On Income Inequality and Green Preferences

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We derive conditions on individual preferences and technology that give rise to a negative correlation between income inequality and environmental protection. We present a class of models (which captures a static model as well as an overlapping-generations model) in which individuals differ in earning abilities, and where a majority elected representative takes decisions over a pollution tax and a redistributive tax. We show that, if private consumption goods and the environment are non-inferior goods, then if the decisive individual has lower ability than the average, she will prefer a higher redistributive tax and a lower pollution tax.

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1. INTRODUCTION

Empirical evidence suggests that individual income is positively correlated with public support for environmental protection.\(^1\) Nevertheless, a theoretical explanation of this relationship has been attracted little attention in the literature. In a tax competition framework Oates and Schwab (1988) develop a static model in which individuals are distinguished in wage and non-wage earners and the median voter takes decisions over a capital tax and a standard for local environmental quality. If the decisive individual is a wage earner, she will choose a negative capital tax and a higher environmental standard than the first-best optimal level. If the decisive individual is a non wage-earner, she will clearly prefer a positive capital tax (for redistributive reasons). However, whether the environmental standard is higher or lower than the first-best optimum is not clear cut.

In this paper we explore the conditions which are needed to generate a negative relationship between income inequality and environmental protection. By using general preferences and technology, we show that, in a neighbourhood of no inequality, only non-inferiority of consumption goods and of the environmental good is needed to achieve this result. Our finding is robust to a class of models from a static model to an overlapping-generations economy model. In a companion paper (Marsiliani and Renström, 2000b), we are able to derive the same result globally but for specific preferences. In this paper, individuals differ in income-earning abilities, and the decisive individual (median voter or median candidate) implements her preferred policy choice over a pollution tax and a capital tax (used for redistributive reasons), which is in turn influenced by her characteristics such as income-earning ability.

The main finding is that there are two driving forces at work. First, environmental policy results in loss of production possibilities. Different individuals evaluate the production loss differently. Individuals with a higher marginal utility of consumption (the poorer ones) have a lower marginal rate of substitution between environment and private consumption if environment is a non-inferior good. Second, a poorer individual typically wishes to redistribute (using tax instruments on income) from richer individuals. The redistribution

\(^1\) For example, Elliot et al. (1997) find that both socio-demographic and economic factors are influential for individual support on environmental spending in the US, while Kahn and Matsusaka (1997) find that individual income and the price of the environmental good can explain most of the variation in voting on environmental policies in California.
causes the consumption-possibilities frontier to move inwards (due to efficiency losses). In such an equilibrium, if the environment is a normal good, the marginal rate of substitution between environment and private consumption decreases (for all individuals).

The paper is organised as follows. The general model is introduced in section 2. In section 3, the economic equilibrium is solved for. In section 4, individuals’ preferences over taxation are characterised for the general case, in a neighbourhood of no inequality. Section 5 summarises and interprets the results.

2. THE ECONOMY

We will specify a general model that contains three different cases. The first case (case I) is a static economy in which output is produced by labour and pollution. Labour and pollution are taxed at possibly different rates, and the tax receipts are redistributed lump-sum to the individuals. Individuals differ in time endowments. This implies that individuals with less productive time will supply less labour (than those with more productive time) if consumption is a normal good. There will then be a redistributive conflict, since the less endowed individuals gain from taxation of labour. This is similar to the Meltzer-Richard (1981) model, but augmented for pollution.

The second case (case II) is a sequence of two-period economies. Individuals live for two periods, consuming in both periods, but only working when they are young. Generations of different ages never co-exist. This is the same set-up used by Persson and Tabellini (1994), but augmented to allow for pollution. The period-one good is produced by labour (exogenous in supply), and the period-two good by capital (saved from the previous period) and pollution. Taxes are levied on capital income and on pollution, and a lump-sum transfer is given when the individuals are old.

The third case (case III) is an overlapping-generations economy (similar to Renström, 1996, but augmented for pollution). Output in each period is produced by labour (inelastically supplied by the young), capital (supplied by the old), and pollution. The decision about taxes is taken one period in advance (the young decide on taxes to be implemented when they are old). Taxes are levied on capital income and on pollution, and the transfer is given to the old generation.

