z' \in V_{i-1}$. But now, by Claims (1) and (3), $G[x, z, z', v_{i-1}, v_{i+2}, y]$ is isomorphic to $K_{1,3} + K_2$, a contradiction. In order to see that G_0 is of bounded clique-width, we complement the edges between every pair of consecutive large sets. From Claim (4) and the discussion above, it follows that the resulting graph is of vertex degree at most 2. From Fact 1 it follows that this graph has bounded clique-width, and therefore applying Fact 4, G_0 has bounded clique-width.

Case 1 allows us to assume that G contains a large set such that the opposite sets are small. Without loss of generality we let V_1 be large, and V_3 and V_4 be small. The rest of the proof is based on the analysis of the size of the sets V_2 and V_5.

Case 2: V_2 and V_5 are large. Then by Claims (1),(3),(6) and (7), G_0 is a bipartite graph with bipartition $(V_1, V_2 \cup V_5 \cup V_0)$. Therefore by Fact 2, G_0 has bounded clique-width.

Case 3: V_2 and V_5 are small. Then, by Claims (1) and (6), G_0 is a bipartite graph with bipartition (V_1, V_0), and therefore, by Fact 2, G_0 has bounded clique-width.

Case 4: V_2 is large and V_5 is small, i.e. G_0 is induced by $V_0 \cup V_1 \cup V_2$. Consider a vertex $w \in V_0$ that is adjacent to some vertex $x \in V_1$ (resp. $y \in V_2$). We claim that

(9) *w is complete to all the non-neighbours of x in V_2 (resp. of y in V_1).* By symmetry we let x belong to V_1 and for contradiction, suppose that w is non-adjacent to a non-neighbour $z \in V_2$ of x. Since V_1 is large, it follows from Claims (4) and (5) that V_1 contains three vertices x_1, x_2, x_3 adjacent to z and non-adjacent to w. But now, by Claim (1), $G_0[z, x_1, x_2, x_3, x, w]$ is isomorphic to $K_{1,3} + K_2$, a contradiction.

In order to see that G_0 has bounded clique-width, we complement the edges between V_1 and V_2. Let us denote the resulting graph by G'_0. From Facts 4 and 5, it follows that it is enough to show that each connected component of G'_0 has bounded clique-width. Let C^* be a component of G'_0. If C^* has maximum vertex degree at most two, then C^* has bounded clique-width by Fact 1. So we may assume that C^* contains a vertex x of degree at least three.

First suppose that $x \in V_1 \cup V_2$. By symmetry, we may assume $x \in V_1$. We know that in the graph G'_0 vertex x has at most two neighbours in V_0 (Claim (8)) and at most two neighbours in V_2 (Claim (4)). Therefore, x is adjacent to some vertex $y \in V_2$ and to some vertex $w \in V_0$ in the graph G'_0. Since in the graph G_0 vertex y is a non-neighbour of x, it follows from Claim (9) that y, w are adjacent. Repeating this argument, we conclude that w is complete to $V(C^*) \cap (V_1 \cup V_2)$. By Claim (5), we obtain that $|V(C^*) \cap (V_1 \cup V_2)| \leq 4$. Since each vertex in $V_1 \cup V_2$ has at most two neighbours in V_0 (Claim (8)), we finally conclude that $|V(C^*)| \leq 12$ and therefore the clique-width of C^* is at most 12.

Now suppose that $x \in V_0$ and all vertices of C^* in $V_1 \cup V_2$ have degree at most 2. Since V_0 is an independent set, all neighbours of x are in $V_1 \cup V_2$. Let z, z', z'' denote three neighbours of x. Without loss of generality we may assume that $z, z' \in V_1$ and $z'' \in V_2$ (Claim (5)). Since G is K_3-free, it follows that in C^*, vertex z'' is adjacent to both z, z'. But now $z'' \in V_2$ has degree at least three, contradicting our assumption. □

From Lemmas 3, 4, 5, and 6, we derive the main result of this section.

Theorem 3. *The clique-width of $(K_3, S_{1,1,3})$-free graphs and $(K_3, K_{1,3} + K_2)$-free graphs is bounded by a constant and therefore the* VERTEX COLOURING *problem is polynomial-time solvable in these classes of graphs.*

5. $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$-free graphs

In this section we prove polynomial-time solvability of the problem in the class of $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$-free graphs. It is not difficult to see that both $S_{1,2,3}$ and $S_{1,1,2} + P_2$ contain $P_4 + P_2$ as an induced subgraph. Therefore, our result generalizes a recent solution of the problem in the class of $(K_3, P_4 + P_2)$-free graphs [6]. Our result is based on a sequence of lemmas.

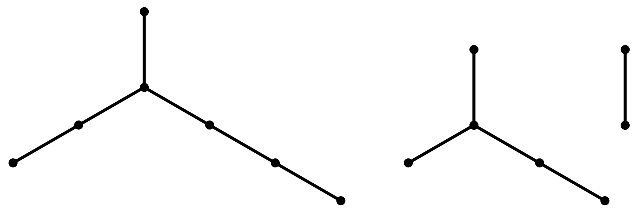


Figure 1: The graphs $S_{1,2,3}$ and $S_{1,1,2} + P_2$.

