Uninsurable Risk and the Determination of Real Interest Rates: An Investigation using UK Indexed Bonds

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Abstract

This paper investigates the empirical performance of a new class of uninsurable risk models in the context of UK indexed bond market. Using closed form expressions for pricing kernels, we test the ability of three consumption-based models to price indexed bonds in the UK, and find that the standard general equilibrium, complete markets model is soundly rejected in favour of two uninsurable-risk models. Of the latter, a model which prohibits all insurance appears to perform better than a model which permits partial insurance. Using the estimated bond price equation, impulse response analysis is undertaken to understand the effects of various macroeconomic fundamental shocks on the real interest rates. In contrast to the estimates that typically arise in equity markets, the estimated coefficient of relative risk aversion and the resulting bond risk premia are found very small in this class of models with uninsurable risk.

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1 Introduction

The influence of uninsurable risk on financial risk premia has generated increasing interest among financial economists in recent years. Among the most recent papers, Kocherlakota and Pistaferri (2007, 2009) look at two uninsurable-risk settings: (i) an incomplete market environment (INC henceforth) in which idiosyncratic consumption risks are entirely uninsurable, and (ii) an alternative market environment in which these private risks are insurable but only partly so, due to incentive constraints on agents’ truth revelation about their private shocks. Kocherlakota and Pistaferri label this market structure ‘private information pareto optimal’ (PIPO), since its consumption allocation is constrained Pareto optimal. They also demonstrate that the PIPO structure performs much better in terms of explaining the equity premium and real exchange rate puzzles than the representative agent complete market and INC models.

The superior performance of the PIPO model is also apparent in Basu, Semenov and Wada (2007), in which the same three models are used in an attempt to reconcile the equity premium, international risk sharing and currency premium puzzles.

In this paper we test the ability of the uninsurable-risk models to price UK inflation-indexed government bonds. We focus on the UK indexed bond for several reasons. First, it has a long history and a simple coupon structure without option feature. Second, the issue amount of indexed bond in the UK has been increasing making it an important financial instrument. Third, indexed, or ‘real’, bonds are an attractive test bed because their prices provide the real risk free rate, which is common to the discount rates of all other assets. Consequently, if a model cannot perform well in this market it is reasonable to expect that it will face difficulties in other markets as well.\footnote{In practice, indexed bonds are not quite ‘real’ bonds since there remains a small inflation risk due to unavoidable imperfections in the indexation process. In our model, this inflation risk interacts with the uninsurable consumption risk in determining the bond risk premia.}

In addition, one can also undertake useful investigation about how the implied real interest rates depend on economic fundamentals.

Our paper makes three principal contributions. First, it models the market prices of "indexed" bonds in a general equilibrium setting with uninsurable risks, and is, to the best of our knowledge, the first paper to do so. Second, we derive closed form expressions for "indexed" bond prices based
on uninsurable risk as measured by the cross sectional, time-varying, variance of consumption. Finally, using this "indexed" bond price formula we derive implied real risk premia, and investigate the response of real rates to changes in the model's state variables.

Our estimation uses an affine form bond price equation in which the pricing kernels are derived using a lognormal process for cross sectional distribution of consumption which has strong empirical validity. This bond price is a loglinear function of three state variables whose expected future values are constructed from a vector autoregression. The lognormality of the consumption process yields nice analytical solution to bond pricing equation in the tradition of affine yield curve models. Two of the macroeconomic state variables, namely aggregate consumption growth and the growth rate of the cross sectional log variance of consumption (which represents the uninsurable risk) come directly from the underlying equilibrium model. A third factor, inflation, is included because we are fitting the market prices of indexed bonds, which are likely to depend on expected inflation due to the imperfect nature of their indexation. The estimated VAR is then used to provide the expectations proxies that allow us to estimate the structural parameters (agent's risk aversion and the rate of time preference) by fitting the closed form price equation to market data.

Our results are consistent with the common finding that the standard complete market model is not consistent with the data. The models with uninsurable risk fare much better. The estimates of the coefficient of risk aversion and the resulting bond risk premia are found to be small but these are consistent with the low returns earned on UK indexed bonds during our sample period. The impulse response analysis with the estimated bond price

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2 Battistin et al. (2007, 2009) establish that the cross-sectional consumption distribution in both the US Consumer expenditure Survey (CEX) and the UK British Family Expenditure Survey (FES) is approximately log normal within demographically homogeneous groups. This is due to the fact that the Gibrat's law applies to consumption. Brzozowski et al. (2009) provide further empirical evidence that the cross-sectional distribution of consumption within cohort groups in Canada may be approximated by a log-normal distribution. Blundell and Lewbel (1999) also provide powerful empirical evidence of log-normality of the cross-sectional distribution of consumption in a variety of data sets. Attanasio et al. (2004) assume log-normality of the cross-sectional distribution of household consumption when studying the evolution of inequality in consumption in the US both within cohorts and for the all population.

3 See for example, Campbell et al (1997, Ch. 11) for an excellent exposition of this class of affine yield models.
equations reveals that a rise in inflation lowers real interest rates of nearly all maturities. A rise in economic growth raises real interest rates consistent with the permanent income hypothesis. Greater uninsurable consumption risk, on the other hand, lowers the real interest rates driven by a precautionary saving motives.

The paper is organized as follows. The following section lays out the basic setup for the three pricing kernels. Section 3 presents the applications of these pricing kernels to UK "indexed" bond prices. Section 4 discusses the estimation methods and the data. Section 5 presents the estimation results. Section 6 ends with concluding comments.

2 Basic Setup

2.1 Three Pricing Kernels

Our benchmark case is the traditional complete market model with homogeneous agents. With a power utility function (with risk aversion parameter $\gamma$), the stochastic discount factor is given by:

$$M_{t+1}^{RA} = \frac{\beta c_{t+1}^{-\gamma}}{c_t^\gamma}$$

(1)

where $\beta$ is the subjective discount factor and $c_t$ is the aggregate consumption at date $t$.

