Some results on the Oberwolfach problem

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Abstract

The well-known Oberwolfach problem is to show that it is possible to 2-factorize $K_n$ ($n$ odd) or $K_n$ less a 1-factor ($n$ even) into predetermined 2-factors, all isomorphic to each other; a few exceptional cases where it is not possible are known. In this paper we introduce a completely new technique which enables us to show that there is a solution when each 2-factor consists of $k$ $r$-cycles and one $(n - kr)$-cycle for all $n \geq 6kr - 1$. Solutions are also given (with three exceptions) for all possible values of $n$ when there is one $r$-cycle, $3 \leq r \leq 9$, and one $(n - r)$-cycle, or when there are two $r$-cycles, $3 \leq r \leq 4$, and one $(n - 2r)$-cycle.

1 Introduction

Let $K^*_n$ be the complete graph $K_n$ if $n$ is odd and $K_n$ less a 1-factor if $n$ is even. The problem of determining whether there is a 2-factorization of $K^*_n$ in which each 2-factor is isomorphic to the same specified graph is known as the Oberwolfach problem. The notation $\text{OP}(r_1^{a_1}, r_2^{a_2}, \ldots, r_s^{a_s})$ represents the case in which each 2-factor must consist of $a_i$ $r_i$-cycles, $1 \leq i \leq s$. The problem was formulated by Ringel and is first mentioned in [6]. Many cases have now been solved; see, for example, [1, 2, 7, 11, 12, 13, 14], or, for a summary of known results, [3]. In this paper we obtain some further solutions by a method that is completely novel in the context of the Oberwolfach problem. The method is an adaptation of the outline/amalgamation technique used, in particular, in [8] concerning Hamiltonian decompositions of complete graphs. Our main result is:

**Theorem 1** Let $r \geq 3$, $k \geq 1$ and $n \geq 6kr - 1$ be integers. Then $\text{OP}(r^k, n - kr)$ has a solution.

Theorem 1 is just a specialization of the following more detailed result.
Theorem 2 Let \( r \geq 3, k \geq 1 \) and \( n \) be integers. Then a solution to \( OP(r^k, n-kr) \) exists if

1. for even \( r \), either \( n \geq 6kr - 3 \), or \( n \in \{2r(2k+i)-3, 2r(2k+i)-2 \mid i = 1, \ldots, k-1\} \),

2. for odd \( r \), even \( k \), either \( n \geq 6kr - 3 \), or \( n \in A \cup B \), where \( A = \{2r(2k+i+1)-1, 2r(2k+2i+1), 2r(2k+2i+2)-3, 2r(2k+2i+2)-2 \mid i = 0, 1, \ldots, \lfloor (k-3)/2 \rfloor\} \), \( B = \{2r(3k-1)-1, 2r(3k-1)\} \),

3. for odd \( r \), odd \( k \), either \( n \geq 6kr - 1 \), or \( n \in A \) (where \( A \) is as in part 2),

4. \( n = 4kr - 2 \).

Theorem 2 is a consequence of Lemmas 3 and 4, except in the cases \( (r,k) = (3,1), n = 27 \) or \( n = 28 \). These cases are covered by Theorem 5 below.

Lemma 3 Let \( r \geq 3, k \geq 1, m \geq 2k+1 \) and \( n \) be integers with \( (r,m) \neq (3,4) \). Then \( K_n^* \) has a 2-factorization in which each 2-factor contains \( r \) \( r \)-cycles and an \((n-kr)\)-cycle for the following values of \( n \):

1. \( 2(rm-1)+1 \leq n \leq \left\lfloor \frac{m}{k} \right\rfloor (rm-1)+2 \), if \( rm \) is odd, and

2. \( 2(rm-2)+1 \leq n \leq \left\lfloor \frac{m}{k} \right\rfloor (rm-2)+2 \), if \( rm \) is even.

Lemma 4 Let \( r \geq 3, k \geq 1 \) and \( n = 2(2kr-2)+2 \) be integers with \( (r,k) \notin \{(3,1),(3,2)\} \). Then \( K_n^* \) has a 2-factorization in which each 2-factor contains \( r \) \( r \)-cycles and an \((n-kr)\)-cycle.

Figure 1 shows the values of \( n \) which are obtained with the above lemmas for small \( k \) and \( r \).

We also have the further result:

Theorem 5 Let \( 3 \leq r \leq 9, n \geq r+3 \) be integers. Then \( OP(r,n-r) \) has a solution except for the cases \( OP(3,3) \) and \( OP(4,5) \). Let \( 3 \leq r \leq 4, n \geq 2r+3 \) be integers. Then \( OP(r,r,n-2r) \) has a solution except for the case \( OP(3,3,5) \).

Many of these cases follow from Lemmas 3 and 4, or are proved in [12, 13, 14]. We have solutions to all the remaining cases. In the final section we describe our method for obtaining these solutions and, as an example, give solutions to all the outstanding cases of \( OP(5,n-5) \).