Lemma 7. *Let G be a $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$-free graph. Then the chromatic number of G is at most 4 and a 4-colouring of G can be found in polynomial time.*

Proof. We may assume G is connected and contains an edge ab. Note that since G is K_3-free, $G[N(a) \cup N(b)]$ is a bipartite graph. Let $X = V(G) \setminus (N(a) \cup N(b))$. We will now show that $G[X]$ is bipartite, in which case G is 4-colourable. Indeed, suppose for contradiction that $G[X]$ is not bipartite. Then, since it is K_3-free, it must contain an induced odd cycle $v_1 - \cdots - v_{2k+1} - v_1$ with $k \geq 2$.

Let $w_1, w_2, \ldots, w_q$ be a shortest path from this cycle to a, with $w_q = a$ and $w_1 = v_i$ for some $i \in \{1, \ldots, 2k+1\}$. If $q = 3$ then $w_2 \in N(a) \setminus \{b\}$. In this case let $w_4 = b$.

Vertex w_2 cannot be adjacent to v_{i-1} or v_{i+1} since G is K_3-free. But now w_2 must be adjacent to v_{i+2} otherwise $G[v_i, v_{i-1}, v_{i+1}, v_{i+2}, w_2, w_3, w_4]$ would be isomorphic to $S_{1,2,3}$. Since vertex v_i was chosen arbitrarily, we can repeat this argument k times to find that w_2 must be adjacent to 2 consecutive vertices in the cycle. But this cannot happen, since G is K_3-free. This contradiction completes the proof. □

Lemma 7 reduces VERTEX COLOURING in the class of $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$-free graphs to 3-COLOURABILITY. We now prove some lemmas to help solve this problem.

Lemma 8. *Let G be a connected $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$-free graph containing an odd induced cycle C of length at least 9. Then $G = C$.*

Proof. Let $C = v_1 - v_2 - \cdots - v_{2k+1}$ be an induced odd cycle of length at least 9 in G. Let x be adjacent to some vertex v_i on C. Then obviously it is adjacent to neither v_{i-1} nor v_{i+1}, since the graph is K_3-free. If in addition it is non-adjacent to v_{i-2}, then the subgraph of G induced by $v_i, v_{i+1}, v_{i-1}, v_{i-2}, x, v_{i+3}, v_{i+4}$ is either isomorphic to $S_{1,2,3}$ (if x has a neighbour in $\{v_{i+3}, v_{i+4}\}$) or to $S_{1,1,2} + P_2$ (if x has no neighbour in $\{v_{i+3}, v_{i+4}\}$). Therefore, x is adjacent to v_{i-2}. But v_i was an arbitrary vertex of the cycle, so as in the proof of Lemma 7, by iterating this argument k times, we find that G must contain a K_3, which is a contradiction. □

Lemma 9. *Let G be a connected $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$-free graph containing an induced cycle C of length 7. Then C is dominating.*

Proof. Suppose G is connected and contains an induced cycle $C = v_1 - v_2 - v_3 - v_4 - v_6 - v_7 - v_1$. If C is not dominating then there must exist vertices x and y such that y is not adjacent to any vertex of the cycle and x is adjacent to both y and some vertex of the cycle, say v_1. x is non-adjacent to v_2 and v_7 since G is K_3-free. So x must be adjacent to v_4 or v_5, otherwise $G[v_1, v_2, v_7, x, y, v_4, v_5]$ would be isomorphic to $S_{1,1,2} + P_2$. Without loss of generality, assume that x is adjacent to v_4. Since G is K_3-free, x is non-adjacent to v_3 and v_5. Now, x must be adjacent to v_6, otherwise $G[v_1, x, v_2, v_3, v_7, v_6, v_5]$ would be isomorphic to $S_{1,2,3}$. But then $G[v_6, v_5, v_7, x, y, v_2, v_3]$ is isomorphic $S_{1,1,2} + P_2$. This contradiction leads to the conclusion that such vertices x and y cannot exist and thus C is dominating. □

Let B be a connected bipartite induced subgraph of a graph G with at least 3 vertices. We say that the vertices in one part of B are *odd* and those in the other part are *even*. If two vertices are in the same part of B, we say they have the same *parity*. The following lemma is an easy observation.

Lemma 10. *Suppose a graph G has a connected bipartite induced subgraph B, $|V(B)| \geq 3$, and that for every vertex $x \notin B$, x is either complete or anticomplete to the odd vertices in B and is either complete or anticomplete to the even vertices in B. Then all vertices of B except any two adjacent vertices can be deleted from G and the new graph has a 3-colouring if and only if G does.*

Lemma 11. *Let G be a connected $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$-free graph containing an induced cycle C of even length $k \geq 8$. If a vertex x has a neighbour on the cycle, then x is adjacent to all vertices of the same parity with respect to C.*

Proof. Let x be adjacent to a vertex v_i on the cycle. Then obviously it is adjacent to neither v_{i-1} nor v_{i+1}, since the graph is K_3-free. If it is also non-adjacent to v_{i-2}, then the subgraph of G induced by $v_i, v_{i+1}, v_{i-1}, v_{i-2}, x, v_{i+3}, v_{i+4}$ is either isomorphic to $S_{1,2,3}$ (if x has a neighbour in $\{v_{i+3}, v_{i+4}\}$) or to $S_{1,1,2}+P_2$ (if x has no neighbour in $\{v_{i+3}, v_{i+4}\}$). Therefore, x is adjacent to v_{i-2}. Since vertex v_i was chosen arbitrarily, x must be adjacent to all vertices which have the same parity as v_i. □

Notice that we may assume that G satisfies the following property:

(*) for any two non-adjacent vertices u and v, there exists a neighbour of u which is non-adjacent to v and there exists a neighbour of v which is non-adjacent to u.