In order to clearly understand how inequality may affect the pollution tax, we use a general utility specification and a general (constant returns-to-scale) production technology.
In this paper, we will only look at one situation: this is when inequality is marginally increased from a situation with full equality.\textsuperscript{2} Using general preferences and technologies to derive global results makes the problem untractable.

Denote the two consumption goods (consumed by individual $i$) as $c_{1i}$ and $c_{2i}$, respectively. The individual may transfer some of commodity 1 ($k_{1i}$) into commodity 2 at the after-tax rate $p$. The individual has an endowment of commodity 1, $w_{0i}$, and receives a transfer of commodity 2, $S$. In case (I) (the static model), $c_{1i}$ is leisure, $c_{2i}$ is consumption, $k_{1i}$ is labour supply, $p$ is the after-tax wage, and $w'$ is the individual's time endowment. In cases (II) and (III) (the dynamic economy with and without separation across generations), $c_{1i}$ and $c_{2i}$ are period 1 and 2 consumption respectively, $k_{1i}$ is savings, $p$ is the after-tax return on savings, and $w'$ is period-1 labour income. We assume that $w_{0i} = \gamma w_{0}$, and that the distribution of $\gamma$ (denoted $\Gamma(\gamma)$) is continuous and, for cases (II) and (III), stationary over time. $\Gamma(\gamma)$ is also normalised so that the average $\gamma$ equals unity, and so that averages equals aggregates. We will denote averages/aggregates by omitting superscript $i$.

Throughout we will make one separability assumption: the pollution externality enters the individuals’ utility functions in a weakly separable way. This will make the individuals’ marginal rates of substitutions between private consumption units independent of the pollution externality. This will make the private consumption decisions independent of pollution; without such a separation, the problem becomes intractable and one would have to resort to computation. The weak separability will \textit{not}, however, make the individuals’ evaluation of the environment independent of their private consumption, and, consequently, we may explore this interaction in the analysis. We next state the assumptions made.

2.1 Assumptions

A1 Individuals’ preferences
We assume weak separability between private consumption and pollution

$$V^i = V(u(c_{1i}, c_{2i}), x)$$

(1)

where $V$ and $u$ are strictly concave, and $V_1 > 0, V_2 < 0, u_1 > 0, u_2 > 0$.

\textsuperscript{2} This is analogous to the optimal-tax literature. A situation in which one solves for an optimal tax system, and evaluates it at zero tax rates, is a situation in which one marginally introduces the second best from a first-best situation (no taxes).
A2 Individuals’ constraints
The individuals’ budget constraints are
\[ c_1^i + k^i = w_0 y^i \quad (2) \quad c_2^i = p k^i + S, \quad p = (1 - \tau^k) R \quad (3) \]

A3 Production
A large number of firms are operating under identical constant-returns-to-scale technologies. Therefore aggregate production, \( y_t \), can be calculated as if there was a representative firm employing the aggregate quantity of the factors supplied by the individuals, \( k = \int k^i d \Gamma(\gamma^i) \) and, in case (III) \( l = \int l^i d \Gamma(\gamma^i) \), and the polluting factor.\(^3\) For case (I) and (II)
\[ F(k, x) \quad (4a), \quad \text{and for case (III)} \quad F(k, x, l). \quad (4b) \]
Firms take the factor prices of labour \((w)\), capital \((R)\), and the pollution tax \( \tau\), as given.\(^4\)

A4 Government’s constraint
The tax receipts are fully used for the lump-sum transfer
\[ S = \tau^k R k + \tau^x x \quad (5) \]

A5 Representative democracy
The tax rates, \( \tau^k \) and \( \tau^x \), are determined by a majority-elected representative one period in advance. We assume that one candidate of each type runs for office, and that candidacy is costless.

3. ECONOMIC EQUILIBRIUM
In this section, the individual and aggregate economic behaviour are solved for any given arbitrary sequences of tax rates.

