Inflation, Human Capital and Tobin’s $q^*$

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Abstract

A less well-known empirical finding for the US and UK is a pronounced low frequency negative relationship between inflation and Tobin’s $q$, a normalized market price of capital. This stylized fact is explained within a dynamic stochastic general equilibrium model using three key features: (i) a Lucas and Prescott (1971) physical capital adjustment cost with a rising marginal cost of investment, (ii) production of human capital with endogenous growth and (iii) an inflation tax cash-in-advance economy. The baseline endogenous growth model matches the US inflation and $q$ long term correlation, while comparable exogenous growth are unable to do this, and it outperforms the exogenous growth models in explaining business cycle volatilities of $q$ and of stock returns.

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1 Introduction

The negative association between firm value and inflation in general equilibrium has been the focus of work at least since Danthine and Donaldson (1986), who use a money-in-the-utility function and an endowment economy. This focus is motivated for example by Figure 1, in which US postwar data illustrate a negative correlation between the inflation rate and Tobin’s $q$, a normalized market price of capital.\footnote{The Figure 1 data for $q$ comes from the Smithers & Co (http://www.smithers.co.uk/). The Figure 1 negative correlation also holds using the Tobin’s $q$ estimates of Hall (2001).} This association remains to be explained within a production-based dynamic stochastic general equilibrium (DSGE) model economy. This paper explains the empirical link as resulting from inflation causing less growth, lower human and physical capital accumulation rates, a lower marginal cost of physical capital investment, and subsequently a lower $q$.

![Figure 1: q and Inflation, 1960Q1-2007Q4](image)

Figure 1 has a sample period of 1960:Q1 to 2007:Q4; its negative correlation is particularly pronounced starting in the mid 1960’s. Tobin’s $q$ bottoms out around the early 1980’s when inflation peaks. The subsequent rise of $q$ coincides with an era of disinflation and high economic growth.
Then $q$ reaches an all time high before the stock market crash when inflation falls to about 1%. Since then the relationship is less clear cut although $q$ remains on a declining path while inflation rises somewhat.

Using a Baxter and King (1999) band-pass filter, with the low frequency component having a periodicity of longer than 32 quarters, Figure 2 plots the low frequency components of the US inflation and $q$ series, for the same sample period as in Figure 1, and with a window of 12 quarters that loses the first and last three years. The negative correlation is stark, at a $-0.76$ that is statistically significant at a 5% level using Newey and West (1987) heteroskedasticity-adjusted standard errors. Note that the negative relationship between inflation and $q$ is particularly a low frequency phenomenon; at a business cycle frequency, with a periodicity of 6 to 32 quarters, the correlation coefficient between inflation and $q$ is $-0.07$ and insignificant at a 5% level.

UK quarterly data exhibits a similar low frequency negative correlation of $-0.77$, also significant at a 5% level. Figure 3 plots this low frequency $q$ and inflation relation over the sample period 1989:Q1 to 2009:Q4. Given the closeness of the US and UK correlation coefficient, the calibration is based on US data with the idea that the results may also be suggestive for the UK.

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2UK quarterly CPI data is from the Office of National Statistics; Tobin’s $q$ data is from the Bank of England, in which the methodology for computing $q$ is described in Price and Schleicher (2005).
Our DSGE model has three features which give rise to such a negative relation between $q$ and inflation, namely (i) a Lucas and Prescott (1971) physical capital adjustment cost with a rising marginal cost of investment, (ii) human capital investment that endogenizes the balanced growth path equilibrium ($BGP$) growth rate, and (iii) a cash-in-advance inflation tax
economy. A higher inflation rate, as a result of the model’s shocks, induces agents to take more leisure since the proceeds from work are subject to the inflation tax (Gomme, 1993, and Gillman and Kejak, 2005). This reduces human capital utilization, the $BGP$ growth rate, the rate of accumulation of both human and physical capital, the marginal cost of physical capital investment, and so also $q$.

A related paper is by McGrattan and Prescott (2005), who argue that the rise until 2000 of the stock price to GDP ratio is due to lower taxes on corporate distributions to shareholders. McGrattan and Prescott do not explore the inflation-stock price linkage; our paper can be viewed as an inflation tax counterpart that is related to the corporate tax results of McGrattan and Prescott (2005). The mechanism in our paper is different, in that the inflation acts as a tax on goods that induces less time being used productively and thereby a lower utilization rate of human capital. Second, McGrattan and Prescott focus on the role of intangible capital in determining stock price behaviour; we base our model on human capital, which is one form of such intangible capital.

The fusion of the Beckerian (1975) time allocation aspect of human capital acquisition in an endogenous growth, costly physical capital adjustment, setting with asset prices and cash-in-advance is the novelty of the model. In a sense it is a straightforward combination of four of Bob Lucas’s papers: Lucas and Prescott (1971), Lucas (1978), Lucas (1980), and Lucas (1988). Such a key role for human capital based endogenous growth is reasonable given the empirical support, ranging from US-UK times series work such as Kocherlakota and Yi (1997), to a $DSGE$ setting with shocks to human capital productivity as in Maffezzoli’s (2000) explanation of international business cycle facts.

Instead of a quadratic cost of physical capital adjustment, the specific adjustment cost function is the case of the Lucas and Prescott (1971) func-

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3 While McGrattan and Prescott (2005) look at the behavior of stock price/GDP ratio, we examine the Tobin’s $q$. The Tobin’s $q$ is the ratio of stock price index to the capital stock which can be alternatively written as stock price/GDP multiplied by GDP/capital stock. Given that the output/capital ratio is stable, as pointed out by McGrattan and Prescott (2005), the stock price/GDP also reflects the behavior of Tobin’s $q$. 


tion that is found in Basu (1987) and Hercowitz and Sampson (1991), except now allowing for less than 100% depreciation. Combined with an endogenous growth setting, this allows \( q \) to be expressed as a simple function of the growth rate (Proposition 4), with a rising marginal cost of investment, for which Belo et al. (2010) find empirical support.