Indeed if a pair of vertices does not satisfy Property (*), then the neighbourhood of one of the vertices u, v is included in the neighbourhood of the other. In this case the first vertex can be deleted from the graph G and it is easy to see that the new graph has a 3-colouring if and only if the original graph does.

Lemma 12. *Let G be a $(K_3, S_{1,2,3}, S_{1,1,2}+P_2)$-free graph with Property (*) and let P a be an induced path in G with at least 8 vertices. If a vertex x is adjacent to a vertex of degree 2 in P, then x is adjacent to all vertices of the same parity in P.*

Proof. Let P be the path $v_1 - v_2 - \cdots - v_k$ with $k \geq 8$. Suppose, for contradiction that x has a neighbour v_i with $2 < i \leq k-1$, such that x is not adjacent to v_{i-2} (the case where x is not adjacent to v_{i+2} is symmetric). Clearly x cannot be adjacent to v_{i-1} or v_{i+1} since G is K_3-free.

If $i < k-3$, then $G[v_i, x, v_{i+1}, v_{i-1}, v_{i-2}, v_{i+3}, v_{i+4}]$ is either isomorphic to $S_{1,2,3}$ (if x has a neighbour in $\{v_{i+3}, v_{i+4}\}$) or to $S_{1,1,2}+P_2$ (if x has no neighbour in $\{v_{i+3}, v_{i+4}\}$). Thus we may assume $i \geq k-3$.

But now if $k \geq 9$ or $k = 8, i \geq k-2$, then $G[v_i, x, v_{i+1}, v_{i-1}, v_{i-2}, v_{i-4}, v_{i-5}]$ is either isomorphic to $S_{1,2,3}$ (if x has a neighbour in $\{v_{i-5}, v_{i-4}\}$) or to $S_{1,1,2}+P_2$ (if x has no neighbour in $\{v_{i-5}, v_{i-4}\}$). This contradiction proves that if $k \geq 9$ or $k = 8, i \neq k-3$, then x must be adjacent to v_{i-2}.

Now let us analyse the case when $k = 8$ and $i = k-3 = 5$. By the above argument for $k = 8, i = 3$, we conclude that x is adjacent to v_7. Since G satisfies Property (*), vertex v_6 must have a neighbour y which is non-adjacent to x. From the first part of the proof, we know that y must be adjacent to v_8 and v_4 and therefore to v_2. But x cannot be adjacent to v_2, since then it would have to be adjacent to v_4, contradicting the fact that G is K_3-free. If x is adjacent to v_1, then $G[y, v_6, v_8, v_4, v_3, v_1, x]$ is an $S_{1,1,2}+P_2$. If x is non-adjacent to v_1, then $G[y, v_4, v_2, v_1, v_6, v_7, x]$ is an $S_{1,2,3}$. This final contradiction completes the proof of the lemma. □

We may also assume that G satisfies the following property (otherwise we can apply Lemma 10):

(**) For any induced path P in G on 6 or 7 vertices, there is a vertex $x \in V(G) \setminus V(P)$ which has both a neighbour and a non-neighbour of the same parity in P.

Let $\mathcal{G}$ denote the subclass of $(K_3, S_{1,2,3}, S_{1,1,2}+P_2, C_7, C_8, P_8)$-free graphs with Properties (*) and (**).

Lemma 13. *Any connected graph $G \in \mathcal{G}$ containing an induced P_6 has chromatic number at most 3 and a 3-colouring of G can be found in polynomial time.*

Proof. Let Q denote the graph obtained from a C_6 by adding a vertex which has exactly one neighbour on the cycle. We split the proof into two cases.

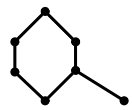


Figure 2: The graph Q.

Case 1: G contains an induced subgraph isomorphic to Q. Say Q is induced by vertices $a, b, c, d, e, f, g \in V(G)$ where $a-b-c-d-e-f-a$ is a chordless cycle and the only neighbour of g on the cycle is e. The vertices of G outside the set $\{a, b, c\}$ can be partitioned into at most 5 non-empty subsets in the following way:

V_a is the set of vertices adjacent to a and non-adjacent to b and c,

V_b and V_c are defined by analogy with V_a,

V_{ac} is the set of vertices adjacent to a and c and non-adjacent to b,

W is the set of vertices anticomplete to $\{a, b, c\}$.