3.1 Individual economic behaviour

\(^3\) The polluting factor is provided at no cost. Thus, in absence of a government taxing or regulating it, this factor would be used up until the satiation point.

\(^4\) Note that \( w_0 \) is labour income in the previous period, and \( w \) is labour income earned by the next generation to come.
Maximisation of (1) subject to (2)-(3) gives the individuals’ optimal decision over \( k \). The first-order condition forms an implicit function

\[
\phi = -u_1^i(w_0^i - k^i, pk^i + S) + pu_2^i(w_0^i - k^i, pk^i + S) = 0
\]  

(6)

which differentiated gives the following partial derivatives

\[
\frac{\partial k^i}{\partial w_0^i} = N_2^i \quad (7)
\]
\[
\frac{\partial k^i}{\partial p} = \frac{u_2^i}{D^i} - \frac{k^i}{p} N_1^i \quad (8)
\]
\[
\frac{\partial k^i}{\partial S} = -\frac{N_1^i}{p} \quad (9)
\]

where

\[
N_1^i = \frac{pu_{12}^i - p^2u_{22}^i}{D^i} \quad (10)
\]
\[
N_2^i = -\frac{u_{11}^i - pu_{12}^i}{D^i} \quad (11)
\]
\[
D^i = -u_{11}^i + 2pu_{12}^i - p^2u_{22}^i > 0 \quad (12)
\]

\( N_1 \) is positive (negative) if \( c_1 \) is a normal (inferior) consumption good, and \( N_2 \) is positive (negative) if \( c_2 \) is a normal (inferior) consumption good. Furthermore, \( N_1 + N_2 = 1 \), implying that, at most, one of the goods can be inferior. We will see later what role the normality of the private consumption goods plays in the analysis.

### 3.3 Firms’ behaviour

Firms take prices as given. Profit maximisation implies that the before-tax prices are given by \( r = F_k \) (in cases (I), (II), and (III)), and \( w = F_l \) (in case (III)). Notice that in case (III), \( w \) is the wage received by the next generation (the present generation receives \( w_0 \), which is labour’s marginal product in the previous period). The first-order condition for the use of factor \( x \), \( F,(k,x,l) = \tau \), gives (aggregate/average) \( x \) as a function of (aggregate/average) \( k \) and \( \tau \) (and of \( l \) which, however, is fixed), with the following property

\[
dx = d\tau \cdot (F_{xx} - (F_{xk}F_{xx})) dk
\]

(13)

### 3.4 Government’s budget

The budget may alternatively be written as

\[
S = F - F_1l - pk
\]

(14)

We will define environmental strictness as the level of \( \tau \), which implies that if the government operates an emissions standard, the strictness measure is the (equilibrium) marginal product of pollution, \( F_x \).
4. PREFERENCES OVER POLICY

We assume that policy will be chosen by a representative. In a political-economy framework this individual would be the majority elected one. Since individuals differ in only one dimension one can construct a median-voter equilibrium where the median individual cannot lose against any other candidate.5

In this paper we will not model the political equilibrium, we will just examine the preferences of a hypothetical representative and see how her optimal choice changes as we change the ability of this individual. We do so for the case in which a candidate is "close" to the average. In this way we analyse a situation where we move from the first best (all individuals being the same) to the second best. The aim is to understand the driving forces behind inequality and environmental policy.

The problem of the decision maker is to

\[
\max_{p,s,x} V[(w^t - k^t, pk^t + S), x] + \lambda [F(k, x, l) - pk - F(l - S)]
\]

The problem is written as if the individual was to choose \( x \) directly (for example, imposing an emissions standard); however, it is just an equivalent representation of the situation where the pollution tax is chosen. The first-order conditions are

\[
V_1 u_x = \lambda \left( F_k - F_{lk} l - p \right) \frac{\partial k}{\partial p} - k = 0
\]

\[
V_1 u_x + \lambda \left( F_k - F_{lk} l - p \right) \frac{\partial k}{\partial S} - 1 = 0
\]

\[
V_2 + \lambda [F_x - F_{lx} l] = 0
\]

We may observe the following. Since the pollution tax is pollution’s marginal product, (18) may be written as \( \tau = F_{lk} l + (-V_2)/\lambda \). Everything being equal, an increase in \( \lambda \) (the decisive individual’s marginal utility of lump-sum income at the optimum) reduces the pollution tax.