There are two types of structural shocks, real productivity shocks, one in each the goods and human capital investment sectors, and a money supply growth rate shock. Both productivity shocks tend to induce a negative correlation between inflation and \( q \) as well as between inflation and growth, but the human capital sector shock has such an effect that is an order of magnitude stronger than that of the goods sector. The monetary shock induces a Tobin (1965) type effect of an increase in physical capital accumulation that initially weakens the negative \( q \)-inflation correlation, but then marginally strengthens this correlation over an extended period.

A comparison is made of the baseline endogenous growth model to an alternate endogenous growth model differing by only one parameter, and to two versions of the exogenous growth model. The baseline model best matches the low frequency correlation between \( q \) and inflation (Table 5), and the business cycle volatility of both \( q \) and a measure of the stock return (Table 6). The alternative endogenous growth model best fits the volatilities of the growth rate and the investment rate. A qualification is that all of the models overstate the inflation rate volatility, as the model was kept as simple as possible without price adjustment factors.

Section 2 sets out the model, Section 3 the analytic BGP equilibrium \( q \), and Section 4 the calibration and impulse response analysis. Section 5 presents the low frequency correlation and business cycle volatility results, while Section 6 concludes.

2 The Model

2.1 The Representative Household

The representative household allocates time between leisure \( (x_t) \), work in the goods sector \( (l_{Gt}) \) at a nominal wage \( W_t \), and work in the human capital
investment \((I_{Ht})\). Households own the human capital \((ht)\) and augment it through human capital investment. Firms own the physical capital \((kt)\) and accumulate it through physical investment \((it)\).

At time \(t\), households first trade in goods with the cash held in advance, \(Mt\), and then they visit the asset markets to trade in stocks at the ex-dividend prices \(V_t\) and in nominal bonds at the price \(P_{bt}\). Nominal bonds \(B_t\) held at date \(t\) pay 1 unit of currency with certainty in the following periods. Money is used to buy goods, and is augmented by the central bank through a stochastic nominal lump-sum transfer \(N_t\), which with market clearing in equilibrium equals \(\mu_tM_{t-1}; \mu_t\) is the stochastic growth rate of money supply.\(^4\)

At date \(t\), the revenues of the household are nominal dividends per ownership share in the goods producer, \(D_t\), factored by the shares \(z_t\), plus wages \(W_tl_{Gt}h_t\) and the lump sum transfer \(N_t\). Expenses are investment in bonds, \(P_{bt}B_{t+1}-B_t\), in cash, \(M_t-M_{t-1}\), and in stocks, \(V_t(z_{t+1}-z_t)\), plus consumption purchases \(P_tc_t\).

The household maximizes the following life time utility function:

\[
\text{Max } E_0 \sum_{t=0}^{\infty} \beta^t \{U(c_t) + \psi \Gamma(x_t)\}
\]  

where \(U(.)\) and \(\Gamma(.)\) are monotonically increasing and strictly concave functions, with the parameter \(\psi \geq 0\), subject to the flow budget constraint facing the household,

\[
D_tz_t+W_tl_{Gt}h_t+N_t-(P_{bt}B_{t+1}-B_t)-V_t(z_{t+1}-z_t)-(M_t-M_{t-1})-P_tc_t = 0,
\]

\(1 = x_t + l_{Gt} + l_{Ht}\),

and human capital accumulation and exchange constraints.

Human capital investment is linear in effective labor time \(l_{Ht}h_t\) as in Lucas (1988), with a depreciation rate of \(\delta_h\) and with \(A_{Ht}\) the exogenous

\(^{4}\)Gillman et al. (2007) demonstrate how a related endogenous growth economy implies an equilibrium Taylor (1993) condition so that interest rate (“speed-limit”) rules and exogenous money supply growth rate targets are synonymous.
sectoral productivity shock, giving the accumulation constraint of

\[ h_{t+1} = (1 - \delta_h)h_t + A_{Ht}l_{Ht}h_t. \]  

(4)

The exchange constraint requires money to purchase consumption such that

\[ P_t c_t \leq M_{t-1} + N_t. \]  

(5)

The household first order conditions are found in Appendix A.1. The standard stochastic discount factor \( m_{t+1} \) facing the household is given by

\[ m_{t+1} = \frac{\beta E_{t+1} \left[ U'(c_{t+2}) \frac{c_{t+2}}{c_{t+1}} \frac{1}{1 + \mu_{t+2}} \right]}{E_t \left[ U'(c_{t+1}) \frac{c_{t+1}}{c_t} \frac{1}{1 + \mu_{t+1}} \right]}. \]  

(6)

Using the equations (A.13) and (A.14) in Appendix A.1, the stock price and bond price equations can be written typically as

\[ 1 = E_t m_{t+1} \left\{ v_{t+1} + d_{t+1} \right\}, \]  

(7)

and

\[ p_t^b = E_t m_{t+1} \]  

(8)

where \( v_t \) is the real share price, \( v_t \equiv V_t/P_t \) and \( p_t^b \) is the real price of bond that satisfies \( p_t^b \equiv \frac{p_t^b}{P_t} \).

2.2 The Firm’s Problem

The firm produces output \( y_t \) with a Cobb-Douglas production function \( A_{Gl}F(k_t, l_{Gl}h_t) \) in physical capital \( k_t \) and effective labor \( l_{Gl}h_t \), with \( A_{Gl} \) the stochastic total factor productivity (TFP) at date \( t \), and \( \alpha \in (0, 1) \), such that

\[ y_t = A_{Gl}F(k_t, l_{Gl}h_t) = A_{Gl}k_t^\alpha (l_{Gl}h_t)^{1-\alpha}. \]  

(9)
The firm costs are wages and the nominal physical capital investment \( P_t i_t \). With \( \lambda_t \) the shadow price of the consumer’s nominal income in equation (2), \( A_{Gt} \) the stochastic total factor productivity (TFP) at date \( t \), and \( \alpha \in (0,1) \), the firm solves

\[
\max_{i_{Gt}} \quad E_0 \sum_{t=0}^{\infty} \lambda_t \left[ P_t A_{Gt} k_t^\alpha (l_{Gt} h_t)^{1-\alpha} - W_t l_{Gt} h_t - P_t i_t \right]
\]

subject to the physical capital accumulation constraint, for \( \delta_k \in (0,1) \) and \( \theta \in (0,1) \) of

\[
k_{t+1} = k_t \left[ 1 - \delta_k + \frac{i_t}{k_t} \right]^\theta,
\]

as in Basu (1987) and Hercowitz and Sampson (1991). The parameter \( \theta \) represents the extent of adjustment cost; with \( \theta = 1 \) there is no adjustment cost.