Note that V_a, V_b, V_c and V_{ac} are independent sets, since G is K_3-free. We will split W into independent sets. We will investigate the possible edges between all these independent sets and finally, we will show how to obtain a 3-colouring of G.

(i) *For any edge uv in $G[W \setminus \{e, g\}]$, at least one of u, v has a neighbour in $\{e, g\}$.* Suppose not. Then since $G[e, d, g, f, a, u, v]$ cannot be isomorphic to $S_{1,1,2} + P_2$, it follows that at least one of u, v is adjacent to one of d, f. Without loss of generality, we may assume that u is adjacent to f. But then $G[f, u, e, g, a, b, c]$ would be an $S_{1,2,3}$, a contradiction.

We may now partition W into two sets W_0 and W_1, where $G[W_1]$ is the connected component of $G[W]$ containing e and g. Notice that $W_0 = W \setminus W_1$ is an independent set (by (i)).

(ii) *For every edge uv in $G[W_1]$, exactly one of u, v has a neighbour in $\{d, f\}$.* This is trivially true for every edge incident to e. Now consider an edge ug in $G[W_1]$, where $u \neq e$. Notice that g is non-adjacent to d, f. If u is non-adjacent to d, f, then $G[e, f, g, u, d, c, b]$ is isomorphic to $S_{1,2,3}$, a contradiction. Thus u is adjacent to at least one of d, f. Now consider an edge uv in $G[W_1]$ such that $u, v \neq e, g$. Since G is (K_3, C_7)-free, at most one of u, v can have a neighbour in $\{d, f\}$. Suppose that u, v are non-adjacent to d, f. From the previous case, we may assume that u, v are non-adjacent to g. It follows from (i) that one of u, v is adjacent to e. Without loss of generality we may assume that u is adjacent to e. But then $G[e, g, u, v, f, a, b]$ would be isomorphic to $S_{1,2,3}$, which is a contradiction.

(iii) *$G[W_1]$ is complete bipartite.* First let us show that every vertex $u \in W_1 \setminus \{e, g\}$ is adjacent to exactly one of e, g. Clearly no vertex can be adjacent to both e and g since G is K_3-free. Now let $u \in W_1 \setminus \{e, g\}$ and suppose that u is non-adjacent to e, g. If u is adjacent to f (resp. d) then $G[f, u, e, g, a, b, c]$ (resp. $G[d, u, e, g, c, b, a]$) is isomorphic to $S_{1,2,3}$, a contradiction. Now let v be a neighbour of u in W_1. It follows from (ii) that v is adjacent to at least one of d, f. We may assume that v is adjacent to f. But now $G[f, e, v, u, a, b, c]$

is isomorphic to $S_{1,2,3}$, a contradiction. Thus every vertex $u \in W_1 \setminus \{e, g\}$ is indeed adjacent to exactly one of e, g. Let $W_1(g)$ be the vertices in W_1 which are adjacent to e and let $W_1(e)$ be the vertices adjacent to g. Notice that $e \in W_1(e)$ and $g \in W_1(g)$. Now we only need to show that $W_1(e)$ is complete to $W_1(g)$. Suppose not. Let $w \in W_1(g)$ and $w' \in W_1(e)$ be non-adjacent. Since g is non-adjacent to d, f, it follows from (ii) that w' is adjacent to at least one of d, f. Without loss of generality we may assume that w' is adjacent to f. But now $G[f, w', e, w, a, b, c]$ is isomorphic to $S_{1,2,3}$, a contradiction.

Notice that since e is adjacent to d, f, (ii) implies that $W_1(g)$ must be anticomplete to $\{d, f\}$ and that every vertex in $W_1(e)$ is adjacent to at least one of d, f.

(iv) *Let $v \in V_a \cup V_c$ with $v \neq d, f$. Then for every edge ww' in $G[W_1]$, exactly one of w, w' is adjacent to v.* Suppose not. Without loss of generality, assume $v \in V_c$, $w \in W_1(e)$ and $w' \in W_1(g)$. But then $G[c, v, b, a, d, w, w']$ is isomorphic to $S_{1,2,3}$ (if $dw \in E(G)$) or to $S_{1,1,2} + P_2$ (if $dw \notin E(G)$), which is a contradiction.

(v) *There exist no two vertices $u, v \in W_1(e)$ such that $uf, vd \in E(G)$ and $ud, vf \notin E(G)$.* Suppose, for contradiction, that such two vertices exist. Notice that $u, v \neq e$. But then $G[d, v, c, b, e, f, u]$ is isomorphic to $S_{1,2,3}$, a contradiction.

Thus either d or f is complete to $W_1(e)$. Without loss of generality, we may assume f is complete to $W_1(e)$. Then by (iii) and (iv) it follows that we may partition V_a into $V_a = V_a^1 \cup V_a^2$ such that V_a^1 is complete to $W_1(e)$ and anticomplete to $W_1(g)$ and V_a^2 is complete to $W_1(g)$ and anticomplete to $W_1(e)$. From (iii) and (iv) it also follows that we may partition V_c into $V_c = V_c^1 \cup V_c^2$ such that every vertex in V_c^1 has a neighbour in $W_1(e)$ and is anticomplete to $W_1(g)$ and every vertex in V_c^2 has a neighbour in $W_1(g)$ and is anticomplete to $W_1(e)$. Since G is K_3-free, V_a^1 must be anticomplete to V_c^1 and V_a^2 must be anticomplete to V_c^2.