Environmental policy comes at the expense of production possibilities. This tends to make poorer individuals (with lower marginal rate of substitution between environment and private

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5 The elected individual would choose policy so as to maximise her own utility. This policy would be a function of the type of the individual, say \( \gamma \). Substituting this policy into any other individual’s utility function one obtains an indirect utility function of \( \gamma \) only. It is clear, since individuals differ only in one dimension, the political equilibrium would be of the median-voter type. This is a political equilibrium if individuals’ indirect utilities over \( \gamma \) are single peaked. This is modelled in Marsiliani and Renström, 2000b, for specific utility and production functions.
consumption) wanting a lower pollution tax. Furthermore, \( \lambda \) is also evaluated at equilibrium production.

The argument put forth above is just to illustrate what we believe are the driving forces. We need to prove that \( \lambda \) is larger for a poorer individual if she was to choose policy than it would be for a richer individual if the latter were to choose policy. We also need to take into account how individuals evaluate the environment. If \( V \) is not additively separable, then \( V^2 \) depends on the private consumption of the decisive individual (at the optimum) as well. For example, it could be the case that a poorer individual values the environment more (for example, \(-V^2 \) could be larger for poorer individuals). Furthermore, there is also an effect (in case (III)) regarding the return to labour of the young generation, which the present decisive individual does not care about, but would rather use the tax system so as to reduce the next generation’s labour income. In order to formally prove the link between the income of the decisive individual and environmental protection, we need to take into account the whole system (16)-(18). We will do so by performing comparative statics, by changing \( \gamma \) of the decision maker, and evaluating the consequences on \( \tau \) in a situation with no inequality. We can then see the consequences of making the decision maker (marginally) poorer or richer than average.

Combining (16) and (17) gives

\[
 k - k^i = (F_k - F_k^i p) \partial k / \partial p - k^i \partial k / \partial \delta \tag{19}
\]

In the RHS of (19), the second expression in round brackets is unambiguously positive because \( k \) is very close to \( k^i \) (from equations (8) and (9)). We need to evaluate the first term in round brackets in the RHS of (19). First, in case (I) and (II), \( F_{lk} \) is zero (production technology (4a)). Then the capital tax is positive (zero/negative) if the decisive individual supplies less (equal/more) of \( k \) than the average.\(^6\) In case (II), the future generation will earn wage income, and by choosing a larger capital tax, labour income is reduced (if \( F_{lk} > 0 \)) and an implicit transfer from the future young is accomplished. Thus, here, even if the decisive individual owns capital exactly equal to the average (e.g., if full equality), capital would be taxed. Equation (19) forms an implicit function in \( \gamma, p, k, \) and \( x \) (the latter two being functions of policy). Differentiating and evaluating at \( k^i = k \) (full equality) gives

\(^6\) If Engel curves are linear, and period-two consumption normal, this is the case if the individual has \( \gamma \) smaller (equal/greater) than unity (i.e., average).
Using (8)-(9) to substitute for the expression in parenthesis, and using (13) to substitute for $dx$ yields

$$u_2^{-1} N_2 D w_o^t = \sigma dk + \eta \tau^x + dp$$

(21)

where

$$\sigma = -(F_{kk} - F_{l_k l}) + (F_{k x} - F_{l_k x}) F_{x x} / F_{x x}$$

(22)

$$\eta = -(F_{k x} - F_{l_k x}) / F_{x x}$$

(23)

Equation (21) gives the after-tax return $p$ as a function of the decisive individual’s endowment $w_0^i$, of the pollution tax $\tau^i$, and of the level of $k$ (in turn a function of $p$ and $\tau^i$). If we consider cases (I) and (II), (production technology (4a)), $\sigma=0$. Then for each level of $\tau^i$, the after tax return $p$ is increasing in $w_0^i$ if commodity 2 is a normal good. If factor $k$’s marginal product increases with pollution ($F_{k x} > 0$), $\eta > 0$, then an increase in $\tau^i$ (everything else equal) reduces $p$. The reason is that an increase in $\tau^i$ reduces $x$ and thereby reduces $F_k$, and it is not optimal to reduce the tax on $k$ so as to leave $p=(1-\tau^k)F_k$ unaffected.