We must show that Theorem 2 is a consequence of Lemmas 3 and 4. For each valid pair \( (r,k) \), Lemma 3 provides a series of intervals such that if an integer \( n \) lies in one of these intervals a solution to \( OP(r^k, n-kr) \) can be found. For example, consider when \( r \) is even: the intervals are given by the last line of
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Figure 1: Some cases of $\text{OP}(r^k, n - kr)$ solved by Lemmas 3 and 4.

the lemma. The intervals abut or overlap if the upper bound of one interval, \(\left\lfloor \frac{m}{k} \right\rfloor (rm - 2) + 2\), is greater than or equal to the lower bound of the next, \(2(r(m + 1) - 2) + 1 = 2rm + 2r - 3\); that is, if \(m \geq 3k\). Therefore for all values of \(n\) greater than \(2(r(3k) - 2) + 1 = 6kr - 3\), $\text{OP}(r^k, n - kr)$ has a solution. For \(2k + 1 \leq m \leq 3k - 1\), as \(\left\lfloor \frac{m}{k} \right\rfloor = 2\) we can find solutions if \(2(rm - 2) + 1 \leq n \leq 2(rm - 2) + 2\); that is, if \(n \in \{2rm - 3, 2rm - 2 \mid m = 2k + 1, \ldots, 3k - 1\}\), which is the set described in the first part of Theorem 2. The next two parts of the theorem can also be seen to follow from Lemma 3 using similar arguments. The final part follows from Lemma 4.

We shall prove Lemma 3 by showing that an edge-colouring of $K_{rm}$ that satisfies certain conditions can be extended to obtain a 2-factorization of $K_n^*$. The method of extending edge-colourings is introduced in the next section. In Section 3 we present some results which we shall use to find the initial edge-colouring of $K_{rm}$.

3
First, some further definitions and notation are presented. An $r$-cycle is denoted $[v_1, \ldots, v_r]$. An edge joining vertices $v_i$ and $v_j$ is denoted $(v_i, v_j)$, and a sequence of adjacent edges $(v_1, v_2), (v_2, v_3),\ldots, (v_{n-1}, v_n)$ will be abbreviated $(v_1, v_2, \ldots, v_n)$. The degree of a vertex $v$ in a graph $G$ is denoted $d_G(v)$. The maximum degree of the vertices in a graph $G$ is denoted $\Delta(G)$. An edge-colouring of a graph $G$ is a function $f: E(G) \to C$, where $C$ is a set of colours. If $G$ is an edge-coloured graph, then $G(c_i)$ is the subgraph induced by edges coloured $c_i$. A cycle is $c_i$-coloured if all its edges are coloured $c_i$ and is $(c_i, c_j)$-coloured if all its edges are coloured either $c_i$ or $c_j$ and it contains at least one edge of each colour.

2 Extending Edge-Colourings

Theorem 6 is an adaptation of the outline/amalgamation result obtained for Hamiltonian decompositions of complete graphs in [8]. It also generalizes results in [5, 9]. It concerns a $K_m$ edge-coloured with $t$ colours, $c_1, \ldots, c_t$, and a set of associated parameters, $(s_1, \ldots, s_t)$, where each $s_i \in \{1, 2\}$ and $\sum_{i=1}^t s_i = n - 1$. We give necessary and sufficient conditions for such an edge-colouring to be extendible to an edge-colouring of $K_n$ in which $K_n(c_i)$ is an $s_i$-factor, and if $s_i = 2$, $K_n(c_i)$ contains just one more cycle than $K_m(c_i)$. The case in which each $s_i = 2$ was proved in [9].

Theorem 6 Let $m$ and $n$ be integers, $1 \leq m < n$. Let $(s_1, \ldots, s_t)$, $s_i \in \{1, 2\}$, $1 \leq i \leq t$, be a composition of $n - 1$. Let $K_m$ be edge-coloured with $t$ colours $c_1, \ldots, c_t$. Let $f_i$ be the number of edges coloured $c_i$. This colouring can be extended to an edge-colouring of $K_n$ in which $K_n(c_i)$ is an $s_i$-factor, $1 \leq i \leq t$, and when $s_i = 2$, $K_n(c_i)$ contains exactly one more cycle than $K_m(c_i)$ if and only if

\begin{align*}
(A1) & \quad f_i \geq s_i \left( m - \frac{n}{2} \right) \\ (A2) & \quad s_i n \text{ is even} \\ (A3) & \quad \Delta(K_m(c_i)) \leq s_i
\end{align*}

Proof of necessity in Theorem 6: $K_m$ contains $f_i$ edges coloured $c_i$. Each of the $(n - m)$ further vertices is incident with $s_i$ edges coloured $c_i$. In $K_n$ there must be exactly $s_i n/2$ edges coloured $c_i$. Hence

$$f_i + s_i(n - m) \geq \frac{s_i n}{2}.$$ 

Rearranging, (A1) is obtained.