(vi) *W_0 is anticomplete to $V_a \cup V_c$.* Let $u \in W_0$ and suppose that u is adjacent to some vertex v in $V_a \cup V_c$. Consider an edge ww' in $G[W_1]$. It follows from (iv) that exactly one vertex of w, w' is adjacent to v. We may assume without loss of generality that w is adjacent to v. But now $G[v, u, w, w', a, b, c]$ is isomorphic to $S_{1,2,3}$, a contradiction.

(vii) *$W_1(g)$ and W_0 have no common neighbours in V_{ac}.* Suppose that $w \in W_1(g)$ and $u \in W_0$ have a common neighbour $v \in V_{ac}$. Since G is K_3-free, e is non-adjacent to v. But then $G[v, u, a, b, w, e, d]$ is isomorphic to $S_{1,2,3}$, a contradiction.

Let X denote the subset of vertices of V_{ac} that have a neighbour in $W_1(g)$ and let Y denote the remaining vertices of V_{ac}. Notice that X is anticomplete to $W_1(e)$ since G is K_3-free. From the above and the fact that G is K_3-free, we conclude that each of the following three sets is independent: $V_a^2 \cup V_c^2 \cup W_1(e) \cup W_0 \cup \{b\} \cup X$, $V_a^1 \cup V_c^1 \cup W_1(g) \cup Y$, $V_b \cup \{a, c\}$. Therefore G is 3-colourable and such a colouring can be found in polynomial time.

Case 2: G contains no induced subgraph isomorphic to Q. Suppose that the vertices a, b, c, d, e, f induce a P_6 with edges $\{ab, bc, cd, de, ef\}$ (we know that G contains an induced P_6). The vertices outside the set $\{b, c, d, e\}$ can be partitioned into at most 8 non-empty sets as follows:

V_b is the set of vertices adjacent to b and non-adjacent to c, d, e,

V_c, V_d, V_e are defined by analogy with V_b,

V_{bd} is the set of vertices adjacent to b and d and non-adjacent to c and e,

V_{ce} and V_{be} are defined by analogy with V_{bd},

W is the set of vertices anticomplete to $\{b, c, d, e\}$.

(i) *V_b is anticomplete to V_e.* Note that $a \in V_b$ and $f \in V_e$. We know that $af \notin E(G)$. Suppose a has a neighbour $u \in V_e \setminus \{f\}$. Then $G[a, b, c, d, e, u, f]$ is isomorphic to Q, a contradiction. Therefore a is anticomplete to V_e. Now suppose that there exist two adjacent vertices $u \in V_b \setminus \{a\}, v \in V_e$. Then $G[b, c, d, e, v, u, a]$ is isomorphic to Q. This contradiction shows that V_b is anticomplete to V_e.

(ii) *Every vertex in W is either complete to V_b (resp. V_e) or anticomplete to V_b (resp. V_e).* Suppose there exists a vertex $w \in W$ which is adjacent to some vertex $u \in V_b$ and non-adjacent to some other vertex $v \in V_b$. Then $G[b, v, u, w, c, d, e]$ is isomorphic to $S_{1,2,3}$, a contradiction. Thus the claim holds for V_b and by symmetry we conclude that it holds for V_e as well.

(iii) *No vertex in W is complete to both V_b and V_e.* Suppose a vertex $w \in W$ is complete to $V_b \cup V_e$. Then $G[a, b, c, d, e, f, w]$ is isomorphic to C_7, a contradiction.

It follows from the above that we may partition W into three sets W_b, W_e, W_0, where W_b is complete to V_b and anticomplete to V_e, W_e is complete to V_e and anticomplete to V_b, and W_0 is anticomplete to $V_b \cup V_e$. Notice that W_b and W_e are both independent sets.

(iv) *At most one of W_b, W_e is nonempty.* Indeed if both W_b and W_e are nonempty, say $u \in W_b$ and $v \in W_e$, then $G[u, a, b, c, d, e, f, v]$ is either isomorphic to C_8 or P_8, a contradiction.

It follows from (iv) that we may assume without loss of generality that $W_e = \emptyset$. Thus W is anticomplete to V_e. Furthermore, $|W_b| \leq 1$, since if $u, v \in W_b$, then $G[a, u, v, b, c, e, f]$ is isomorphic to $S_{1,1,2} + P_2$, a contradiction.

(v) *W is an independent set.* Suppose W contains an edge uv and that $u \in W_b$. Since G is K_3-free, it follows that v is non-adjacent to a. But now $G[v, u, a, b, c, d, e, f]$ is isomorphic to P_8. This contradiction shows that neither u nor v has neighbours in V_b, hence $u, v \in W_0$.

We let P denote either the induced path $P_6 = \{ab, bc, cd, de, ef\}$ (if $W_b = \emptyset$) or the induced path $P_7 = \{ya, ab, bc, cd, de, ef\}$ (if $W_b = \{y\}$). We label the vertices of P by natural numbers $1, 2, \ldots, 6$ or $1, 2, \ldots, 7$ and let k be the number of vertices in P.