In two influential papers (2007, 2009) Kocherlakota and Pistaferri (K-P hereafter) introduce consumer heterogeneity and uninsurable risk for two distinct market environments: (i) incomplete market ($INC$) where private skill shocks are uninsurable, (ii) partial insurance environment where the private skill shocks are partially insured by an insurance company who stipulate long term contracts with agents subject to a truth revelation constraint for eliciting efforts and private skill shocks. The latter environment is constrained Pareto efficient and K-P call it private information Pareto optimal (PIPO) environment.

Using the law of large numbers K-P demonstrate that the pricing kernels for these two market environments can be written as:

$$M_{t+1}^{INC} = \frac{\beta \sum_i c_{it+1}^{-\gamma} prob(i)}{\sum_i c_{It}^\gamma prob(i)}$$

(2)
\[ M_{t+1}^{PIPO} = \frac{\beta \sum_i c_{it} \text{prob}(i))}{\sum_i c_{it+1} \text{prob}(i)} \]  

(3)

where \( c_{it} \) is the consumption of individual \( i \) at date \( t \) and \( \text{prob}(i) \) is the cross section probability of the occurrence of the \( i \)th household in the population. In the absence of any information frictions and heterogeneity \( (x_t = 0 \text{ for all } t) \), both (2) and (3) reduce to (1).

### 2.2 Lognormal Parameterization of Consumption Processes

In a similar spirit as in Sarkissian (2003), we consider a lognormal parameterization of the post-trade consumption process. We represent the post-trade allocation of consumption as follows.\(^4\) The \( i \)th investor’s consumption is:

\[ c_{i,t} = \delta_{i,t} c_t \]  

(4)

where \( \delta_{i,t} \) is the \( i \)th investor’s share in aggregate consumption, \( c_t \). This specification basically means that the log of individual consumption is the sum of the log of aggregate per capita consumption and the uninsurable consumption due to the idiosyncratic uninsurable skill shock. De Santis (2007) also assumes this log-additive specification to estimate the welfare cost of business cycles.

We assume the following lognormal process for \( \delta_{i,t} \):

\[ \delta_{i,t} = \exp(u_{i,t} \sqrt{x_t} - \frac{x_t}{2}) \]  

(5)

where \( u_{i,t} \) is standard normal i.i.d shock, and \( x_t \) is the cross sectional variance of log consumption.

The \( s^{th} \) raw moment of the cross sectional distribution of consumption is

\(^4\)Sarkissian (2003) writes the post trade allocation in terms of consumption growth rate while we write here in terms of level of consumption. The motivation for doing this is to apply this post-trade allocation to the Kocherlakota-Pistaferri (2007, 2009) discounting methodology. The Kocherlakota-Pistaferri incomplete market discount factor is based on the growth rates of the cross sectional moments of consumption in level while Sarkissian (2003) and also Semenov (2004) use the Constantinides-Duffie (1996) discount factor which is based on the cross sectional average of the intertemporal marginal rates of substitution.
given by:

\[ E_t(c_t^s) = c_t^s \exp \left( \frac{(s^2 - s)x_t}{2} \right) \]  

(6)

Note that, by construction, aggregate consumption is the sum of individual consumption, which can be checked by setting \( s = 1 \). We now address the issue of whether or not it satisfies the optimality conditions. We follow the same reverse engineering approach as in Basu, Semenov and Wada (2008): If there is a unique pricing kernel that supports this allocation of consumption, then it must also support the individual optimality conditions.

### 2.3 Lognormal Pricing Kernels

Substituting (6) into (2), and evaluating at \( s = -\gamma \), gives the following pricing kernel for the \( INC \) environment:

\[ M_{t+1}^{INC} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \exp \left( \frac{\gamma(\gamma + 1)}{2} (x_{t+1} - x_t) \right) \]  

(7)

Similarly, substitution of (6) into (3), and evaluating at \( s = \gamma \), gives:

\[ M_{t+1}^{PIPO} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \exp \left( \frac{-\gamma(\gamma - 1)}{2} (x_{t+1} - x_t) \right) \]  

(8)

### 3 Application to UK Indexed Bonds

#### 3.1 Pricing pure-real zero-coupon bonds

We start by considering the real price, \( P_{nt}^R \), of a zero-coupon bond with maturity \( n \), and develop this into the nominal price of the imperfectly indexed coupon bonds that are traded in the UK. \( P_{nt}^R \) can be written as follows for each of the market environments, \( h = RA, INC, PIPO \):

\[ P_{nt}^R = E_t \left[ P_{n-1,t+1}^R M_{t+1}^h \right] \]  

(9)

Assuming log normality we get the following expression for the log real price of a perfectly indexed zero-coupon bond of maturity \( n \),

\[ p_{nt}^R = E_t[m_{t+1}^h + p_{n-1,t+1}^R] + \frac{1}{2} \text{Var}_t[m_{t+1}^h + p_{n-1,t+1}^R] \]  

(10)
where

\[ m^{RA}_{t+1} = \ln(\beta) - \gamma g_{t+1} \]  
(11)

\[ m^{INC}_{t+1} = \ln(\beta) - \gamma g_{t+1} + \left( \frac{\gamma(\gamma + 1)}{2} \right) v_{t+1} \]  
(12)

\[ m^{PIPO}_{t+1} = \ln(\beta) - \gamma g_{t+1} - \left( \frac{\gamma(\gamma - 1)}{2} \right) v_{t+1} \]  
(13)

and \( g_{t+1} \equiv c_{t+1} - c_t \), \( v_{t+1} \equiv x_{t+1} - x_t \).^5

### 3.2 Pricing imperfectly indexed coupon bonds

UK indexed bonds are indexed to the change in goods prices^6 over a base period starting 8 months before their issue date, and ending 8 months before their redemption date.^7 We approximate this eight-month lag by 3 calendar quarters because we are using quarterly data. Thus the total inflation compensation for an \( n \)-period zero-coupon indexed bond is \( Q_{t+n}/Q^* \), where \( Q^* \) is the goods price for the bond’s base period, which leaves the bond’s real price exposed to inflation over final 3 periods of its life. Thus equation (9) becomes

\[
P^{R}_{nt} = E_t \left[ \left( \prod_{s=1}^{n} M^h_{t+s} \right) \frac{Q_{t+n-3}}{Q^*} \frac{1}{Q_{t+n}} \right] \]

(14)

from which we get the nominal price of the bond as,

\[
P^{Nom}_{nt} = \frac{Q_t}{Q^*} E_t \left[ \left( \prod_{s=1}^{n} M^h_{t+s} \right) \frac{Q_{t+n-3}}{Q_{t+n}} \right] \]

(15)

^5 In K-P’s (2007, 2009) setup, there are both aggregate and individual shocks and the former are completely hedged by a set of aggregate-shock contingent claims. In our bond economy, if these contingent claims do not exist in addition to bonds, PIPO market environment is not constrained pareto optimal and the use of the PIPO discount factor is not justified. In order to avoid this problem, we assume that there are both these contingent claims and bonds traded but for brevity we focus on bonds and do not present a fully specified model which is available from the author upon request.