An $s_i$-factor of $K_n$ has $s_i n/2$ edges, and each vertex is incident with $s_i$ edges. Hence (A2) and (A3) are necessary. \qed
Before we can prove sufficiency in Theorem 6 we require a result concerning edge-colourings of bipartite multigraphs.

Given an edge-colouring of a loopless multigraph $G$ with colours $c_1, \ldots, c_n$, for each $v \in V(G)$, let $C_i(v)$ be the set of edges incident with $v$ of colour $c_i$. An edge-colouring is equitable if, for all $v \in V(G)$,

$$\max_{1 \leq i < j \leq n} ||C_i(v)| - |C_j(v)|| \leq 1.$$ 

The following result is due to de Werra [16, 17, 18]. A straightforward proof can be found in [4].

**Proposition 7** (de Werra) For each positive integer $k$, any finite bipartite multigraph has an equitable edge-colouring with $k$ colours.

**Proof of sufficiency in Theorem 6:** The greater part of this proof is devoted to showing that if $m < n - 1$, the edge-colouring of $K_m$ can be extended to an edge-colouring of $K_{m+1}$ in such a way that (A1), (A2) and (A3) remain satisfied, with $m$ replaced by $m+1$, and, if $s_i = 2$, $K_{m+1}(c_i)$ contains no more cycles than $K_m(c_i)$. By repeating this argument a finite number of times an edge-colouring of $K_{n-1}$ that satisfies (A1), (A2) and (A3), with $m$ replaced by $n-1$, can be found.

We first show that such a colouring of $K_{n-1}$ can be used to find the required factorization of $K_n$.

First note that each vertex in $K_{n-1}$ has degree $n-2$, that $(s_1, \ldots, s_t)$ is a composition of $n-1$, and that (A3) is satisfied. Therefore the edge-coloured $K_{n-1}$ has the property (P): Each vertex is incident with $s_i$ edges of colour $c_i$ for $t-1$ values of $i$, and with $s_i - 1$ edges of colour $c_i$ for one value of $i$.

From $K_{n-1}$ we obtain $K_n$ by adding a vertex $v_n$ and joining it by one edge to each existing vertex. With $m = n-1$, (A1) becomes $f_i \geq s_i(n/2 - 1)$ $(1 \leq i \leq t)$. Therefore

$$\sum_{i=1}^t f_i \geq \frac{(n-1)(n-2)}{2},$$

and, as there are $(n-1)(n-2)/2$ edges in $K_{n-1}$, each $f_i$ must be exactly $s_i(n/2-1)$. If $s_i = 1$, then there are $n/2 - 1$ edges coloured $c_i$. Since, by (A3), these edges are independent, there is just one vertex not incident with an edge coloured $c_i$ in $K_{n-1}$. The edge joining this vertex to $v_n$ is coloured $c_i$. Thus $K_n(c_i)$ is a 1-factor. If $s_i = 2$, then, by (P), each vertex must be incident with at least one edge coloured $c_i$. Thus the $n-2$ edges coloured $c_i$ cannot all lie in cycles; there must be exactly one path of edges coloured $c_i$. The vertices at either end of this path are joined to $v_n$ by edges coloured $c_i$. Hence $K_n(c_i)$ is a 2-factor containing one more cycle than $K_{n-1}(c_i)$. We have described how to colour $n-1$ edges incident to $v_n$, and, by (P), these edges must be distinct.

Now consider the case when $m < n - 1$. Construct a bipartite multigraph $B$ with vertex sets $\{c'_1, \ldots, c'_t\}$ and $\{v'_1, \ldots, v'_m\}$. For $1 \leq i \leq t$, $1 \leq j \leq m$, join
with \( v_i \) by \( x \) edges, \( x \in \{0, 1, 2\} \), if there are \((s_i - x)\) edges of colour \( c_i \) incident with \( v_j \) in \( K_m \). (Consider the bipartite graph with the same vertex sets as \( B \) in which each \( c_i' \) is joined to each \( v_j' \) by \( s_i \) edges. If for each edge \((v_j, v_k)\) of colour \( c_i \) in \( K_m \) we delete \( c_i'v_j' \) and \( c_i'v_k' \), then \( B \) will be obtained.) Notice that

\[
d_B(v_j') = \left( \sum_{i=1}^p s_i \right) - (m - 1) = n - m \quad (1 \leq j \leq m).
\]

Also notice that \( d_B(c'_i) = s_i m - 2f_i \). Then, considering (A1), we find that

\[
d_B(c'_i) \leq s_i m - 2s_i(m - n/2) = s_i(n - m) \quad (1 \leq i \leq t).
\]