Suppose a vertex z outside P has a neighbour in P. Then it must be adjacent to a vertex i of degree 2 in P. Note that W_0 and P are anticomplete, so $z \neq u, v$.

This implies that z is adjacent to $i-2$ (if $i > 2$), since otherwise $G[i, i+1, i-1, i-2, z, u, v]$ induces either an $S_{1,2,3}$ (if z has a neighbour in $\{u, v\}$) or an $S_{1,1,2} + P_2$ (if z has no neighbour in $\{u, v\}$). Similarly z must be adjacent to $i + 2$ if $i < k - 1$. As a result z is adjacent to all vertices of the same parity in P. Therefore, if W is not an independent set, then G does not have Property (**). This contradiction implies that W is an independent set.

(vi) *W_b is anticomplete to V_d.* Let $W_b = \{y\}$. Suppose that y is adjacent to $u \in V_d$. Then $G[a, b, c, d, u, y, e]$ is isomorphic to Q, a contradiction.

(vii) W_0 *is anticomplete to* $V_c \cup V_d$. By symmetry it is enough to show that W_0 is anticomplete to V_c. Suppose that a vertex $w \in W_0$ is adjacent to some vertex $u \in V_c$. Then u must be adjacent to f otherwise $G[c, b, u, w, d, e, f]$ would be isomorphic to $S_{1,2,3}$, a contradiction. Now we claim that u is adjacent to a. Suppose not, then $G[u, w, f, e, c, b, a]$ would be isomorphic to $S_{1,2,3}$, a contradiction. But now $G[u, w, a, b, f, e, d]$ is isomorphic to $S_{1,2,3}$, a contradiction.

(viii) *One of* W_b, V_{be} *is empty.* Indeed, suppose $W_b = \{y\}$ and $u \in V_{be}$. If y is non-adjacent to u then $G[b, c, a, y, u, e, f]$ is isomorphic to $S_{1,2,3}$, a contradiction. On the other hand, if y is adjacent to u, then $G[e, f, d, c, u, y, a]$ is isomorphic to $S_{1,2,3}$, a contradiction.

(ix) *If* $W_b = \emptyset$*, then* G *is* 3*-colourable.* First, suppose that W_0 is anticomplete to V_{be}. Then it is easy to see that the following are independent sets: $W_0 \cup V_b \cup V_e \cup V_{be} \cup \{c\}$, $V_{bd} \cup V_d \cup \{e\}$, $\{b, d\} \cup V_{ce} \cup V_c$. So we may now assume that there exists a vertex $w \in W_0$ which has a neighbour $v \in V_{be}$. We claim that v must be complete to $V_c \cup V_d$. Suppose that v is non-adjacent to some vertex $u \in V_c$. Then f is adjacent to u, since otherwise $G[v, w, e, f, b, c, u]$ would be isomorphic to $S_{1,2,3}$, a contradiction. But now $G[c, d, u, f, b, v, w]$ is isomorphic to $S_{1,2,3}$, a contradiction. Thus v is complete to V_c and by symmetry we conclude that v is complete to V_d as well. Hence V_c and V_d are anticomplete. Now we obtain a 3-colouring as follows: $V_b \cup V_{be} \cup V_{bd} \cup \{c\}$, $\{b, e\} \cup V_c \cup V_d \cup W_0$, $\{d\} \cup V_e \cup V_{ce}$.

It follows from (ix) that we may now assume that $W_b = \{y\}$ and hence $V_{be} = \emptyset$. We claim that V_e is complete to V_d. Suppose some vertex $u \in V_d$ is non-adjacent to some vertex $v \in V_e$. Then u must be adjacent to a, otherwise $G[d, u, e, v, c, b, a]$ is isomorphic to $S_{1,2,3}$, a contradiction. But now $G[d, c, e, v, u, a, y]$ is isomorphic to $S_{1,2,3}$, a contradiction. Thus V_e is complete to V_d. This implies that V_b is anticomplete to V_d. Indeed if a vertex $u \in V_b$ is adjacent to some vertex $v \in V_d$, then $G[u, y, b, c, v, f, e]$ is isomorphic to $S_{1,2,3}$, a contradiction. Now we obtain a 3-colouring as follows: $V_b \cup V_{bd} \cup V_d \cup \{c, e\}$, $\{b, d\} \cup V_e \cup W$, $V_{ce} \cup V_c$.

This completes the proof that any connected graph $G \in \mathcal{G}$ containing an induced P_6 has chromatic number at most 3. From the above, it is easy to see that a 3-colouring of G can be found in polynomial time. □

Theorem 4. *The* VERTEX COLOURING *problem is solvable in polynomial time in the class of* $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$*-free graphs.*

Proof. Since we can solve the problem component-wise in G, we may assume that G is connected. It follows from Lemmas 2, 7, 8 and 9 that the problem reduces to 3-COLOURABILITY of $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$-free graphs which contain no odd induced cycle of length at least 7. Also, we only need to consider graphs that satisfy Property (*). Lemmas 10, 11 and 12 further reduce the problem in polynomial time to those graphs that contain no induced paths or induced even cycles of length at least 8. The reduction is as follows:

- Check if G contains a P_8 or C_8. If G contains a C_8 apply Lemmas 10 and 11. If G contains a P_8 extend it to a maximal (with respect to set inclusion) induced path P. This can obviously be done in polynomial time. If there is a vertex which creates a cycle with P, by Lemma 11, we can apply Lemma 10. Otherwise, every vertex of G which has a neighbour on P must be adjacent to a vertex of degree 2 in P, in which case Lemma 12 tells us we can apply Lemma 10.