^6 Measured by the Retail Prices Index (RPI).

^7 The indexation method for UK bonds changed in 2005 (after the end of our sample). For bonds issued since that date the indexation lag is 3 months.
After log-linearizing (15) and denoting the lower cases as the log of upper cases, gives the nominal price as

\[ p_{nt}^{Nom} = (q_t - q^*) + E_t[z_{n,t+1}] + \frac{1}{2} Var_t[z_{n,t+1}] \] (16)

where

\[ z_{n,t+1} = \sum_{s=1}^{n} m_{t+s} - \sum_{s=0}^{2} \pi_{t+n-s} \] (17)

and \( \pi_{t+s} = q_{t+s} - q_{t+s-1} \).

The nominal price, in natural units, of a bond that pays a quarterly coupon\(^8\) \( C \) can then be expressed as a linear combination of zero coupon log prices as follows:

\[ P_{nt}^{Nom,c} = \sum_{s=1}^{n} \exp(p_{st}^{Nom})C + \exp(p_{nt}^{Nom}) \] (18)

This price is exposed to changes in current inflation to the extent that it influences expectations of future inflation and the consumption components of the stochastic discount factor.

4 Estimation Method and Data

Our focus is on maximum likelihood estimation of the log-linearized bond pricing models described above. We use a ‘panel’ of observed prices consisting of a time-series of a selection of about six bonds in each period. The structural parameters can be estimated from a single cross-section, or from a time-series of prices for a single bond. Subject to parameter stability, the simultaneous use of both cross-sectional and time-series data should increase the efficiency of the estimates and provide a sharper test of the model than we get from either cross-section or time-series estimation alone.

\(^8\)UK indexed coupons are paid 6-monthly. We fit this into our quarterly model by assuming half of the 6-monthly coupon to be paid each quarter. This introduces a small error due to the overvaluation of each coupon that accompanies our assumption that half of it is paid earlier than it is in reality.
4.1 A vector autoregressive model for the state variables

The nominal coupon bond price \( P_{nt}^{\text{Nom,c}} \) in (18) through (16) depends on expectations of the three state variables; consumption growth \((g)\), the change in the cross-sectional variance of consumption \((v)\), and inflation \((\pi)\), which we generate from a separately estimated vector autoregression as explained below.

Let \( w_t \) be a vector of state variables

\[
\begin{bmatrix}
g_t \\
v_t \\
\pi_t
\end{bmatrix}
\]

(19)

where all variables are in logs.

We assume the state vector to be autoregressive

\[
w_{t+1} = A + B w_t + \epsilon_{t+1}
\]

(20)

where

\[
\epsilon_{t+1} \sim N(0, \Omega_t) \quad \forall t
\]

We define a set of coefficient vectors \( \phi_R \) to be consistent with equations (11) to (13) as follows:

<table>
<thead>
<tr>
<th>( \phi_R^{RA} )</th>
<th>( \phi_R^{INC} )</th>
<th>( \phi_R^{PITP} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\gamma)</td>
<td>(-\gamma)</td>
<td>(-\gamma)</td>
</tr>
<tr>
<td>0</td>
<td>(\gamma(\gamma+1))</td>
<td>(-\gamma(\gamma-1))</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

and define a second set \( \phi_L \), to capture the effects of inflation, as

\[
\phi_L = \phi_R + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\]

The log of the pricing kernels can then be written as,

\[
m_{n,t+1}^h = \ln(\beta) + \phi_R^t w_{t+1}
\]

(21)
and, from (17),

$$\begin{align*}
E_t[z_{n,t+1}] &= \sum_{i=1}^{n-3} \phi'_R E_t[w_{t+i}] + \sum_{i=n-2}^n \phi'_L E_t[w_{t+i}] + \ln(\beta) \\
Var_t[z_{n,t+1}] &= \sum_{i=1}^{n-3} \phi'_R \Omega_{t+i} \phi_R + \sum_{i=n-2}^n \phi'_L \Omega_{t+i} \phi_L
\end{align*}$$

(22)

(23)

where

$$\begin{align*}
E_t[w_{t+i}] &= \tilde{B}_i A + B w_t \\
\Omega_{t+i} &= \sum_{j=0}^{i-1} B^j \Omega_e B^{j'} \quad \forall \ t
\end{align*}$$

(24)

(25)

and

$$\tilde{B}_i = \sum_{j=0}^{i-1} B^j$$

After substituting (22) and (23) into (16), the real price of the indexed zero-coupon bond can then be expressed in familiar affine form as:

$$p^R_{nt} = G_n + H_n w_t$$

(26)

where

$$\begin{align*}
G_n &= \ln(\beta) + \left( \sum_{i=1}^{n-3} \phi'_R \tilde{B}_i A + \sum_{i=n-2}^n \phi'_L \tilde{B}_i A \right) + \\
&\quad \frac{1}{2} \left( \sum_{i=1}^{n-3} \sum_{j=0}^{i-1} \left( \phi'_R B^j \Omega_e B^{j'} \phi_R \right) + \sum_{i=n-2}^n \sum_{j=0}^{i-1} \left( \phi'_L B^j \Omega_{t+i} B^{j'} \phi_L \right) \right)
\end{align*}$$

(27)

$$H_n = \sum_{i=1}^{n-3} \phi'_R B + \sum_{i=n-2}^n \phi'_L B$$

(28)

The log nominal price follows as $p^N_{nt} = p^R_{nt} + q_t$ which we substitute...
into (18) to obtain our estimation equation.\(^9\)

\[ p_{nt}^{Nom,c} = \sum_{s=1}^{n} \exp(p_{st}^R + q_t)C + \exp(p_{nt}^R + q_t) \] \hspace{1cm} (29)

We first estimate the vector autoregression for the state variables in order to obtain estimates of \( A, B \) and \( \Omega_e \), and then use maximum likelihood to estimate the parameters (\( \beta \) and \( \gamma \)) of the asset pricing models by fitting equation (29) to market prices. The pricing errors are assumed to be normally and independently distributed, and homoskedastic across both maturities and time.