Let \( B \) be given an equitable edge-colouring with \( n - m \) colours, \( \kappa_1, \ldots, \kappa_{n-m} \). Let \( B^* \) be the multigraph induced by the edges coloured \( \kappa_1 \) and \( \kappa_2 \). Notice that

\[
\begin{align*}
d_{B^*}(v_j') &= 2 \quad (1 \leq j \leq m), \\
d_{B^*}(c'_i) &\leq 2s_i \quad (1 \leq i \leq t), \\
|E(B^*)| &= 2m.
\end{align*}
\]

For \( 1 \leq i \leq t \), if in \( K_m \) two vertices \( v_j \) and \( v_k \) lie at either end of a path of edges coloured \( c_i \), where \( s_i = 2 \), and in \( B^* \) \( v_j' \) and \( v_k' \) are both adjacent to \( c'_i \), then \( v_j' \) and \( v_k' \) form an \( i \)-pair.

From \( B^* \) a further bipartite multigraph \( B^+ \) is constructed. If \( s_i = 2 \), vertex \( c'_i \) is split into two vertices, \( c'_{i1} \) and \( c'_{i2} \). For each edge \((c'_i, v'_j)\) in \( B^* \), if \( s_i = 1 \), then there is an edge \((c'_{i1}, v'_j)\) in \( B^+ \); if \( s_i = 2 \), then there is an edge \((c'_{i2}, v'_j)\), for some \( l \in \{1, 2\} \), in \( B^+ \). Furthermore, these latter edges are constructed such that, for \( 1 \leq i \leq t \), \( 1 \leq l \leq 2 \), \( d_{B^+}(c'_{il}) \leq 2 \), and if \( v'_j \) and \( v'_k \) are an \( i \)-pair, then in \( B^+ \) there are edges \((c'_{il}, v'_j)\) and \((c'_{il}, v'_k)\) for some \( l \in \{1, 2\} \).

\( B^+ \) is given an equitable edge-colouring with two colours, \( \alpha \) and \( \beta \). This edge-colouring is transferred to \( B^* \). Let \( B^*(\alpha) \) be the subgraph of \( B^* \) induced by edges coloured \( \alpha \). Notice that

\[
\begin{align*}
d_{B^*(\alpha)}(v'_j) &= 1 \quad (1 \leq j \leq m), \\
d_{B^*(\alpha)}(c'_i) &\leq s_i \quad (1 \leq i \leq t), \\
|E(B^*(\alpha))| &= m,
\end{align*}
\]

and, if \( v_j' \) and \( v_k' \) are an \( i \)-pair, then exactly one of the edges \((c'_i, v_j')\) and \((c'_i, v_k')\) is in \( B^*(\alpha) \).

The edge-colouring of \( K_m \) can be extended to an edge-colouring of \( K_{m+1} \) by adding a vertex \( v_{m+1} \) which is joined to each existing vertex by one edge. For \( 1 \leq j \leq m \), if \((c'_i, v_j')\) is an edge of \( B^*(\alpha) \), then \( v_{m+1}v_j \) is coloured \( c_i \). By (1), the colour of each new edge is precisely determined, and, by (2), \( v_{m+1} \) is incident with no more than \( s_i \) edges of colour \( c_i \). The construction of \( B \) ensures that no other vertex is incident with more than \( s_i \) edges of colour \( c_i \) in \( K_{m+1} \), so (A3)
remains satisfied. For $1 \leq i \leq t$, $K_{m+1}(c_i)$ contains no more cycles than $K_m(c_i)$ as we ensured, by the creation of $i$-pairs, that if there is a path of edges coloured $c_i$, then $v_{m+1}$ cannot be joined by edges coloured $c_i$ to both ends of this path.

We must check that (A1) remains satisfied with $m$ replaced by $m + 1$.

If $s_i = 1$, and initially $f_i \geq m - n/2 + 1$, then (A1) will remain satisfied; if initially $f_i = m - n/2$, then $d_B(c'_i) = n - m$, $d_{B^*}(c'_i) = 2$ and $d_{B^*(a)}(c'_i) = 1$, so one further edge in $K_{m+1}$ is coloured $c_i$ and (A1) is still satisfied.

If $s_i = 2$, and initially $f_i \geq 2(m - n/2) + 2$, then (A1) will remain satisfied; if initially $f_i = 2(m - n/2) + 1$, $d_B(c'_i) = 2(n - m) - 2$, $d_{B^*}(c'_i) \geq 2$ and $d_{B^*(a)}(c'_i) \geq 1$, so at least one further edge in $K_{m+1}$ is coloured $c_i$; if initially $f_i = 2(m - n/2)$, $d_B(c'_i) = 2(n - m)$, $d_{B^*}(c'_i) = 4$ and $d_{B^*(a)}(c'_i) = 2$, so two further edges in $K_{m+1}$ are coloured $c_i$. In both of the latter two cases (A1) remains satisfied.