The marginal cost of investment, \( MC_t \), can be expressed by solving for \( i_t \) in equation (11), and differentiating with respect to next period capital, as

\[
MC_t \equiv \frac{\partial i_t}{\partial k_{t+1}} = \frac{1}{\theta} \left( \frac{k_{t+1}}{k_t} \right)^{\frac{1-\theta}{\theta}} = \frac{1}{\theta} \left( 1 - \delta_k + \frac{i_t}{k_t} \right)^{1-\theta},
\]

which is rising in \( k_{t+1} \), or in \( i_t \). Figure 4 graphs the \( MC_t \) function for a varying \( \frac{i_t}{k_t} \), given \( \theta = 0.8 \), and \( \delta_k = 0.03 \), as in the baseline calibration below. The marginal cost rises as the investment rate rises.
This $MC_t$ is almost linear in a range that is specified for reasonable growth rates, since on the $BGP$ it is true that $\frac{i_t}{k_t} = (1 + g)^{\frac{1}{\nu}} - 1 + \delta_k$, which equals $g + \delta_k$ as $\theta \to 1$. This standard near-linearity holds for most values of $\theta$, such as for $\theta \in (0.15, 1)$, within the growth rate range, while for very low values of $\theta$ some concavity is evident.

In comparison, Belo et al (2010) use a related adjustment cost function, whereby investment plus their adjustment cost, with the sum denoted by $C_t$, is given with their parameters of $\nu$ and $a$ by

$$C_t \equiv i_t + \frac{a}{\nu} \left( \frac{i_t}{k_t} \right)^\nu k_t,$$

with a marginal cost of $\frac{\partial C_t}{\partial k_{t+1}} = 1 + a \left( \frac{i_t}{k_t} \right)^{\nu-1}$. Like Figure 4, this gives a rising marginal cost, but one that can be quite convex, mainly through the curvature parameter, $\nu$. They find empirical support for significant convexity in their GMM estimation; however these interesting results are based on a partial equilibrium model that is not directly comparable to our DSGE, endogenous growth setting.

The endogenous growth setting in particular distinguishes our model for example from the DSGE model of Christiano et al (2007). Their comparable
equation to our equation (11) is

\[
\frac{k_{t+1}}{k_t} = 1 - \delta + \frac{i_t}{k_t} - z \left( \frac{i_t}{k_t} - \eta \right)^2,
\]

where \( z \) is a parameter and \( \eta \) is the steady state investment to capital ratio. Their adjustment cost is therefore zero in the steady state, as in Lucas (1967), while in our specification in equation (11), the adjustment cost is positive along the balanced growth path equilibrium.

2.3 Forcing Processes

The exogenous variables \( A_{Gt}, A_{Ht}, \mu_t \) follow the processes:

\[
A_{Gt} - \bar{A}_G = \rho_G (A_{Gt-1} - \bar{A}_G) + \epsilon_t^G \tag{13}
\]

\[
A_{Ht} - \bar{A}_H = \rho_H (A_{Ht-1} - \bar{A}_H) + \epsilon_t^H \tag{14}
\]

\[
\mu_t - \bar{\mu} = \rho_\mu (\mu_{t-1} - \bar{\mu}) + \epsilon_t^\mu \tag{15}
\]

where \( \epsilon_t^G, \epsilon_t^H, \epsilon_t^\mu \) are white noises with standard deviations \( \sigma_G, \sigma_H \) and \( \sigma_\mu \) respectively. We assume zero contemporaneous covariances between these three shocks. Letters with a bar represent steady state values.

2.4 Characterization of Equilibrium

(E.1): Given the processes \( \{P_t\}, \{W_t\}, \{D_t\}, \{A_{Ht}\}, \{V_t\}, \{P^b_t\}, \) and \( \{N_t\} \), the household maximizes utility in equation (1) subject to equations (2) to (5).

(E.2): Given the processes \( \{P_t\}, \{W_t\}, \{A_{Gt}\} \), the goods producer maximizes (10) subject to (11).

(E.3) : Spot assets, goods, and money markets clear: \( z_t = 1, B_t = 0, \) and \( N_t = u_t M_{t-1} \).
3 Tobin’s q

The shadow price of physical capital investment normalized by the shadow price of consumption gives a standard expression for Tobin’s q. Using the first order condition with respect to physical capital investment and equation (A.17) of Appendix A.2, one gets the expression for Tobin’s q.

**Proposition 1**

\[ q_t = \frac{\omega_t}{P_t \lambda_t} = \frac{1}{\theta} \left[ 1 - \delta_k + \frac{i_t}{k_t} \right]^{1-\theta}. \]  

**(Proof.** This follows directly from the first order condition with respect to physical capital investment, equation (A.17) of Appendix A.2, where the shadow price of consumption \( P_t \lambda_t \) is the shadow price of nominal income in equation (2) of the household problem as multiplied by the nominal price level \( P_t \). ■

**Corollary 2** Tobin’s q equals the marginal cost of investment, which is rising in \( k_{t+1} \).

**(Proof.** By equations (16) and (11), \( q_t = \frac{1}{\theta} \left( \frac{k_{t+1}}{k_t} \right)^{\frac{1-\theta}{\theta}} \), which by equation (12) is the marginal cost of investment; and \( \frac{\partial q_t}{\partial k_{t+1}} > 0. \) ■

As in any standard q model of investment ((Obstfeld and Rogoff, 1996), the marginal cost of investment equals the average q based on the stock market valuation equation. In other words,

**Proposition 3** The marginal and average q are the same; in that

\[ q_t = \frac{v_t}{k_{t+1}}. \]

**(Proof.** See Appendix. ■

As investment increases its marginal and average cost rise. And this cost is closely connected to the economy’s growth rate. Hereafter, log utility is specified, with \( U(c_t) = \ln c_t \) and \( \Gamma(x_t) = \ln x_t \). Along the BGP, the q depends positively on the growth rate, and in turn on the return to capital.
Proposition 4 Along the balanced growth path, Tobin’s $q$ is a simple rising function of the growth rate and a falling function of the adjustment cost parameter $\theta$ whereby

$$q = \frac{1}{\theta} (1 + g)^{1-\theta}, \quad (17)$$

and this can be expressed through $g$ in terms of either the return on physical or on human capital.