Using all of the available data in this way greatly increases the number of degrees of freedom, but does so at the cost of imposing parameter constancy over the sample. Some degree of persistence in the parameter values seems reasonable, so our approach offers a potential efficiency gain over the familiar approach of estimating the yield curve parameters for each period independently. To allow for the possibility that the parameters change with changes in the policy regime we also estimate the model over a number of sub samples, as discussed below.

### 4.2 Data

We use bond price data from the UK Debt Management Office. Since all indexed bonds with a maturity of 8 months or less, are pure nominal bonds we select only bonds with a residual maturity of 2 years or more. The number of indexed bonds in the market in any quarter is very small, ranging from 7 to 9. We select 6 bonds in each period, aiming for as even a spread as possible across the maturities from 1 to 25 years. When choosing between bonds with similar maturities, we select the one with the largest issue size.

Aggregate real consumption data are from the Office for National Statistics, and the cross sectional variances of the log of real consumption are from the Family Expenditure Survey (FES).\(^{10}\) Data are quarterly for the period 1983Q1 to 2004Q4 and are seasonally unadjusted.\(^{11}\)

\(^9\)The details of the derivation are presented in Appendix B.

\(^{10}\)The FES was replaced by the Expenditure and Food Survey, which also covered the National Food Survey, in April 2001.

\(^{11}\)The details of the computation of the cross sectional variances are presented in Appendix A.
4.3 Sub-samples and Monetary Policy Regimes

We estimate the model over the full sample 1983Q2 to 2004Q4, and over the following sub-samples:

<table>
<thead>
<tr>
<th>Sub sample</th>
<th>Monetary policy regime</th>
</tr>
</thead>
<tbody>
<tr>
<td>1983Q2 to 1992Q3</td>
<td>Monetary-growth and exchange-rate targets.</td>
</tr>
</tbody>
</table>

From 1983 to 1992 the UK sought to anchor inflation first with control of monetary aggregates, then by using an informal combination of monetary and exchange rate targets, and finally with a 2-year membership of the European Exchange Rate Mechanism (ERM). In the post-ERM period inflation was targeted directly by the Treasury, and then, from 1997, by the newly independent Bank of England. sub-sample estimation provides an informal check of the model’s robustness. While it is unlikely that the preference parameters $\beta$ and $\gamma$ would change as a result of monetary regime changes, we might expect some instability in their estimates if the model is misspecified.

5 Results

5.1 Maximum Likelihood Estimates of $\beta$ and $\gamma$

Estimates of the coefficients $\beta$ and $\gamma$ are presented in Table 1, along with likelihood values, and t-statistics in parentheses. The traditional representative agent model does not perform well. While the estimates of $\beta$ are reasonable (implying a discount rate of about 1.5%) and highly significant, the estimates of $\gamma$ are generally imprecise, with small t values, and are sometimes negative.

The estimates for INC and PIPO are substantially better, with statistically significant estimates of $\beta$ and $\gamma$ throughout. The estimates for $\gamma$ are rather small, however, at about 0.2.\textsuperscript{12}

In terms of the likelihood values, the RA model does not perform as well as the other two, but the results for the latter are too close to each other to

\textsuperscript{12}Results of GMM estimation, following Hansen and Singleton (1982), and using the VAR factors as instruments, produced similar results for $\gamma$, with full-sample estimates of $-0.48, 0.18, 0.23$ for RA, INC and PIPO respectively.
allow us to choose between them. To the extent that a choice can be reached, it seems that the INC specification performs slightly better in the first half of the sample, while the PIPO specification is slightly better in the second. In terms of explaining the prices of indexed bonds however, it is clear that the incomplete-markets models have something to offer over the representative agent model.

We measure the goodness of fit using Nagelkerke’s (1991) generalized $R^2$,

$$R^2 = 1 - \left( \frac{L(0, 0)}{L(\hat{\beta}, \hat{\gamma})} \right)^{\frac{2}{n}}$$

where $n$ is the number of bonds in the sample, and $L$ is the value of the likelihood function.

The results are very similar for each model at around 0.05 for the full sample, and ranging from 0.1 to 0.25 for the sub-samples. The PIPO model has the highest of the three $R^2$s in each case. The measures suggest that there is a lot of variation in the prices of indexed bonds that is not accounted for by the factors underlying our models. Nevertheless, the models can explain 5% to 25% of the variation in prices, without letting the parameters of the model change from one period to the next, as is standard practice in market applications of no-arbitrage models.

### 5.2 Risk premia.

The expected 1-period return on an $n$-period pure real bond is

$$E_t[r_{n,t+1}] = -E_t[m_{t+1}] - \frac{1}{2} Var_t[m_{t+1} + p_{n-1,t+1}]$$

from which we get the expected excess return ($\tilde{r}$) of an $n$-period bond over that of an $l$-period bond as

$$E_t[\tilde{r}_{n,l,t+1}] = -(Cov_t[m_{t+1}, p_{n-1,t+1}] - Cov_t[m_{t+1}, p_{l-1,t+1}])$$

$$= -(Cov_t[\phi_R w_{t+1}, (H_{n-1} - H_{l-1}) w_{t+1}])$$

In the case of a one-period, and therefore riskless, bond we have $H_{1-1} =$
\( H_0 = 0 \) from which we get the \textit{ex ante} risk premium on an \( n \) period bond as

\[
E_t[\tilde{r}_{n,1:t+1}] \equiv \rho_n = -H_{n-1} \Omega_{n} \varphi_R
\]

Our assumption that the errors in the VAR are homoskedastic (i.e. that \( \Omega_2 \) is constant) results in \textit{ex ante} risk premia that are time invariant. Table ?? shows that the point estimates of these \textit{ex-ante} risk premia are almost zero.