## 3 Resolvable Cycle Systems

In the next section Lemma 3 will be proved by edge-colouring a graph and applying Theorem 6. In this section we introduce some results that will help us find the initial edge-colouring. A fuller description of these results can be found in [15].

An $r$-cycle system of order $rm$ is an edge-disjoint collection of $r$-cycles that partitions $K_{rm}$. A set of $m$ cycles within a system forms a parallel class if each vertex of $K_{rm}$ is incident with exactly one of the cycles. A cycle system is resolvable if the cycles can be partitioned into parallel classes. Clearly each parallel class is a 2-factor of $K_{rm}$ comprising $m$ $r$-cycles, and therefore $rm$, and hence also $r$ and $m$, must be odd. Alspach, Schellenberg, Stinson and Wagner [2] have proved the following result.

**Proposition 8** Let $r \geq 3$ and $m \geq 1$ be positive odd integers. Then there exists a resolvable $r$-cycle system of $K_{rm}$.

The analogous structure for even $rm$ is a nearly resolvable $r$-cycle system of order $rm$. This is a partition of $K_{rm}$ less a 1-factor into 2-factors each comprising $m$ $r$-cycles. The following result was proved for all cases except $m = 4$ by Alspach, Schellenberg, Stinson and Wagner [2]; the remaining case was proved by Hoffman and Schellenberg [10].

**Proposition 9** Let $r \geq 3$ and $m \geq 1$ be positive integers. Then there exists a nearly resolvable $r$-cycle system of $K_{rm}$ if and only if $rm$ is even and $(r, m) \notin \{(3, 2), (3, 4)\}$.
4 Proof of Lemma 3

Let $p$ and $q$ be the number of 2-factors in the 2-factorizations of $K^*_{rm}$ and $K^*_n$ respectively. Then

$$p = \left\lfloor \frac{rm-1}{2} \right\rfloor, \quad q = \left\lfloor \frac{n-1}{2} \right\rfloor.$$ 

We know from Propositions 8 and 9 that $K^*_{rm}$ has a 2-factorization in which the 2-factors $F_1, \ldots, F_p$ each consist of $m$ $r$-cycles. Very roughly, the idea of the proof is to separate out $q$ sets of $k$ $r$-cycles, each such set lying in one of $F_1, \ldots, F_p$, and to colour these sets of cycles with the colours $c_1, \ldots, c_q$. Then the remaining edges of $K^*_{rm}$ are coloured using the colours $c_1, \ldots, c_q$, but without creating any further cycles in these colours. Theorem 6 is used to extend each colour class to a 2-factor of $K^*_n$, where the 2-factor contains $k$ $r$-cycles and an $(n-kr)$-cycle, and where the set of all such 2-factors forms a 2-factorization of $K^*_n$.

Suppose that $rm$ and $n$ are both even. Our assumption is that

$$2(rm-2) + 1 \leq n \leq \left\lfloor \frac{m}{k} \right\rfloor (rm-2) + 2.$$ 

Since $n$ is even, this is equivalent to

$$2(rm-2) + 2 \leq n \leq \left\lfloor \frac{m}{k} \right\rfloor (rm-2) + 2,$$

which in turn is equivalent to

$$2 \left( \frac{rm-2}{2} \right) \leq \frac{n-2}{2} \leq \left\lfloor \frac{m}{k} \right\rfloor \left( \frac{rm-2}{2} \right).$$

That is

$$2p \leq q \leq \left\lfloor \frac{m}{k} \right\rfloor p.$$ 

Very similar arguments show that this inequality holds in all the other cases as well.

In the edge-colouring of $K^*_{rm}$ which we shall obtain from the 2-factorization $F_1, \ldots, F_p$ of $K^*_{rm}$, each colour $c_i, 1 \leq i \leq q$, will be used on the edges of $k$ $r$-cycles all belonging to the same 2-factor. Let us show that this is possible. Since each of the $p$ 2-factors of $K^*_{rm}$ contains $m$ $r$-cycles, it is possible to select $\left\lfloor \frac{m}{k} \right\rfloor$ sets of $k$ $r$-cycles from any 2-factor of $K^*_{rm}$, and so it is possible to pick out altogether $\left\lfloor \frac{m}{k} \right\rfloor p$ sets of $k$ $r$-cycles, each of the cycles in each set lying in the same 2-factor of $K^*_{rm}$. Thus, since $q \leq \left\lfloor \frac{m}{k} \right\rfloor p$, it is possible to colour the edges of $q$ sets of $k$ $r$-cycles so that the edges in each $r$-cycle of each set receive the same colour, and no vertex has more than two edges of any colour incident with it. Therefore, for
1 \leq i \leq q$, the colour $c_i$ is used on the edges of $k$ $r$-cycles of the 2-factor $F_j$ if $i \equiv j \mod p$.