Proof. From Corollary 1, and given that $\frac{k_{t+1}}{k_t} = 1 + g$ along the $BGP$, then $q = \frac{1}{\theta} (1 + g)^{1-\theta}$ and $\frac{\partial q}{\partial g} > 0$, and $\frac{\partial q}{\partial \theta} < 0$. Further, as shown in Appendix B, the balanced growth rate in this economy is given in terms of the physical capital net return $A_G F_1 - \delta_k$ by

$$1 + g = \left[ \frac{\beta \theta (1 + A_G F_1 - \delta_k)}{1 - \beta (1 - \theta)} \right]^\theta, \quad (18)$$

and by in terms of the human capital net return of $\bar{A}_H (1 - x) - \delta_h$ by

$$1 + g = \beta [1 + \bar{A}_H (1 - x) - \delta_h]. \quad (19)$$

And so

$$q = \frac{1}{\theta} \left[ \frac{\beta \theta (1 + A_G F_1 - \delta_k)}{1 - \beta (1 - \theta)} \right]^{1-\theta} = \frac{1}{\theta} \left( \beta \left[ 1 + \bar{A}_H (1 - x) - \delta_h \right] \right)^{1-\theta}. \quad (20)$$

A higher $BGP$ return on capital, with the return on human and physical capital equal along the $BGP$, causes a higher growth rate and a higher $q$. A persistent shock that lowers the growth rate on the $BGP$ is likely to cause a low frequency decrease in $q$. For example, an increase in the productivity factors $A_G$ and $\bar{A}_H$ cause the $BGP$ $q$ to rise. A persistent positive money supply rate increase, of $\bar{m}$, causes higher inflation over time, substitution from goods to leisure, a lower human capital utilization rate of $1 - x$, and
a lower return on both human and physical capital. This cause Tobin’s $q$ to fall over time, which should be reflected in low frequency data. With exogenous growth, or without an adjustment cost of physical capital (if $\theta = 1$ and $q = 1$), there is no interaction between growth, the capital return and $q$ that produces the low frequency inflation and $q$ correlation found in the data.

4 Calibration

In calibrating a standard DSGE growth model, typically only business cycle properties are matched, using exogenous growth models. Endogenous growth also allows examination of long run, low frequency, properties of the simulated model relative to the data. This additional step involves setting the structural parameters to calibrate the growth component of the model, along with low frequency and business cycle aspects. Here, the focus is on the pronounced low frequency fluctuation in two endogenous variables, Tobin’s $q$ and inflation.

4.1 Data

Following Baxter and King (1999), the low frequency component of a series has a periodicity of longer than 32 quarters, the business cycle component a periodicity of 6 to 32 quarters, and the high frequency component a periodicity of 2 to 6 quarters, given a minimum duration of a cycle as being 2 quarters. Therefore the low frequency component is identified using a band pass filter to filter out the periodicity of 2 to 32 quarters.

For the target variables below in Table 1, the data are annual averages of quarterly post-1960 US data, from the National Income and Product Accounts, except $q$ which is from Hall (2001), and leisure which is from the Bureau of labor Statistics (BLS). For the average values of target variables the data period is 1960 to 1999, since we are constrained by the need to target a plausible historical $q$ that is greater than one; for 1960 to 2007 data, $q$ falls below unity. However for the volatility data, found below in Table 5, the data is quarterly from 1960 to 2007. For the $q$ volatility, the data
is from Smithers and Co. (2007), which is computed using the methodology of Wright (2004). Note that the business cycle and low frequency properties of both the Hall and Smithers and Co. q series are similar.

One exception to the 1960 to 1999 period for the historical averages of target values in Table 1 is leisure since the BLS data starts in 1964 instead of 1960. Here, the average leisure is estimated at 0.55 by following Gomme and Rupert (2007), who have a calibrated value of 0.505. In particular, using the annual average weekly hours of work, with the total daily time of 16 hours, and a 5 day working week, normalized leisure is \(\frac{16 - (\text{average weekly hours of work}/5)}{16}\).

### 4.2 Target Variables and Parameter Values

Table 1 presents the target variables with values given from the data and the steady state calibrated model. The value of \(q\) is 1.26, while the data value of \(g\) and \(\pi\) are 3.4% and 4.01%. The average share in GDP of consumption plus government spending, which is abstracted from in the model and considered as consumption, is 84%. The calibrated model is close to the target values.

<table>
<thead>
<tr>
<th>Target Variables, 1960 – 1999</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDP Growth ((g))</td>
<td>3.4%</td>
<td>3.26%</td>
</tr>
<tr>
<td>Rate of Inflation ((\pi))</td>
<td>4.01%</td>
<td>4.03%</td>
</tr>
<tr>
<td>(c/y)</td>
<td>0.84</td>
<td>0.79</td>
</tr>
<tr>
<td>(i/y)</td>
<td>0.16</td>
<td>0.21</td>
</tr>
<tr>
<td>(q)</td>
<td>1.26</td>
<td>1.26</td>
</tr>
<tr>
<td>Leisure ((x))</td>
<td>0.55</td>
<td>0.52</td>
</tr>
</tbody>
</table>

Table 2 gives the baseline model parameter values. Standard values are chosen for \(\beta\), \(\alpha\), and \(\psi\). The mean money supply growth rate, \(\mu\) is chosen to be consistent with the 4.01% annual average inflation rate of the data. The human capital technology parameters \(\bar{A}_H\) and \(\delta_h\) are fixed to target the 3.4% annual average GDP growth rate and a human capital utilization rate \(1 - x\) equal to 0.45 based on equations (4) and (19). The physical capital depreciation rate is fixed at 0.03 in line with Benk et al (2009). Calibration
of the adjustment cost parameter $\theta$ is novel given the partial depreciation of the model. Substituting into the $q$ equation (17) the average growth rate $g$ and the average $q$, from the data in Table 1, the result is that $\theta = 0.80$, which then made the baseline value of $\theta$. While assuming 100% depreciation of physical capital, Hercowitz and Sampson (1991) for example estimate $\theta = 0.44$.