The low estimated risk aversion parameter, and the resulting small implied \textit{ex ante} and \textit{ex post} risk premia, raises the question of why investors appear to demand so little compensation for term-structure risks. The poor returns on indexed bonds are noted in Dimson \textit{et al} (2002) who find that indexed bonds returned 1.25% p.a. less than Treasury bills for the period 1981 to 2000. They offer 2 possible explanations:

‘...their poor performance stems from unexpected increases in the real rate of interest particularly in the early 1980s.’

and

‘Since most inflation-indexed bonds have low coupons, and since there is no capital gains tax on UK government bonds, they are attractive to high-rate taxpayers relative to most conventional bonds. The low returns on inflation-indexed bonds may therefore also partly reflect the influence of tax clienteles’ (Dimson \textit{et al} (2002), page 86.)

Our results suggest a third possibility i.e. that the low returns may have arisen from a low aversion to risk on the part of indexed-bond investors. This explanation may complement Dimson \textit{et al}’s tax-clienteles argument. It also suggests that, while unexpected increases in real rates may be part of the explanation, they are not a necessary part since low returns could arise from low risk aversion even if interest expectations were, on average, correct.

\section*{5.3 The response of real interest rates to factor shocks.}

\subsection*{5.3.1 Impulse effects.}

We examine the impulse responses of real interest rates to shocks to the factors in the form of 1-period ahead expectations of consumption growth, the change in cross-sectional consumption variance, and inflation.
The system of equations can be represented as,

\[ p_{n,t}^R = G_n + H_n w_t \quad n = 1, 2, \ldots \]  
\[ w_t = A + B w_{t-1} + \epsilon_t \]  

from which we get the prices as functions of the history of the factor shocks \( \epsilon \) as

\[ p_{n,t}^{R,z} = G_n + H_n (I - B)^{-1} A + H_n (I - BL)^{-1} \epsilon_t \]  

Real interest rates at all maturities \( n \) follow directly from this equation.

The \( \epsilon_t \) terms are mutually correlated so we recast these as linear functions of three orthogonal random terms \( \xi_t \) and measure the response of real rates to shocks to the latter. Thus we assume that,

\[ \epsilon_{1t} = c_{11} \xi_{1t} + c_{12} \xi_{2t} + c_{13} \xi_{3t} \]  
\[ \epsilon_{2t} = c_{21} \xi_{2t} + c_{22} \xi_{2t} + c_{23} \xi_{3t} \]  
\[ \epsilon_{3t} = c_{31} \xi_{3t} + c_{32} \xi_{2t} + c_{33} \xi_{3t} \]

This leads to the familiar problem that we cannot identify all 9 \( c_{ij} \) coefficients from the 6 independent coefficient estimates in \( \hat{\Omega}_x \). We deal with this in the usual way with a Cholesky decomposition of the covariance matrix \( \Omega_x \) i.e. we impose zero-restrictions on \( c_{12}, c_{13} \) and \( c_{23} \). This is equivalent to assuming that the shock \( \xi_{1t} \) influences all three variables, \( \xi_{2t} \) influences only the latter two, and \( \xi_{3t} \) influences only the third. Since the ordering of the \( \xi \)s is not unique (we could put the 3 variables in the VAR in any order), and because we have no prior information as to the real-world ordering under these identifying restrictions (if in fact any is correct), we present results for four of the six possible orderings; for the remaining two the reordered \( \Omega \) is not positive definite and, therefore, there is no Cholesky decomposition.

Thus we define

\[ \epsilon_t = C \xi_t \]  
\[ E[\xi_t \xi_t^\prime] = I \]

where \( C \) is lower-triangular.
Hence,

$$\Omega_t = CC'$$  \hspace{1cm} (42)

Substituting (40) into the bond price equations we get:

$$p_{n,t}^R = G_n + H_n(I - B)^{-1}A + H_n(I - BL)^{-1}C\xi_t \hspace{1cm} n = 1, 2, ... \hspace{1cm} (43)$$

In order to give the shocks to the $\xi$ a clearer economic meaning we scale them such that they generate a 1 percentage point increase in each of the factors in turn. For example, in Table 3, the first row shows the effects of a shock to $\xi_{1t}$ such that consumption growth increases by 1%. In line with the ordering of the VAR, this same $\xi_{1t}$ shock also generates a contemporaneous 30.71% increase in the change of the cross-sectional variance of consumption, and a 0.12% decline in inflation. For our estimated models, the qualitative effects on yields of all of the factor shocks turn out to be robust to changes in the order of the factors.

The effect of a shock to inflation that does not cause contemporaneous shocks to the other two factors can be seen from lines 3 and 6 of Table 3. For both the INC and PIPO models, there is no impact on the 1-period (i.e. 3-month) real rate, but there are small falls in real rates at longer maturities. This is as expected: the short real rate does not respond to changes in expected inflation since agents are assumed to optimize their utility over real magnitudes, and there are no changes in the consumption factors in the utility function. At longer maturities however, the effect of a current inflation shock on expectations of future consumption growth and variance do have an impact on real rates by altering the utility value of future real returns. The negative response of longer real rates is consistent with results found in Barr and Campbell (1997) and others, and provide a possible explanation for their results. This negative impact of expected inflation on real rates arises for all of the VAR orderings, although with inflation placed at position 1 or 2 in the VAR the associated contemporaneous shocks to the other factors generate negative responses in the 3-month rate also.

Increases in consumption growth lead to increases in real rates for both models (with the exception of the 2-year rates for the INC model) irrespective of the ordering of the factors in the VAR: higher consumption growth lowers agent’s incentive to save, and financial markets respond by offering a higher
real yield as the demand for real bonds declines.