The above procedure further reduces the problem to 3-COLOURABILITY of $(K_3, S_{1,2,3}, S_{1,1,2} + P_2)$-free graphs with Property (*) that are (C_7, C_8, P_8)-free. Finally, if G does not satisfy

Property (**), we can find a suitable path on 6 or 7 vertices and apply Lemma 10. We may therefore assume G satisfies Property (**).

Note that all of the above reductions work in polynomial time and either solve the 3-COLOURABILITY problem or delete vertices from the graph, so at most $|V(G)|$ such reductions can be applied. We may now assume that G is a connected $(K_3, S_{1,2,3}, S_{1,1,2} + P_2, C_7, C_8, P_8)$-free graph satisfying Properties (*) and (**), i.e. $G \in \mathcal{G}$.

Now if G is P_6-free, we can solve the 3-COLOURABILITY problem in polynomial time by Theorem 1 and if G is not P_6-free, we can solve the problem in polynomial time using Lemma 13. This completes the proof. □

6. Further results

In this section we prove a few additional results. The first two results deal with graph classes $Free(K_3, F)$ where F is a "big" forest of simple structure.

Theorem 5. *For every fixed m, the* VERTEX COLOURING *problem is polynomial-time solvable in the class $Free(K_3, mK_2)$.*

Proof. Obviously, if a graph G is k-colourable, then it admits a k-colouring in which one of the colour classes is a maximal independent set.

It is known that for every fixed m the number of maximal independent sets in the class $Free(mK_2)$ is bounded by a polynomial [1] and all of them can be found in polynomial time [37]. Therefore, given a mK_2-free graph G, we can solve the 3-COLOURABILITY problem for G by generating all maximal independent sets and solving 2-COLOURABILITY for the remaining vertices of the graph. Then by induction on k, we conclude that for any fixed k the k-COLOURABILITY problem can be solved in the class $Free(mK_2)$ in polynomial time. Since the chromatic number of (K_3, mK_2)-free graphs is bounded by $2m - 2$ (see e.g. [3]), the VERTEX COLOURING problem is polynomial-time solvable in the class $Free(K_3, mK_2)$ for any fixed m. □

Theorem 6. *For every fixed m, the* VERTEX COLOURING *problem is polynomial-time solvable in the class $Free(K_3, P_3 + mK_1)$.*

Proof. To prove the theorem, we will show that for any fixed m, graphs in the class $Free(K_3, P_3 + mK_1)$ are either bounded in size, or they are 3-colourable and a 3-colouring can be found in polynomial time.

Let G be a $(K_3, P_3 + mK_1)$-free graph. We start by finding a maximum independent set in G. For each fixed m, this problem is solvable in polynomial time, which can easily be seen by induction on m. Let S be a maximum independent set in G. Let R denote the remaining vertices of G, i.e. $R = V(G) - S$. We may assume that R contains an induced odd cycle $C = v_1 - v_2 - \cdots - v_p - v_1$ with $p \geq 5$. Since S is a maximum independent set, each vertex of C has at least one neighbour in S. Let us call a vertex $v_i \in V(C)$ strong if it has at least 2 neighbours in S and weak otherwise. Since C is an odd cycle, it has either two consecutive weak vertices or two consecutive strong vertices.

If C has two consecutive weak vertices, say v_1, v_2, then jointly they are adjacent to two vertices of S, say v_1 is adjacent to s_1, and v_2 is adjacent to s_2, and therefore, they have $|S| - 2$ common non-neighbours in S. If $|S| - 2 \geq m$, then s_1, v_1, v_2 together with m vertices in $S \setminus \{s_1, s_2\}$ induce a subgraph isomorphic to $P_3 + mK_1$, a contradiction. Therefore $|S| < m + 2$. But then the number of vertices in G is bounded by the Ramsey number $R(3, m + 2)$, since G is K_3-free and contains no independent set of size $m + 2$.

Now suppose C has two consecutive strong vertices, say v_1, v_2. Since the graph is (P_3+mK_1)-free, every strong vertex has at most $m-1$ non-neighbours in S, and since the graph is K_3-free, consecutive vertices of C cannot have common neighbours. Therefore each of v_1 and v_2 has at most $m-1$ neighbours in S. But then $|S| < 2m-1$ and hence the number of vertices of G is bounded by the Ramsey number $R(3, 2m-1)$ by the same argument as before.