Table 2: Baseline Structural Parameter Values

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$\delta_k$</th>
<th>$\delta_h$</th>
<th>$\psi$</th>
<th>$\theta$</th>
<th>$A_G$</th>
<th>$A_H$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.96</td>
<td>0.36</td>
<td>0.03</td>
<td>0.024</td>
<td>1.84</td>
<td>0.8</td>
<td>1.2</td>
<td>0.21</td>
<td>0.0745</td>
</tr>
</tbody>
</table>

4.3 Shock Process Parameters

Table 3 reports the baseline values of the shock processes. The three forcing processes described in (13) through (15) involve six parameters, namely three autocorrelation parameters, $\rho_G$, $\rho_H$, $\rho_\mu$, and three standard deviation parameters, $\sigma_G$, $\sigma_H$, $\sigma_\mu$. The money supply parameters $\rho_\mu$ and $\sigma_\mu$ are 0.72 and 0.004, as estimated from an $AR(1)$ regression of quarterly seasonally adjusted currency supply growth from the Federal Reserve Bank of St Louis database for 1960 to 2007.

For the other two shocks, the closest paper may be Maffezzoli (2000) who employs similar stochastic goods and human capital technologies, although Maffezzoli has an international focus, plus human capital spillover and the use of both physical and human capital in the Cobb-Douglas production of human capital. As in Maffezzoli, $\rho_G$ and $\rho_H$ are both set to 0.96. And as in Maffezzoli, $\sigma_G = 0.001$. The human capital shock standard deviation, $\sigma_H$, is set at 0.003 in the baseline, with an alternate endogenous growth model calibration using $\sigma_H = 0.003$ as in Maffezzoli.

Table 3: Baseline Second Moment Parameter Values

<table>
<thead>
<tr>
<th>$\rho_G$</th>
<th>$\rho_H$</th>
<th>$\rho_\mu$</th>
<th>$\sigma_G$</th>
<th>$\sigma_H$</th>
<th>$\sigma_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.96</td>
<td>0.96</td>
<td>0.72</td>
<td>0.001</td>
<td>0.003</td>
<td>0.004</td>
</tr>
</tbody>
</table>
5 Results

Figures 5 to 7 describe the model’s impulse responses, while Table 4 presents low frequency simulations of $q$ under alternative shock assumptions. And then Table 5 presents a fuller comparison across alternative models of both low frequency values and business cycle volatilities.

5.1 Impulse Response Analysis

The impulse responses to orthogonalized shocks to $A_G$, $A_H$ and $\mu$ are based on the log-linearization of the full equation system (A.19) through (A.25) that is given in Appendix A. In Figures 5 to 7, the notation is "$iy" \equiv i/y,$ "$kh" \equiv k/h,$ "$lg" \equiv l_G,$ "$lh" \equiv l_h,$ "$infl" \equiv \pi.$

Figure 5 shows that a positive productivity shock in the goods sector makes agents substitute away from human capital investment time and leisure towards labor. This effort shocks upwards the physical capital investment rate ($iy$), with a consequent gradual increase in the physical capital to human capital ratio ($kh$). The output growth rate ($g$) falls as the physical capital investment rate rises. The greater productivity also raises the real wage and lowers the relative price of output, causing the inflation rate ($infl$) to be initially shocked downwards. The $q$ initially rises, as the investment rate and the labor in the goods sector are shocked upwards, as can be seen in equation (21), which is derived simply by using the average product of capital $\frac{k_t}{\bar{h}_t}$ and equation (16):

$$q_t = \frac{1}{\theta} \left[ 1 - \delta_k + \frac{i_t}{y_t} A_{Gt} \left( \frac{k_t}{\bar{h}_t} \right)^{\alpha - 1} l_{Gt}^{1-\alpha} \right]^{1-\theta}$$  \hspace{1cm} (21)

However as $\frac{k_t}{\bar{h}_t}$ gradually rises, this pushes $q$ down. As $\frac{k_t}{\bar{h}_t}$ begins to fall, the investment rate falls below its baseline and so does $q$. Meanwhile the inflation rate rises over time, moving in negative correlation to the $q$ effects.
Figure 5: Impulse responses with respect to an orthogonalized TFP shock

Figure 6 shows that a positive shock to $A_H$ causes agents to switch from leisure and labor in goods production towards human capital investment time, causing the growth rate to rise. The physical investment rate declines as the consumer shifts towards human capital investment and a lower $k_l/k_h$. A lower $l_G$ and investment rate shock $q$ downwards, again as in equation (21). Inflation falls over time as the increased human capital time leaves less time for goods production, causing a higher wage rate and lower relative price of output. Over time the $q$ and inflation rate effects are relatively strong in their negative correlation, as compared to the $A_G$ shock above.
In Figure 7, a positive monetary shock raises the inflation rate, thereby inducing substitution from goods to leisure and human capital investment, which are not subject to the inflation tax. The initial rise in the investment rate (iy) corresponds to the gradual rise in the physical capital to human capital ratio (kh), and a rise in q. As the capital ratio begins falling, the investment rate (iy) and q fall somewhat, even as inflation is still shocked upwards. This produces some additional negative correlation over time in the q and inflation rate effects.
Figure 7: Impulse responses with respect to an orthogonalized money shock

5.2 Low Frequency Correlation

Table 4 presents the correlations between $q$ and inflation for the baseline model and variations in the standard deviation of the two productivity shocks. The baseline model in the first simulation row does well in reproducing the data. The Table further shows in the last three simulation rows that the productivity shock to human capital investment is critical in affecting the level of the negative correlation. In contrast, changes in the productivity shock to the goods sector, in the first three simulation rows, have negligible effects. This reflects in part that the inflation and $q$ effects of the impulse responses to the productivity shocks, in Figures 5 and 6, are an order of magnitude higher for the human capital shock than for the goods sector shock. To evaluate the model’s performance against the data.
Table 4: Low Frequency Correlation between $q$ and Inflation