Next, we combine (17) and (18) to obtain the optimality condition for $\tau^i$.

$$\frac{V_1 u_2}{-V_2} = \frac{1 - \eta (F_{k x} - F_{k l} l - p) \frac{\partial k}{\partial S}}{\tau^x - F_{l x} l}$$

(24)

We need to know how the marginal rate of substitution between private consumption and the environment changes with the underlying variables. Let $V_j$ denote the derivative of $V$ with respect to argument $j=\{1,2\}$, we then have

$$dV_j = V_{j1} [u_1 d w_0^t + u_2 (k^t dp + dS)] + V_{j2} dx$$

(25)

differentiating (14) gives

$$dS = (F_k - p - F_{l_k l}) dk + (F_{k x} - F_{l_k x}) dx - k dp$$

(26)

using (19) and evaluating at $k'=k$ gives

$$k dp + dS = (F_x - F_{l_k x}) dx = -V_2 (V_1 u_2) dx$$

(27)

where the last inequality follows from (18), evaluated at no inequality (i.e. $\lambda=V_1 u_2$).

Substituting the last equality of (27) into (25) gives

$$dV_j = V_{j1} [u_1 d w_0^t - (V_2 / V_j) dx] + V_{j2} dx$$

(28)

or after rearrangement
where
\[
Q = \frac{V_{11}}{V_1} - 2\frac{V_{12}}{V_2} + \frac{V_1 V_{22}}{V_2} < 0
\]  (30)

Equation (30) is a quadratic form in the Hessian of \( V \) and is negative since \( V \) is strictly concave. Equation (29) gives the change in the individual’s marginal rate of substitution between the private consumption index, \( u \), and the environment. If the environment is a normal consumption good, the first term on the right-hand side is negative, implying, at the optimum, that a richer individual has a lower marginal rate of substitution and thus prefers to substitute less from the environment to private consumption. This effect makes a poorer individual wishing to protect the environment less. The second term on the right-hand side is negative, implying that the marginal rate of substitution is decreasing in pollution (i.e., increasing in the environment). That is, if the level of the environment is large at the optimum, the individual is willing to substitute less private consumption for the environment.

Finally, we need to find \( du_2 \) in order to find the change in the marginal rate of substitution between private consumption of commodity 2 and the environment. We have (details in Appendix)

\[
du_2 = -\tilde{D}\left[pdw_0^t + kdp + dS - u_2 N_1 dp/p\right]  \tag{31}
\]

where
\[
\tilde{D} = (u_{11} u_{22} - u_{12} u_{22})/D > 0.
\]  (32)

\( \tilde{D} \) is positive since \( u \) is strictly concave. From (31) we see that, at the optimum, the marginal utility of commodity 2 is declining in the commodity-1 endowment (a richer individual has a lower marginal utility of commodity 2 at the optimum). The rest of the terms reflect the income effect of the tax-transfer system. An increase in the return on factor \( k \), and in the transfer, reduces the marginal utility of commodity 2.

Equation (13) gives \( dx \) as a function of \( dk \) and \( d\tau^i \); equation (21) gives \( dp \) as a function of \( dk \), \( d\tau^i \), and \( dw_0^i \). Since \( dk \) can be written as a function of \( dp \) and \( dx \), we have a system of three equations that gives us \( dk, dx, dp \) as functions of \( d\tau^i \) and \( dw_0^i \). Substituting for those in (29) and (31) gives \( d(\ln[V_i u_i /(-V_j)]) \) as a function of \( d\tau^i \) and \( dw_0^i \) (see equation (43) in Appendix). If the environment is non-inferior (i.e., \( V_{ii}/V_{i} - V_{jj}/V_{j} \leq 0 \), commodity 1 and 2 non-inferior \( (N_1 \geq 0, N_2 \geq 0) \), [and \( F_{ii} \geq 0, \sigma \geq 0, \text{ and } \eta \geq 0 \)], then the marginal rate of substitution
between the environment and commodity-2 consumption is unambiguously decreasing in $w_0^i$. This implies that, at the optimum, a poorer individual will have a lower value of the environment as compared to private commodity-2 consumption. If we analyse cases (I) and (II) (production \((4a)\)), then $F_{\delta}>0$, $\sigma=0$, and $\eta=x/k>0$, then it is sufficient that the environment is non-inferior, and the two private goods are non-inferior. Thus, the key is the non-inferiority of goods.