Thus, if R has an odd cycle, then the number of vertices in G is bounded by a constant. If R has no odd cycles, then $G[R]$ is bipartite, and hence G is 3-colourable. Finding a maximum independent set in a (P_3+mK_1)-free graph can be done in polynomial time, so any (K_3, P_3+mK_1)-free graph is either bounded in size, or can be 3-coloured in this way in polynomial time. Thus VERTEX COLOURING of (K_3, P_3+mK_1)-free graphs can be solved in polynomial time. □

We conclude the paper with an alternative proof of the fact that every (K_3, H)-free graph is 3-colourable which is much shorter than the original proof in [30].

Theorem 7. *Every (K_3, H)-free graph is 3-colourable and a 3-colouring can be found in polynomial time.*

Proof. Let G be a (K_3, H)-free graph and S be any maximal (with respect to set inclusion) independent set in G. We assume that S admits no augmenting $K_{1,2}$ (i.e. a triple x, y, z such that x and y are non-adjacent vertices outside S with $N(x) \cap S = N(y) \cap S = \{z\}$), since finding an augmenting $K_{1,2}$ can be done in polynomial time. (If such an augmenting $K_{1,2}$ exists, we can just replace S by $\{x, y\} \cup S \setminus \{z\}$, which increases the size of S.)

Assume that the graph $G[V \setminus S]$ is not bipartite, and let vertices $x_1, \ldots, x_k$ induce a cycle C of odd length $k \geq 5$ in $G[V \setminus S]$. By maximality of S, every vertex outside S has a neighbour in S.

Suppose that each vertex of C has exactly one neighbour in S, and let $y_2 \in S$ and $y_3 \in S$ be the neighbours of x_2 and x_3, respectively. Then $x_1, x_2, x_3, x_4, y_2, y_3$ induce a copy of the graph H (by lack of triangles and augmenting $K_{1,2}$s). Thus, C must contain vertices with at least two neighbours in S. Assume without loss of generality that x_2 is of this type. If C has two consecutive vertices each of which has at least two neighbours in S, then an induced H can be easily found. Therefore, each of x_1 and x_3 has exactly one neighbour in S. If $y_2 \in S$ is a neighbour of x_2 and $y_3 \in S$ is a neighbour of x_3, then x_4 is adjacent to y_2, since otherwise $x_1, x_2, y_2, x_3, y_3, x_4$ would induce a copy of H. Therefore, $N(x_2) \cap S \subseteq N(x_4) \cap S$, and by symmetry, $N(x_4) \cap S \subseteq N(x_2) \cap S$, i.e. x_2 and x_4 have the same neighbourhood in S. This in turn implies that x_5 has exactly one neighbour in S. Continuing inductively, we conclude that the even-indexed vertices of C have the same neighbourhood in S consisting of at least two vertices, and each of the odd-indexed vertices of C has exactly one neighbour in S. But then $x_1, x_2, x_k, x_{k-1}, y_1, y_k$ induce a copy of the graph H, where $y_1 \in S$ and $y_k \in S$ are the neighbours of x_1 and x_k, respectively. □

7. Concluding remarks and open problems

In this paper we studied the complexity of the VERTEX COLOURING problem in subclasses of triangle-free graphs obtained by forbidding forests and proved polynomial-time solvability of the problem in many classes of this type. In particular our contribution, combined with some previously known results listed in Table 1, provides a complete description of the complexity status of the problem in subclasses of triangle-free graphs obtained by forbidding a forest with at most 6 vertices (Tables 2 and 3 summarize results of this type obtained in the present paper). Very little is known about the status of the problem in subclasses of triangle-free graphs defined by forbidding forests with more than 6 vertices, and this creates a challenging research direction.

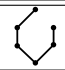
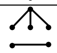
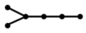
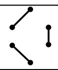
Graph	Graph Name	Complexity	Reference
	P_6	P	Theorem 1
	$K_{1,3} + P_2$	P	Theorem 3
	$S_{1,1,3}$	P	Theorem 3
	$3P_2$	P	Theorem 5

Table 3: Forests F on six vertices, none of which is isolated, for which the complexity of VERTEX COLOURING in the class Free(K_3, F) is contributed in this paper.

One more natural direction of research is investigation of the problem in extensions of triangle-free graphs. Let us observe that all results on triangle-free graphs can be extended, with no extra work, to so-called paw-free graphs, where a paw is the graph obtained from a triangle by adding a pendant edge. This follows from two facts: first, the problem can obviously be reduced to connected graphs, and second, according to [28], a connected paw-free graph is either complete multipartite (i.e. $\overline{P}_3$-free), in which case the problem is trivial, or triangle-free.

Further extensions make the problem much harder. For instance, by adding a pendant edge to each vertex of a triangle, we obtain a graph known in the literature as a net, and according to [35] the problem is NP-hard even for $(net, 2K_2)$-free graphs and $(net, 4K_1)$-free graphs. An interesting intermediate class between paw-free and net-free graphs is the class of bull-free graphs, where a bull is the graph obtained by adding a pendant edge to two vertices of a triangle. Recently, the class of bull-free graphs received much attention in the literature (see e.g. [8, 9, 13, 23]). In particular, [8] provides a structural characterisation of bull-free graphs which may be helpful in designing algorithms for various graph problems, including the vertex colouring problem.

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