<table>
<thead>
<tr>
<th>US (UK) Data</th>
<th>$\sigma_H$</th>
<th>$\sigma_G$</th>
<th>Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.003</td>
<td>0.001</td>
<td>$-0.75$</td>
</tr>
<tr>
<td></td>
<td>0.003</td>
<td>0.002</td>
<td>$-0.76$</td>
</tr>
<tr>
<td></td>
<td>0.003</td>
<td>0.003</td>
<td>$-0.78$</td>
</tr>
<tr>
<td></td>
<td>0.002</td>
<td>0.001</td>
<td>$-0.69$</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>0.001</td>
<td>$-0.50$</td>
</tr>
</tbody>
</table>

Note: Other parameters as in Table 3

5.3 Alternative Model Comparison

Table 5 presents low frequency correlation and business cycle volatility results for the baseline plus three alternative models, which comprise the rows of the table. The first data column of numbers is the inflation - $q$ low frequency correlation. The next four number columns are the standard deviation of four of the six target variables in Table 1, at a business cycle frequency. And the last column is the standard deviation of the stock return, derived in the next subsection.

The alternative models are one alternative endogenous growth model that differs only by the value of one parameter, plus two exogenous growth versions of the model. For the alternative endogenous growth model, the standard deviation of the human capital shock innovation is set to 0.001 as in Maffozolli (2000), instead of 0.003 as in the baseline. The two exogenous growth versions of this model are (i) a fixed labour supply model where $l_H$ and $x$ are fixed at their steady state levels as in the baseline growth model, and (ii) a variable labour supply model where only $l_H$ is fixed at its steady level. Also in these exogenous growth variants, the standard deviation of the $A_H$ shock is set to zero, with the human capital productivity parameter fixed at its steady state level $\bar{A}_H$.

Since both $l_H$ and $A_H$ are fixed at their steady state levels, the endogenous growth channel is shut down, with human capital growing exogenously at a balanced growth rate of 3.26% as in the baseline endogenous growth...
model. Model (i) is observationally equivalent to a standard exogenous growth model with inelastic labour; model (ii) reduces to an exogenous growth model with variable labour supply. Both have comparable $BGP$ properties to the baseline model.

Both exogenous growth models have two forcing processes, $\{A_Gt\}$ and $\{\mu_t\}$ which evolve according to the $AR(1)$ representations in equations (13) and (15). On the other hand, the endogenous growth model has three shock processes. Since exogenous growth model lacks one shock process compared to the endogenous growth model, the calibration for these models reverts to the higher levels of their standard deviations, for example as in Prescott (1986) and Hansen (1985), in order to make a fair comparison. Here $A_G$ is set at a higher level than the baseline, and $\sigma_G = 0.008$.

Table 5 shows that the endogenous growth baseline model clearly outperforms the exogenous growth models with respect to the low frequency correlation of inflation and $q$, in the first column. The baseline model also is closest to the data’s standard deviation of $q$, in the second data column, but still falls short by an order of magnitude. The variable labor exogenous growth model comes closest to the data for the standard deviation of $g$ and of $i/y$, in the third and fourth data columns, while the baseline model overstates these. The alternative endogenous growth model does best in simulating the standard deviation of the inflation rate in the fifth data column.


<table>
<thead>
<tr>
<th>Low Freq Correlation: $\pi : q$</th>
<th>Business Cycle Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma(q)$</td>
</tr>
<tr>
<td>US Data</td>
<td>$-0.76$</td>
</tr>
<tr>
<td>Baseline Endog Growth</td>
<td>$-0.75$</td>
</tr>
<tr>
<td>Alternate Endog Growth</td>
<td>$-0.50$</td>
</tr>
<tr>
<td>Exog: Fixed Labour</td>
<td>$-0.0011$</td>
</tr>
<tr>
<td>Exog: Variable Labour</td>
<td>$-0.13$</td>
</tr>
</tbody>
</table>

Note: For Exog models, $\sigma_G = 0.008$. Other parameters are fixed as in Table 3

21
5.4 Stock Returns

The baseline model does better than the two versions of the exogenous growth models in replicating the $q$ volatility at a business cycle frequency, getting 36% of its magnitude. However, this happens at the cost of overstating the magnitude of the volatility of growth, the investment ratio and inflation, while exogenous growth models underestimate the volatility of $q$ and $g$, and also overestimate the volatility of inflation.

A related facet is the business cycle volatility of stock returns. The last data column of Table 5 shows data and simulation results for the standard deviation of the stock price return, which can be derived within the baseline model. The real stock return, denoted by $R_{mt}$, is typically defined by:

$$R_{mt+1} = \frac{v_{t+1} + d_{t+1}}{v_t}. \tag{22}$$

By deflating the numerator and denominator of equation (22) by the capital stock and using equation (16), the following relationship between $q$ and stock return results.

$$R_{mt+1} = \frac{(1 - \theta)q_t^{\alpha} a_{t+1}^{\gamma} + 1 + MPK_{t+1} - \delta_k}{q_t} \tag{23}$$

Appendix B.3 presents this derivation.

Equation (23) shows that the adjustment cost parameter $\theta$ drives a wedge between stock return and gross marginal product of capital. In the absence of adjustment cost, with $q_t = 1$, the stock return equals to the gross marginal product of capital, $1 + MPK_{t+1} - \delta_k$. With $q_t > 1$, volatility in $q$ would impact on the volatility of the stock return.

The last column of Table 5 reports the standard deviation of stock returns for data and the model. Data in the last column of Table 5 for stock return is from Robert Shiller’s online databank, with monthly series converted to quarterly. The baseline model does very well in matching the data, while the other models do not. For variants of the endogenous growth model, this points out that when the key parameter of the standard deviation of the human capital shock innovation, $\sigma_H$, is set so that the model
matches the inflation-$q$ correlation data, the result is that the model also nearly matches the stock return volatility data. And note that the failure of exogenous growth models in matching such stock volatility data is pointed out by Gomme et al (2008).