Positive shocks to the cross-sectional variance of consumption cause real rates to fall in all cases, which is consistent with the Euler equations (2) and (3) given that the estimated $\gamma < 1$ in both the INC and PIPO models. A higher cross-sectional variance in consumption, means that consumers face greater uninsurable risk. In both INC and PIPO environments with zero or partial insurance, consumers increase their saving for precautionary reasons and this increase in the supply of loanable funds drives down real interest rates. In a PIPO environment, with partial insurance, an opposing effect on saving will be at work. Greater consumption inequality (higher variance of consumption) lowers the agency cost because it is cheaper to provide incentives to poor people. This lower agency cost may create a wealth effect that lowers the saving incentive. Thus we expect that the effect on real rate will be weaker in the PIPO environment.

To summarize the main results: A positive shock to inflation lowers real rates as a consequence of its effects on consumption growth and consumption variance; a positive shock to consumption growth raises real interest rates as markets compete for reduced savings, and a positive shock to uninsurable consumption risk lowers real rates as the precautionary motive drives agents to save more.

6 Summary and conclusions

This paper tests three consumption-based asset pricing models applied to indexed bonds in the UK. We employ a three factor model of log normal bond pricing. Our novelty lies in deriving closed form expressions for the pricing kernels of the new class of uninsurable risk models and integrating this with a lognormal affine form bond pricing function. This innovation allows us to derive the price function of indexed coupon bonds in an estimable form with a convenient marriage between VAR based representation of the state variables and the bond price equation. Our central equation is a log linear bond price equation in which expected values of the state variables are constructed from a parsimonious VAR involving three macroeconomic variables, namely the growth rate of aggregate consumption, cross section variance of consumption and the rate of inflation.

Not surprisingly we find that the standard complete markets model with homogenous agents do not fare well with inflation-indexed bonds. Our
results appear to lend support to the new class of uninsurable risk models. These uninsurable-risk models can account for about 20% of the variation in indexed bond prices in sub-samples of our data. The impulse response analysis with the estimated bond price equation reveals that a rise in inflation lowers the real interest rates of almost all maturities while rise in aggregate consumption growth rate raises real interest rates. An increase in uninsurable risk, on the other hand, lowers the real interest rates. These impulse response results agree with our basic economic intuition thus lending further support to these models.

Our results, however, give rise to new questions and challenges. Why are the estimated coefficient of relative risk aversion and the resulting bond risk premia so small in the UK indexed bond market? One possible explanation is that UK bond market is extremely segmented and populated with near risk neutral institutional investors. An alternative explanation may be that our utility function could be misspecified. A more general function, combining the uninsurable risk features with the separation between risk aversion and intertemporal substitution in consumption as in Epstein and Zin (1991) may lead to larger estimates of the degree of risk. To the best of our knowledge however, there is as yet no theory that integrates these incomplete market models with nonexpected utility maximization, and is likely to be an interesting avenue for further research.
References


A Construction of the Cross Sectional Distribution of Consumption

We construct the cross sectional variance of real consumption using the records of daily expenditure from the Family Expenditure Survey (FES) conducted by the Office for National Statistics (ONS). The data we use are based on the expenditure of approximately 6,500 households for a period of 2 weeks in every quarter.

Our procedure mimics Kocherlakota and Pistaferri (2009, 2007). First, the household-wide consumption of nondurables and services is calculated by adding the nominal consumption of nondurables and services for each individual in the household. We follow the definition of nondurable and services of Attanasio and Weber (1995). Second, since the household consumption data are two week durations only, we multiply them by 6.5 to arrive at quarterly frequency. Third, we divide this quarterly consumption expenditure of each household by the number of people in the household in that quarter to arrive at the quarterly nominal, consumption of nondurables and services per member of each household unit. Fourth, by dividing the quarterly data by the quarterly CPI for all items (not seasonally adjusted) (the CPI is from the OECD main economic indicators) with the basis of 2005:Q1, we arrive at the quarterly real per capita consumption for all the relevant households.

A.1 Measurement errors

KP (2009) alert us to measurement errors from the use of cross section expenditure data. In our context, if these measurement errors appear multiplicatively they do not impact the pricing kernels. To see this define the measured consumption as:

$$\hat{c}_{i,t} = c_{i,t} \exp(\xi_{i,t})$$

where the measurement error $\xi_{i,t}$ is stationary, i.i.d. across households, and uncorrected with $z_t$, we get:

$$\hat{x}_t - \hat{x}_{t-1} = x_t - x_{t-1}$$

Since we work with the first difference of the variance of log consumption, the measurement error is not an issue.
B Derivation of the estimated price equations.

B.1 From zero-coupon to coupon bonds.

We price a coupon bond as the sum of the prices of its coupons and redemption payment. I.e.

\[ P_{nt}^{N,C} = \sum_{s=1}^{n} P_{st}^{Nom} C + P_{nt}^{Nom} \]  

(44)

hence, since

\[ P_{st}^{Nom} = \exp(p_{st}^{Nom}) \]  

(45)

etc, we have the log price of a coupon bond as

\[ P_{nt}^{N,c} = \sum_{s=1}^{n} \exp(p_{st}^{Nom})C + \exp(P_{nt}^{Nom}) \]  

(46)

where \( p_{st}^{Nom} \) and \( p_{nt}^{Nom} \) have the forms in the previous section.

B.2 Incorporating the macroeconomic factors.

First, we introduce a vector notation for the factors i.e.

\[ w_t = \begin{pmatrix} g_t \\ v_t \\ \pi_t \end{pmatrix} \]  

(47)

where \( g_t = c_t - c_{t-1} \), \( v_t = x_t - x_{t-1} \) and \( \pi_t = q_t - q_{t-1} \) and \( q_t \) is the RPI.

Now introduce 2 selection vectors to allow us to pick out different combinations of the factors i.e.

\[ \phi_R = \begin{pmatrix} \pm \gamma \\ \frac{\gamma(\gamma \pm 1)}{2} \\ 0 \end{pmatrix} \]  

(48)

\[ \phi_L = \begin{pmatrix} \pm \gamma \\ \frac{\gamma(\gamma \pm 1)}{2} \\ -1 \end{pmatrix} \]  

(49)
B.3 The stochastic discount factors.