Since $q \geq 2p$ it follows that, in particular, the two colours $c_j$ and $c_{p+j}$ are used on the edges of the 2-factor $F_j$ of $K_{rm}^*$. If $rm$ and $n$ are both even, then we actually consider $K_{rm}$ with a further colour, $c_{q+1}$, used on a 1-factor. In the cases when $rm \equiv n \mod 2$ and when $n$ is even and $rm$ is odd, then, from any given 2-factor, the selection of the sets of $k$ $r$-cycles to be coloured the same can be made arbitrarily. Any remaining cycles in a 2-factor $F_j$ can be $(c_j, c_{p+j})$-coloured.

We now show that we can apply Theorem 6 to obtain the required 2-factorization of $K_n^*$. We consider the four cases separately.

Case 1: $n \equiv rm \equiv 1 \mod 2$.

Clearly (A2) and (A3) are satisfied. To verify (A1), that $f_i \geq 2rm - n$ ($1 \leq i \leq q$), we note that, since $n \geq 2(rm - 1) + 1$, (A1) follows from $f_i \geq 1$. This is clearly true since each colour $c_i$ is used on all the edges of some $r$-cycle.

Case 2: $n \equiv rm \equiv 0 \mod 2$.

In this case we now have an edge-colouring of $K_{rm}$ with a further colour, $c_{q+1}$, occurring on a 1-factor. We need to extend this to an edge-colouring of $K_n$ in such a way that $c_{q+1}$ occurs on a 1-factor of $K_n$, and the other colours each form a 2-factor with $k$ $r$-cycles and an $(n - kr)$-cycle. (A2) and (A3) are clearly satisfied. For (A1) we have to show that

$$f_i \geq 2rm - n \quad (1 \leq i \leq q), \quad \text{and} \quad f_{q+1} \geq rm - n/2.$$

Since $n \geq 2(rm - 2) + 2$ and $n$ is even, these inequalities follow from the inequalities $f_i \geq 2$, $1 \leq i \leq q$, (which is true since $c_i$ occurs on all the edges of at least one $r$-cycle) and $f_{q+1} = rm/2 = p + 1 \geq 1$.

(Notice that in Lemma 4 we can let $m = 2k$ and then we also have $n \equiv rm \equiv 0 \mod 2$. The proof is essentially the same as for Case 2 of Lemma 3. We can put $p = rk - 1$, $q = 2(rk - 1)$, and then the argument follows easily.)

Case 3: $n \equiv rm + 1 \equiv 0 \mod 2$.

In this case the 2-factorization of $K_n^*$ that we require is equivalent to a factorization of $K_n$ with one further colour, say $c_{q+1}$, occurring on a 1-factor. Since $n \geq 2(rm - 1) + 1$ and $n$ is even, we have $n \geq 2rm$, so (A1) reduces to the vacuous condition $f_i \geq 0$ ($1 \leq i \leq q + 1$).

Case 4: $n \equiv rm + 1 \equiv 1 \mod 2$.  

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In this case the 2-factorization of $K^*_r$ is equivalent to a decomposition of $K^*_r$ into $p = \frac{1}{2}(rm - 2)$ 2-factors $F_1, \ldots, F_p$ and a 1-factor, say $F_{p+1}$. Also in this case $K^*_n = K_n$. As before we shall select $q$ sets of $k$ $r$-cycles, each set lying in exactly one of $F_1, \ldots, F_p$, where, if $i + sp \leq q$, one set of $k$ $r$-cycles in $F_i$ is coloured $c_{i+sp}$.

The remaining $r$-cycles in $F_i$ are $(c_i, c_{i+p})$-coloured. The extra difficulty in this case is that the edges of the 1-factor $F_{p+1}$ need to be coloured with the colours $c_1, \ldots, c_q$ so that no vertex has more than two edges of any colour incident with it and no further cycles of any colour are created. In order to carry out this task, it is convenient not to select the $k$-sets of $r$-cycles arbitrarily, but instead to proceed more cautiously. We shall colour two edges of $F_{p+1}$ with the colours used on the cycles of $F_1$, and, for $2 \leq i \leq p$, we shall colour one edge of $F_{p+1}$ with the colour used on a cycle of $F_i$. Since $F_{p+1}$ has $rm/2 = \frac{1}{2}(rm - 2) + 1 = p + 1$ edges, this will deal with all the edges of $F_{p+1}$.

Let $S$ be a set of $k$ $r$-cycles in $F_1$. At most $kr$ edges of $F_{p+1}$ are incident with vertices in $S$, so at least $\frac{1}{2}rm - kr \geq \frac{1}{2}(2k + 1)r - kr = \frac{1}{2}r$ edges of $F_{p+1}$ are not incident with vertices in $S$. Therefore if $r \geq 4$, there are at least two edges of $F_{p+1}$ that are not incident with vertices in $S$. If $r = 3$ then $m$ is even, so $m \geq 2k + 2$, and the same conclusion may be drawn. Let $e^*$ and $e^+$ be edges of $F_{p+1}$ that are not incident with cycles in $S$.