6 Exogenous Growth

Impulse responses of the exogenous growth models give additional insight into the relative performance of the models in Table 5. Table 5 shows that the exogenous growth models (i) and (ii) give rise to a slight-to-modest negative correlation between $q$ and inflation. Figures 8 and 9 present the $TFP$ and money supply impulse responses in model (i), with fixed labor, and Figures 10 and 11 give the impulse responses in model (ii) with variable labor. The results show that there are mostly similar effects amongst the growth rate, $q$, and the inflation rate, but the difference that stands out in comparison to the endogenous growth model is the low order of magnitude of these effects. The main missing ingredient is that seen in Figure 6, in which a human capital shock causes a relatively big growth rate response, and inflation and $q$ negatively correlated response over time.

Figure 8: Effects of an orthogonalized TFP shock: Fixed Labour Supply Model
Figure 9: Effects of an orthogonalized money shock: Fixed Labour Supply Model

Figure 10: Effects of an orthogonalized TFP shock: Variable Labour Supply Model
7 Conclusion

The paper contributes to an explanation of the empirical stylized negative correlation between Tobin’s $q$ and inflation through a DSGE endogenous growth model that identifies plausible fundamentals. The importance of this study is that while there is an emerging literature that shows how monetary policy, for example, affects the stock market through sticky wages and inflation targeting (Christiano et al, 2007), less is known of the effect of inflation taxes on the stock market via human capital-driven growth.

The paper develops closed form expressions for Tobin’s $q$ with physical capital adjustment cost to understand the relationship between inflation, $q$ and human capital utilization along the balanced growth path. The impulse response analysis helps reveal a novel transmission mechanism of the productivity and monetary shocks through a "human capital channel". The simulation results then provide an explanation in particular for the observed low frequency negative correlation between $q$ and inflation. Comparison of the baseline to alternative models including exogenous growth variants highlights the success of the baseline in this respect, while showing an ability to
capture a good portion of \( q \) business cycle volatility as well as most of the stock return volatility. And this indicates that the human capital sector and its productivity shock is key to the overall results.

Extensions could involve introducing convexity into the model in at least two key ways. The \( q \) function itself can be made convex through factors such as those introduced as in Belo et al (2010). And the effect of inflation on growth can be made significantly more convex by introducing an exchange credit alternative to money for making transactions, as in Gillman and Kejak (2005). These factors can strengthen the translation of inflation effects onto \( q \). In low inflation economies the negative growth effect of inflation would be marginally stronger, causing a bigger fall off of \( q \); and this \( q \) decline could be even more pronounced with \( q \) convexity as the investment rate declines by more when the long term growth rate is decreased by the inflation tax.

Such convexities may combine to allow for an even smaller variance of the human capital shock standard deviation to be specified, in order to replicate the low frequency data correlation between inflation and \( q \). And the convexities may allow for distinguishing between developed low inflation economies and developing high inflation economies in terms of the strength of the inflation - \( q \) correlation. Further, the convexities should improve the business cycle volatility results.

A second direction in extensions would be introducing explicit financial intermediation, both for exchange credit and for intertemporal credit via savings and investment intermediation (Gillman, 2010). This would allow for bank crisis effects through a stochastic bank productivity factor that could help lower simulated inflation rate volatility and lesson the need to introduce sticky prices. The financial intermediation effect on \( q \) during bank crises may cause a less negative inflation - \( q \) correlation, as inflation and growth both fall during bank crises, and \( q \) also falls because of the bank crisis effect on equity markets. But during normal times, low inflation and high growth can combine with a rising bank productivity to cause \( q \) to be even higher and the low frequency inflation - \( q \) correlation to be more negative. As a third type of extension, the ability to explain \( q \) through the current model’s shocks could be illustrated further by backing out the implied shocks of the
model over time using data series as in Nolan and Thoenissen (2010).

A Appendix: Equilibrium Conditions

Define the lagrange multipliers associated with the consumer’s flow budget constraint (2) as \( \lambda_t \), the human capital technology (4) as \( \eta_t \) and the cash-in-advance constraint (5) as \( \gamma_t \). The consumer’s first order conditions are:

\begin{align}
  c_t : \beta U'(c_t) - P_t(\lambda_t + \gamma_t) &= 0, \\
  M_t : -\lambda_t + E_t\{\lambda_{t+1} + \gamma_{t+1}\} &= 0, \\
  z_{t+1} : -\lambda_t V_t + E_t\lambda_{t+1}\{V_{t+1} + D_{t+1}\} &= 0, \\
  B_{t+1} : -P^b_t \lambda_t + \lambda_{t+1} &= 0, \\
  h_{t+1} : -\eta_t + E_t\lambda_{t+1}l_{Gl+1}W_{t+1} + E_t\eta_{t+1}(1 - \delta_h + A_{Ht+1}l_{ht+1}) &= 0, \\
  l_{Gl} : -\psi \Gamma'(1 - l_{Gl} - l_{Ht})\beta^t + \lambda_t W_t h_t &= 0, \\
  l_{Ht} : -\psi \Gamma'(1 - l_{Gl} - l_{Ht})\beta^t + A_{Ht}\eta_t h_t &= 0.
\end{align}

Using (A.1) and (A.2)

\[ \lambda_t = \beta^{t+1}E_t \frac{U'(c_{t+1})}{P_{t+1}}, \]  

which upon substitution in (A.3) and (A.4) yields

\begin{align}
  V_tE_t \left[ \frac{U'(c_{t+1})}{P_{t+1}} \right] &= \beta E_t \left[ E_{t+1} \left[ \frac{U'(c_{t+2})}{P_{t+2}} \right] \{V_{t+1} + D_{t+1}\} \right]; \tag{A.9} \\
  P^b_tE_t \left[ \frac{U'(c_{t+1})}{P_{t+1}} \right] &= \beta E_t \left[ E_{t+1} \left[ \frac{U'(c_{t+2})}{P_{t+2}} \right] \right]. \tag{A.10}
\end{align}
A binding cash in advance constraint means that (5) reduces to

\[ \frac{M_t}{P_t} = c_t, \]  
(A.11)

which implies that

\[ \frac{P_t}{P_{t+1}} = \frac{c_{t+1}}{c_t} \frac{1}{1 + \mu_{t+1}}. \]  
(A.12)

Upon substitution into (A.9) and (A.10) it results that

\[ v_t E_t \left[ U'(c_{t+1}) \frac{c_{t+1}}{c_t} \frac{1}{1 + \mu_{t+1}} \right] = \beta E_t \left[ E_{t+1} \left[ U'(c_{t+2}) \frac{c_{t+2}}{c_{t+1}} \frac{1}{1 + \mu_{t+2}} \right] \right] \{v_{t+1} + d_{t+1}\}, \]
(A.13)

and

\[ p^b_t E_t \left[ U'(c_{t+1}) \frac{c_{t+1}}{c_t} \frac{1}{1 + \mu_{t+1}} \right] = \beta E_t \left[ E_{t+1} \left[ U'(c_{t+2}) \frac{c_{t+2}}{c_{t+1}} \frac{1}{1 + \mu_{t+2}} \right] \right], \]
(A.14)

where \( v_t = \) real share price \( (V_t/P_t) \), \( p^b_t = \frac{p_t^b}{P_t} \), and \( w_t \) denotes the real wage \( (W_t/P_t) \).