For the stochastic discount factors (12) and (13),

\[ M_{t+i} = \beta G_{t+i} \exp \left( \frac{\gamma \pm 1}{2} (x_{t+i} - x_{t+i-1}) \right) \]  \hspace{1cm} (50)

\[ \Rightarrow m_{t+i} = \ln \beta - \gamma g_{t+i} \pm \left( \frac{\gamma \pm 1}{2} (x_{t+i} - x_{t+i-1}) \right) \]  \hspace{1cm} (51)

\[ = \ln \beta - \gamma g_{t+i} \pm \left( \frac{\gamma \pm 1}{2} v_{t+i} \right) \]  \hspace{1cm} (52)

\[ = \ln \beta + \phi'_R w_{t+i} \]  \hspace{1cm} (53)

Now substitute this expression for \( m \) into the price equation (??) to get,

\[ p_{nt}^n - (q_t - q^*) = E_t \left( \phi'_R w_{t+1} + ... + \phi'_R w_{t+n-3} + \phi'_L w_{t+n-2} + \phi'_L w_{t+n-1} + \phi'_L w_{t+n} \right) + \frac{1}{2} \text{Var}_t(...) + n \ln \beta \]  \hspace{1cm} (54)

The terms \( \phi'_R w_{t+1} + ... + \phi'_R w_{t+n-3} \) come directly from the equation for \( m \) above. The others, \( \phi'_L w_{t+n-2} + \phi'_L w_{t+n-1} + \phi'_L w_{t+n} \), are a combination of the \( m \) terms and inflation, for the last 3 months of the bond’s life i.e. the period after the indexation ends, and the bond’s real value is exposed to inflation.

So a convenient alternative way to write \( z \) is,

\[ z_{n,t+1} = (\phi'_R w_{t+1} + ... + \phi'_R w_{t+n-3} + \phi'_L w_{t+n-2} + \phi'_L w_{t+n-1} + \phi'_L w_{t+n}) \]  \hspace{1cm} (55)

B.4 Time series projections for the factors.

For the case of a VAR(1) we have,

\[ w_{t+1} = A + B w_t + \epsilon_{t+1} \]

\[ = A_n + B_n w_t + \eta_{t+n} \]  \hspace{1cm} (56)

where
\[ A_n = (I + B + ... + B^{n-1})A \]  
\[ B_n = B^n \]  
\[ \eta_{t+n} = \epsilon_{t+n} + B\epsilon_{t+n-1} + ... + B^{n-1}\epsilon_{t+1} \]

It follows that, introducing \( \Omega_{t+n} \equiv \text{Var}_t(\eta_{t+n}) \), which we assume to be constant w.r.t \( t \),

\[
E_t(w_{t+n}) = A_n + B_n w_t \\
\text{Var}_t(w_{t+n}) = \text{Var}_t(\eta_{t+n}) \\
= \Omega + B_1 \Omega B'_1 + ... + B_{n-1} \Omega B'_{n-1} \\
= \Omega_{t+n}
\]

More compactly,

\[ \Omega_{t+i} = \sum_{j=0}^{i-1} B_j \Omega B'_j \quad \forall t, \text{ and } i = 1...n \]

**B.5 Derivations of \( E_t(z) \) and \( \text{Var}_t(z) \).**

Given that

\[
z_{n,t+1} = + (\phi'_R w_{t+1} + ... + \phi'_R w_{t+n-3} + \phi'_L w_{t+n-2} + \phi'_L w_{t+n-1} + \phi'_L w_{t+n})
\]

we get,
\[ E_t(z_{n,t+1}) = E_t[\phi'_R (w_{t+1} + ... + w_{t+n-3}) + \phi'_L (w_{t+n-2} + w_{t+n-1} + w_{t+n})] \]

\[ = \phi'_R ((A_1 + B_1 w_t) + ... + (A_{n-3} + B_{n-3} w_t)) + \]
\[ \phi'_L ((A_{n-2} + B_{n-2} w_t) + (A_{n-1} + B_{n-1} w_t) + (A_n + B_n w_t)) \]

\[ = \phi'_R (A_1 + ... + A_{n-3}) + \phi'_L (A_{n-2} + A_{n-1} + A_n) + \]
\[ \phi'_R (B_1 w_t + ... + B_{n-3} w_t) + \phi'_L (B_{n-2} w_t + B_{n-1} w_t + B_n w_t) \]

\[ = \phi'_R (A_1 + ... + A_{n-3}) + \phi'_L (A_{n-2} + A_{n-1} + A_n) + \]
\[ \phi'_R (B_1 + ... + B_{n-3}) w_t + \phi'_L (B_{n-2} + B_{n-1} + B_n) w_t \]

\[ = [\phi'_R (A_1 + ... + A_{n-3}) + \phi'_L (A_{n-2} + A_{n-1} + A_n)] + \]
\[ [\phi'_R (B_1 + ... + B_{n-3}) + \phi'_L (B_{n-2} + B_{n-1} + B_n)] w_t \]

\[ = \left( \sum_{i=1}^{n-3} \phi'_R A_i + \sum_{i=n-2}^{n} \phi'_L A_i \right) + \]
\[ \left( \sum_{i=1}^{n-3} \phi'_R B_i + \sum_{i=n-2}^{n} \phi'_L B_i \right) w_t \]

(64)

\[ Var_t(z_{n,t+1}) = \phi'_R (\Omega_{t+1} + ... + \Omega_{t+n-3}) \phi_R + \phi'_L (\Omega_{t+n-2} + ... \Omega_{t+n}) \phi_L \]

\[ = \phi'_R \left( \sum_{j=0}^{n-3} B_j \Omega_j \Omega'_j + ... + \sum_{j=0}^{n-1} B_j \Omega_j \Omega'_j \right) \phi_R + \]
\[ \phi'_L \left( \sum_{j=0}^{n-2} B_j \Omega_j \Omega'_j + ... \sum_{j=0}^{n-1} B_j \Omega_j \Omega'_j \right) \phi_L \]

(65)
B.6 The final equation for a zero-coupon indexed bond.