First suppose that $k = 1$, so that we require only one $r$-cycle of each colour. Then $S$ contains just one $r$-cycle, say $C_{i+p}$. Let $e_0 = e^*$ and $e_1 = e^+$, let the remaining edges of $F_{p+1}$ be $e_2, \ldots, e_p$, and let $e_i = (v_i, w_i), 0 \leq i \leq p$. Let $C_1$ be an $r$-cycle in $F_1$ with $|V(C_1) \cap \{v_0, w_0, v_1, w_1\}| \geq 1$. We colour $C_1$ with colour $c_1, C_{i+p}$, with colour $c_{i+p}$, and $e_0$ and $e_1$ with $c_{i+p}$ as well. For each $j$ such that $1 + 2p \leq 1 + jp \leq q$, one $r$-cycle of $F_1$ will be coloured with $c_{1+jp}$. Let $T$ be a set containing the remaining uncoloured cycles of $F_1$. These cycles must be $(c_1, c_{i+p})$-coloured. They may be incident with $e_0$ and $e_1$ so we must find a way to colour them so that no $c_{i+p}$-coloured cycle containing $e_0$ or $e_1$ is formed, no vertex is incident with three $c_{i+p}$-coloured edges, and $c_1$ and $c_{i+p}$ are each used at least once on each of the cycles in $T$. We can assume that $v_1 \in C_1$. If $v_0$ is in a cycle in $T$ then we colour the two edges of that cycle incident with $v_0$ with $c_1$.

Therefore neither $e_0$ nor $e_1$ can be in a $c_{i+p}$-coloured cycle. If $w_0$ is in a cycle in $T$ then we colour one of the edges of that cycle that is incident with $w_0$ with $c_1$. Note that we have not yet coloured all the edges of any cycle in $T$ since we have only coloured three edges and if $v_0$ and $w_0$ are in the same cycle it must contain at least four edges as $v_0$ and $w_0$ cannot be adjacent. If $w_1$ is in a cycle in $T$ then we colour one of the edges of that cycle that is incident with $w_1$ with $c_1$ if we have not done so already ($w_1$ may be adjacent to $v_0$ or $v_1$). We have ensured that no vertex is incident with three $c_{i+p}$-coloured edges and we have still not coloured all the edges of any cycle in $T$ since one of the edges incident with $w_1$ (if it is in a cycle in $T$) was left uncoloured. We complete the colouring of the cycles in $T$ ensuring that $c_1$ and $c_{i+p}$ are each used on at least one edge of each cycle. For
2 ≤ i ≤ p, the cycle in $F_i$ containing $v_i$ is coloured $c_i$, and $e_i$ is coloured $c_{i+p}$. If $w_i$ is in a $(c_i, c_{i+p})$-coloured $r$-cycle, we make sure that one edge incident with $w_i$ is coloured $c_i$.

Now consider the case when $k ≥ 2$. Let $e_0 ∈ F_{p+1}$, $e_0 = (v_0, w_0)$. We can choose $S$, a set of at least two $r$-cycles now, so that the cycles in $S$ contain $v_0$ and $w_0$. Colour the edges of the cycles in $S$ with $c_1$, and colour $e_0$ with $c_{1+p}$. Recall that an edge $e^* ∈ F_{p+1}$ is disjoint from $S$. Let $e_1 = e^* = (v_1, w_1)$. Colour $e_1$ with colour $c_1$. The $r$-cycles incident with $(v_1, w_1)$ form part of a set of $k$ $r$-cycles that we colour $c_{1+p}$. Let the remaining edges of $F_{p+1}$ be $e_2, . . ., e_p$, where, for $2 ≤ i ≤ p$, $e_i = (v_i, w_i)$. For $2 ≤ i ≤ p$, $v_i$ and $w_i$ lie in a set of $k$ $r$-cycles that we colour $c_i$, and we colour $e_i$ with $c_{i+p}$. For $1 ≤ i ≤ p$, the remaining cycles of $F_i$ are $(c_i, c_{i+p})$-coloured.