Next, if a higher pollution tax at the optimum reduces pollution, and if the marginal rate of substitution is non-decreasing in the pollution tax, a richer individual prefers a higher pollution tax. If commodity 1 is non-inferior and $\sigma\geq0$ and $\eta\geq0$, then the marginal-rate of substitution is increasing in the pollution tax (by inspection of the second term in \((43)\) in the Appendix). Again the non-inferiority of commodities plays a role. A richer individual typically wishes a higher environmental tax if she is decisive. The following propositions state sufficient conditions.

**Proposition 1** Assume A1-A5 and $F=F(k,x)$ (i.e., case (I) or (II)), then sufficient for an individual marginally poorer (richer) than average (in a situation in which all individuals are the same) to prefer a lower (higher) pollution tax is that

(i) the environment is non-inferior (i.e., $V_{111}/V_{1}-V_{21}/V_{2} \leq 0$),

(ii) private commodities 1 and 2 are non-inferior (i.e., $N_1\geq0$ and $N_2\geq0$).

Proof: See Appendix.

**Proposition 2** Assume A1-A5 and $F=F(k,x,l)$ (i.e. case (III)), then sufficient for an individual marginally poorer (richer) than average (in a situation where all individuals are the same) to prefer a lower (higher) pollution tax is that

(i) the environment is non-inferior (i.e., $V_{111}/V_{1}-V_{21}/V_{2} \leq 0$),

(ii) private commodities 1 and 2 are non-inferior (i.e., $N_1\geq0$ and $N_2\geq0$).

(iii) $u_{i2} \leq 0$,

(iv) $F_{\tilde{x}k} \geq 0$,

(v) $F_{\tilde{x}k}l - F_{\tilde{x}l}l F_{\tilde{x}k}/F_{xx} \geq 0$

(vi) $1 - F_{iil}/F_{xx} \geq 0$,

(vii) $F_{\tilde{x}k} - F_{i\tilde{x}l} \geq 0$,

(viii) $-F_{\tilde{x}k} + F_{i\tilde{x}l} \geq 0$

Proof: See Appendix.

There are a number of technology assumptions which are sufficient (though not necessary) for a richer individual wanting a higher pollution tax in case (III), (but not in cases (I) and
The reason is that an individual here wishes to redistribute from factor $l$, that is the labour supply by the future young generation, and the available instruments would be set so as to achieve that.

We have now identified the forces at work in a link between inequality and environmental protection. First, it is the period-1 endowment of the decisive individual in relation to the average. Thus, it is inequality in terms of skewness. Second, the non-inferiority of both environmental and private commodities plays a role.

5. CONCLUSIONS

We have found a negative relationship between income inequality (in terms of the distance between the decisive individual and the mean individual) and the stringency of environmental policy (the level of an environmental tax). Sufficient for this result is that consumption goods and the environment are non-inferior.

However, we can only determine this result with general preferences and technologies for marginal changes in skewness from a position of full equality. Extending it globally generally makes the problem intractable. The reason is that when analysing the problem for general inequality, two things may happen.

First, with or without government taxes, the competitive equilibrium may be a function of the distribution. This occurs if Engel curves are non-linear. If one changes the median-mean distance of the distribution, not only the decisive individual’s identity changes, but also the competitive equilibrium prices. It is then difficult to assess the political channel.