Recall,

\[ p_{nt}^{Nom} - (q_t - q^*) = n \ln \beta + E_t(z_{n,t+1}) + \frac{1}{2} Var_t(z_{n,t+1}) \quad (66) \]

we can substitute for the conditional expectations and variances of \( w \) that appear in \( z \). The expectations introduce a series of terms in the constant \( A \), which when added to the constant conditional variance, gives us the constant term in the price equation.

The time-varying element i.e. the terms in the factors \( w_{t+i} \) are all functions of \( w_t \). Hence, the real price, \( p_{n,t}^R \equiv p_{nt}^{Nom} - (q_t - q^*), \) is

\[ p_{n,t}^R = G_n + H_n w_t \quad (67) \]

where

\[ G_n = \ln(\beta) + \left( \sum_{i=1}^{n-3} \phi^'_R A_i + \sum_{i=n-2}^{n} \phi^'_L A_i \right) + \]

\[ \frac{1}{2} \left( \sum_{i=1}^{n-3} \sum_{j=0}^{i-1} (\phi^'_R B_j \Omega_c B_j^' \phi^'_R) + \sum_{i=n-2}^{n} \sum_{j=0}^{i-1} (\phi^'_L B_j \Omega_c B_j^' \phi^'_L) \right) \quad (68) \]

\[ H_n = \sum_{i=1}^{n-3} \phi^'_R B_i + \sum_{i=n-2}^{n} \phi^'_L B_i \quad (69) \]
Table 1: Estimation results.

<table>
<thead>
<tr>
<th></th>
<th>RA</th>
<th></th>
<th>INC</th>
<th></th>
<th>PIPO</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta$</td>
<td>$\gamma$</td>
<td>LF</td>
<td>$\beta$</td>
<td>$\gamma$</td>
<td>LF</td>
</tr>
<tr>
<td>1983-2004</td>
<td>0.9950</td>
<td>-0.01446</td>
<td>0.9958</td>
<td>0.1328</td>
<td>0.9962</td>
<td>0.1727</td>
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<tr>
<td></td>
<td>( 2299. )</td>
<td>( -0.3115 )</td>
<td>( 5592. )</td>
<td>( 11.16 )</td>
<td>( 4565. )</td>
<td>( 8.012 )</td>
</tr>
<tr>
<td>1983-1992</td>
<td>0.9948</td>
<td>-0.007153</td>
<td>0.9955</td>
<td>0.1319</td>
<td>0.9957</td>
<td>0.1607</td>
</tr>
<tr>
<td></td>
<td>( 2218. )</td>
<td>( -0.1666 )</td>
<td>( 4122. )</td>
<td>( 7.613 )</td>
<td>( 3169. )</td>
<td>( 5.423 )</td>
</tr>
<tr>
<td>1992-2004</td>
<td>0.9959</td>
<td>0.05012</td>
<td>0.9962</td>
<td>0.1336</td>
<td>0.9967</td>
<td>0.1897</td>
</tr>
<tr>
<td></td>
<td>( 739.8 )</td>
<td>( 0.3478 )</td>
<td>( 3829. )</td>
<td>( 8.043 )</td>
<td>( 3033. )</td>
<td>( 5.356 )</td>
</tr>
<tr>
<td>1992-1997</td>
<td>0.9964</td>
<td>0.3363</td>
<td>0.9941</td>
<td>0.1371</td>
<td>0.9947</td>
<td>0.2085</td>
</tr>
<tr>
<td></td>
<td>( 781.7 )</td>
<td>( 2.635 )</td>
<td>( 3472. )</td>
<td>( 12.59 )</td>
<td>( 2852. )</td>
<td>( 7.320 )</td>
</tr>
<tr>
<td>1997-2004</td>
<td>0.9975</td>
<td>0.06092</td>
<td>0.9977</td>
<td>0.1353</td>
<td>0.9983</td>
<td>0.2004</td>
</tr>
<tr>
<td></td>
<td>( 762.7 )</td>
<td>( 0.4200 )</td>
<td>( 4006. )</td>
<td>( 4.580 )</td>
<td>( 2150. )</td>
<td>( 3.094 )</td>
</tr>
</tbody>
</table>

LF is the value of the log-likelihood function. Figures in parentheses are t-statistics.
Table 2: Implied *ex-ante* risk premia.

<table>
<thead>
<tr>
<th>Period</th>
<th>RA</th>
<th>INC</th>
<th>PIPO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1992-1997</td>
<td>-5.603e-005</td>
<td>-4.364e-006</td>
<td>-1.416e-005</td>
</tr>
<tr>
<td>1997-2004</td>
<td>-1.839e-006</td>
<td>-4.257e-006</td>
<td>-1.300e-005</td>
</tr>
</tbody>
</table>
Table 3: Impulse responses to factor shocks.

<table>
<thead>
<tr>
<th>VAR order.</th>
<th>Factor shocks.</th>
<th>Yields: INC</th>
<th>Yields: PIPO</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Growth</td>
<td>Variance.</td>
<td>Inflation</td>
</tr>
<tr>
<td>g v π</td>
<td>1.000</td>
<td>30.71</td>
<td>-0.1201</td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
<td>1.000</td>
<td>7.696e-005</td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.000</td>
</tr>
<tr>
<td>v g π</td>
<td>0.004116</td>
<td>1.000</td>
<td>-0.0002494</td>
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<tr>
<td></td>
<td>1.000</td>
<td>0.0000</td>
<td>-0.1224</td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.000</td>
</tr>
<tr>
<td>g π v</td>
<td>1.000</td>
<td>8.116</td>
<td>-0.1201</td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
<td>0.3893</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
<td>1.000</td>
<td>0.0000</td>
</tr>
<tr>
<td>π v g</td>
<td>-0.09571</td>
<td>-1.483</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>0.001237</td>
<td>1.000</td>
<td>0.0000</td>
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<tr>
<td></td>
<td>1.000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

$g$: consumption growth, $v$: cross-sectional variance of consumption, $\pi$: inflation.

All changes are measured in percentage points.