Clearly (A2) and (A3) are satisfied. (A1) is satisfied since $f_i ≥ 3 = 2rm − (2(rm − 2) + 1)$, as each colour $c_i$ is used on all the edges of some cycle. □

5 Proof method for Theorem 5

In this section we describe the method we used to solve the remaining cases of $\text{OP}(r, n−r), 3 ≤ r ≤ 9,$ and $\text{OP}(r, r, n−2r), 3 ≤ r ≤ 4$. We again use Theorem 6. As in the proof of the last section, if $n$ is odd, then $s_i = 2, 1 ≤ i ≤ t$; if $n$ is even, then, for $1 ≤ i ≤ t − 1$, $s_i = 2$, and $s_t = 1$. We choose a value of $m$ and colour the edges of $K_m$. If $s_i = 2$, $K_m(c_i)$ contains $k$ $r$-cycles and no other cycles. Thus, if (A1), (A2) and (A3) are satisfied, then the edge-colouring is equivalent to a solution of $\text{OP}(r^k, n−kr)$. (A1) determines the minimum size of $K_m(c_i)$. (A2) and (A3) are clearly satisfied. We shall see that by recolouring only a few edges many solutions can be found quickly from one initial colouring.

We demonstrate this method by solving the remaining cases of $\text{OP}(5, n−5)$. Solutions are already known for $\text{OP}(5, 3)$, $\text{OP}(5, 5)$, $\text{OP}(5, 7)$, $\text{OP}(5, 9)$ and $\text{OP}(5, 11)$ [3] and for $\text{OP}(5, 13)$ and $\text{OP}(5, n−5)$ for $n ≥ 29$ (Lemmas 3 and 4). It is known that $\text{OP}(5, 4)$ has no solution [3]. We present solutions for the remaining cases. An edge-colouring of $K_m$ is given by describing the subgraph induced by each colour. Recall that $[v_1, . . ., v_r]$ is a cycle, and $(v_1, . . ., v_r)$ is a sequence of adjacent edges where $v_1$ is not adjacent to $v_r$.

$\text{OP}(5, 6)$: $m = 9$; by (A1), we require $f_i ≥ 7, 1 ≤ i ≤ 5$.

$K_9(c_1) = [1, 2, 3, 4, 5] (6, 7, 8)$
$K_9(c_2) = [1, 3, 5, 6, 9] (7, 2, 8)$
$K_9(c_3) = [1, 4, 2, 9, 8] (6, 3, 7)$
$K_9(c_4) = [3, 8, 6, 4, 9] (1, 7, 5, 2)$
$K_9(c_5) = [4, 7, 9, 5, 8] (1, 6, 2)$
OP(5, 8): $m = 9$; by (A1), we require $f_i \geq 5$, $1 \leq i \leq 6$. The solution is as above, except that five edges are recoloured to give

$$K_9(c_6) = [2, 5, 7, 3, 6]$$

OP(5, 10): $m = 13$; by (A1), we require $f_i \geq 11$, $1 \leq i \leq 7$.

$$K_{13}(c_1) = [1, 2, 3, 4, 5] (6, 7, 8) (9, 10, 11, 12, 13)$$

$$K_{13}(c_2) = [1, 3, 5, 2, 4] (7, 12, 8, 13, 11, 9, 6, 10)$$

$$K_{13}(c_3) = [1, 6, 8, 5, 10] (3, 13, 7, 9, 2, 12) (4, 11)$$

$$K_{13}(c_4) = [1, 7, 2, 6, 11] (4, 8, 3, 9, 13, 5) (10, 12)$$

$$K_{13}(c_5) = [1, 8, 10, 4, 9] (2, 13) (3, 6, 12, 5, 11, 7)$$

$$K_{13}(c_6) = [1, 12, 3, 10, 13] (2, 8, 11) (6, 4, 7, 5, 9)$$

$$K_{13}(c_7) = [2, 10, 7, 3, 11] (5, 6, 13, 4, 12, 9, 8)$$

For OP(5, 12), OP(5, 14), OP(5, 16), OP(5, 18), OP(5, 20) and OP(5, 22), the solution is as above, except that we recolour a number of edges for each new solution. For OP(5, 12) we require that each $f_i \geq 9$, for OP(5, 14) we require that each $f_i \geq 7$ and for the remaining solutions it is sufficient that each $f_i \geq 5$.

OP(5, 12): $K_{13}(c_8) = [2, 8, 3, 6, 13] (4, 7, 9, 10, 11)$

OP(5, 14): $K_{13}(c_9) = [2, 9, 5, 7, 12] (8, 13, 11)$

OP(5, 16): $K_{13}(c_{10}) = [3, 9, 7, 11, 13]$)

OP(5, 18): $K_{13}(c_{11}) = [4, 7, 6, 12, 11]$)

OP(5, 20): $K_{13}(c_{12}) = [4, 8, 11, 9, 12]$)

OP(5, 22): $K_{13}(c_{13}) = [5, 11, 10, 12, 13]$)

For OP(5, 15), OP(5, 17), OP(5, 19), OP(5, 21) and OP(5, 23), we can adapt the above solutions. The solution for OP(5, 2k − 5) is the same as for OP(5, 2k − 6), except that three edges are recoloured (a sufficient number to satisfy (A1)) to give

$$K_{13}(c_k) = (4, 13) (5, 12) (6, 9)$$

References


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