The second consequence may be that decisions are not monotone in the individual’s type, and single-peakedness may be violated. Then we may fail to have a median-voter equilibrium. We would then have to look for political institutions that can overcome that problem. That is, however, beyond the scope of this paper and it is left for future research.

7 In general it is desirable to analyse a situation where the competitive equilibrium is invariant with respect to the underlying distribution and only the political channel is at work. This is the case when the individual utility function is such that aggregation occurs. There is a broad class of preferences which allows for that. A special case occurs when utility is additively separable and homothetic (logarithmic), which are the preferences restricted to in Marsilianni and Renström (2000b).
References


APPENDIX

Derivation of (31)

Differentiating $du_2$ gives

$$du_2 = u_2 \left[ dw_0^t - dk^t \right] + u_2^t \left[ pdk^t + k^t dp + dS \right]$$

or

$$du_2 = u_2 \left[ dw_0^t + pd(22 - u_2) \right] \left[ \frac{\partial k^t}{\partial p} dp + \frac{\partial k^t}{\partial S} dS + \frac{\partial k^t}{\partial w_0^t} dS \right] + u_2^t \left[ k^t dp + dS \right]$$

or by using (7)-(9), and $N_1 + N_2 = 1$,

$$du_2 = (u_2 N_2 + u_2 N_1) \left[ pdw_0^t + kdp + dS \right] + D^{-1}(pu_2 - u_2) u_2 dp$$

Using (10)-(11) gives (31).

Proof of Propositions 1-2

Using the last equality in (27) to substitute for $kd + ds$, and combining with (29) gives

$$d \ln \left( \frac{V_{1,2}}{V_2} \right) = \left( \frac{V_{1,1} - V_{2,2}}{V_1} \right) u_1 - p \frac{\vec{D}}{u_2} dw_0^t - \left( \frac{V_2}{V_1} Q - \frac{\vec{D}}{u_2} F_{xx} \pi \right) dx - \frac{N_1}{p} dp$$

where

$$\pi = (F_{xx} - F_{xx})$$

Next, differentiating $k$

$$dk = \frac{\partial k}{\partial p} dp + \frac{\partial k}{\partial S} dS - \frac{u_2}{D} dp + \frac{N_1}{p} (kd + dS)$$

where the last equality follows from (8)-(9). Then, by using the last equality in (27), and the definition (37), we have

$$dk = D^{-1} u_2 dp + p^{-1} N_1 F_{xx} \pi dx$$

Equations (39), (13), (14) form a system, such that $dk$, $dx$, and $dp$ can be solved for. The latter two are

$$F_{xx}(1 + m) dx = \left[ 1 + D^{-1} u_2 (\rho + \eta F_{xx}) \right] d \tau^t - F_{xx} N_2 dw^t$$

and

$$(1 + m) dp = N_2 D u_2^{-1} \left[ 1 + F_{xx} N_1 \pi/p \right] dw^t - \left[ \eta + \frac{N_1}{p} \pi (\sigma + \eta F_{xx}) \right] d \tau^t$$

where
Substituting (40) and (41) into (36) gives

\[ m = \alpha D \gamma u_2 + F_{xk} N_1 \pi / p \]  \hspace{1cm} (42)

Next differentiating the log of the numerator and the denominator of the right-hand side of (24) gives, respectively

\[ d \ln \left( \frac{V_1 u_2}{-V_2} \right) = \left( \frac{V_{11} - V_{21}}{V_1 V_2} \right) u_1 + \frac{V_2 F_{xk} Q N_2}{V_1 F_{xx} 1 + m} - \frac{\tilde{D} p F_{xk} \pi N_2}{u_2 (1 + m) p} - \frac{N_1 N_2 D}{p (1 - m) u_2} \left( 1 + \frac{F_{xk} \pi N_1}{p} \right) \]  \hspace{1cm} (43)

Next, combining (43), (44), and (45), the differential of the log of (24) is

\[ d \ln \left( 1 - (F_k - F_{xk} \pi)^{l - p} \right) \frac{\partial k}{\partial S} = - \frac{\partial k}{\partial S} d \ln \left( 1 - (F_k - F_{xk} \pi)^{l - p} \right) = - \frac{N_1 N_2 D}{p u_2} d w^i (44) \]

\[ d \ln \left( F_k - F_{xk} \pi \right) = \frac{d \tau \gamma F_{xk} \pi}{F_k - F_{xk} \pi} - \frac{(\sigma + \eta F_{xk})}{1 + m} + \frac{N_1 \pi}{p} \eta \frac{(\sigma + \eta F_{xk})}{1 + m} - \frac{1}{F_{xk} \pi} \right) \]  \hspace{1cm} (45)

We have used the relation

\[ - \frac{\tilde{D} \gamma}{u_2} - \frac{\tilde{D} p F_{xk} \pi \gamma N_2}{p (1 + m) u_2} - \frac{N_1 N_2 D}{p (1 + m) u_2} \left( 1 + \frac{F_{xk} \pi N_1}{p} \right) + \frac{N_1 N_2 D}{p u_2} = \]

\[ = - \frac{\tilde{D} \gamma}{u_2 (1 + m)} \left[ 1 + \frac{m + F_{xk} \pi N_2}{p} \right] + \frac{N_1 N_2 D}{p (1 + m) u_2} \left( m - F_{xk} \pi N_1 \right) / p \]  \hspace{1cm} (46)

\[ = - \frac{\tilde{D} \gamma}{u_2 (1 + m)} \left[ 1 + \frac{\sigma u_2}{D} + \frac{F_{xk} \pi}{p} \right] + \frac{N_1 N_2 D}{p (1 + m) u_2} \left( m - F_{xk} \pi N_1 \right) / p \]  \hspace{1cm} (47)
We need to evaluate the last term in (46). Using (40) and (41) in (39) gives

\[(1 + m)dk = N_2 dw^i - \left(\eta D^{-1}u_2 - N_1 \pi/p\right)d\tau^x \quad (48)\]

Since \(F_{ix} \geq 0\), \(dk\) enters with negative sign in (46), and since \(F_{ix} \leq 0\), \(dx\) enters with positive sign. \(dk\) and \(dx\) are positively and negatively related, respectively, to \(dw^i\). Therefore, the last term will add \(dw^i\) negatively, and the function (46) is unambiguously negative in \(dw^i\). The terms in \(\tau^i\) are at first sight ambiguous, and we have to add those terms carefully.

\[\frac{F_{ix}}{F_{xx}}dk + \frac{F_{ix}}{F_{xx}}dx = \frac{1}{F_{xx} \pi(1 + m)}\left[F_{ix} \left(\eta \frac{u_2}{D} - \frac{N_1}{\pi}\right) - \frac{F_{ix}}{F_{xx}} \left(1 + \frac{u_2}{D} (\sigma + \eta F_{sk})\right)\right]d\tau^x - Mdw^i \quad (49)\]

where

\[M = \frac{N_2}{-F_{xx} \pi(1 + m)} \left[F_{ix} - \frac{F_{xx} F_{sk}}{F_{xx}}\right] \quad (50)\]

Substituting into (14) gives

\[\begin{bmatrix} V_1 & V_2 \\ V_1 & -V_2 \end{bmatrix} u_1 + \frac{V_2 F_{sk} Q N_2}{V_1 F_{xx}} + \frac{D p + F_{sk} \pi}{1 + m} + \frac{\sigma u_2}{1 + m} - M d\tau^x \]

\[\begin{bmatrix} -V_2 Q \pi + \frac{D}{V_1 F_{xx}} \frac{u_2}{1 + m} + M d\tau^x \end{bmatrix} \quad (51)\]

where

\[\tilde{M} = \frac{1}{-F_{xx} \pi(1 + m)} \left[1 + \frac{u_2}{D} \left(1 - \frac{F_{xx}}{F_{xx}}\right) + \frac{N_1}{\pi} \frac{F_{sk} - F_{ix}}{\pi} + \eta \frac{u_2}{D} \left(F_{ix} - \frac{F_{xx} F_{sk}}{F_{xx}}\right)\right] \quad (52)\]

Then Propositions 1 and 2 follow from (50), (51), and (52). QED