Using (A.11) and (6), one obtains the following compact expression for \( m_{t+1} \):

\[ \frac{\lambda_{t+1} P_{t+1}}{\lambda_t P_t} = m_{t+1}. \]  
(A.15)

For the goods producer, define \( \omega_t \) as the Lagrangian multiplier associated with the adjustment cost technology (11). The firms’ first order conditions are

\[ l^f_{Gi} : \frac{W_t}{P_t} = A_{Gi} F_2(k_t, l^f_{Gi} h_t) \]  
(A.16)

\[ i_t : \lambda_t P_t = \theta \omega_t \left( 1 - \frac{\delta_k + i_t}{K_t} \right)^{\theta-1} \]  
(A.17)
The model can then be summarized by the following equations.

Tobin’s $q$ equation:

$$q_t = E_t m_{t+1} \left[ \alpha A_{Gl+1}^{1-\alpha} \left( \frac{k_{t+1}}{h_{t+1}} \right)^{\alpha} + 1 - \delta_k + (1 - \theta) \theta^{\alpha/(1-\theta)} \frac{k_{t+1}}{h_{t+1}} \right]$$

(A.19)

$l_G$ equation:

$$\frac{A_{Gl}}{A_{Ht}} l_{Gl} \cdot \left[ \frac{k_t}{h_t} \right]^{\alpha} = E_t \left[ m_{t+1} A_{Gl+1}^{1-\alpha} \left( \frac{k_{t+1}}{h_{t+1}} \right)^{\alpha} \right] + E_t \left[ m_{t+1} l_{Gl+1}^{1-\alpha} \left( \frac{k_{t+1}}{h_{t+1}} \right)^{\alpha} \right] (1 - \delta_h + A_{Ht+1} l_{ht+1}) \frac{A_{Gl+1}}{A_{Ht+1}}$$

(A.20)

$x$ equation:

$$\frac{\psi}{x_t} - (1 - \alpha) \beta E_t \left[ \frac{1}{1 + \mu_{t+1}} A_{Gl+1}^{1-\alpha} \left( \frac{k_t}{h_t} \right)^{\alpha-1} \left( \frac{c_t}{k_t} \right)^{-1} \right] = 0$$

(A.21)

$k/h$ equation:

$$\frac{k_{t+1}}{h_{t+1}} = \frac{\{(1 - \delta_k)(k_t/h_t) + A_{Gl+1}^{1-\alpha}(k_t/h_t)^{\alpha} - (c_t/k_t)(k_t/h_t)\}^{\theta}(k_t/h_t)^{1-\theta}}{1 - \delta_h + A_{Ht}(1 - l_{Gl} - x_t)}$$

(A.22)
Output growth equation:

\[ \frac{y_{t+1}}{y_t} = \left[ \frac{A_{Gt+1}}{A_{Gt}} \right] \left[ \frac{k_{t+1}/h_{t+1}}{k_t/h_t} \right]^\alpha \{ A_{Ht+1} l_{Ht} + 1 - \delta_h \} \cdot \left[ \frac{l_{Gt+1}}{l_{Gt}} \right]^{1-\alpha} \] (A.23)

Inflation equation:

\[ \frac{P_{t+1}}{P_t} = \frac{1 + \mu_{t+1}}{(c_{t+1}/k_{t+1})/(c_t/k_t) \{(k_{t+1}/h_{t+1})/(k_t/h_t)\} \{A_{Ht+1} l_{Ht} + 1 - \delta_h\}} \] (A.24)

The discount factor equation:

\[ m_{t+1} = \beta \left\{ \frac{1 + (1 + \rho)\bar{\mu} - \rho\mu_{t+1}}{1 + (1 + \rho)\bar{\mu} - \rho\mu_t} \right\} \left( \frac{c_t}{k_t} \right) \left( \frac{k_t}{h_t} \right)^{-1} \cdot 1 \] (A.25)

Equation (A.19) follows from (A.18), (16) and (A.15). Equation (A.20) follows from (A.5), (A.6), (A.7), (A.8), (A.15) and (A.16). Equation (A.21) follows from (A.6), (A.8) and (A.16). Equation (A.22) follows by combining (4) (9) and (11). The growth equation (A.23) follows from (4) and (9). To obtain the inflation equation (A.24) rewrite the cash-in-advance constraint (5) using (4) as:

\[ \frac{P_{t+1}}{P_t} = \frac{(1 + \mu_{t+1})(A_{Ht+1} l_{Ht} + 1 - \delta_h)^{-1}}{A_{Ht} l_{Ht}} \] (A.25)

For equation (A.25), use the log utility specification and equation (4) to rewrite this as:

\[ m_{t+1} = \beta \frac{c_t}{k_t} \frac{k_{t+1}}{h_{t+1}} \frac{E_{t+1}}{E_t} \left[ \frac{1}{1 + \mu_{t+2}} \right] \cdot \frac{1}{1 - \delta_h + A_{Ht+1} l_{Ht}} \]
Next take a first order approximation around the steady state and use the forcing process for money supply growth in equation (15) to get the expression in equation (A.25).

The balanced growth equilibrium solution then follows. Based on (4), (18), the resource and time constraints of equations (A.20) and (A.21), the steady state can be represented as

\[ 1 + g = 1 - \delta_h + \bar{A}_H l_h = (1 - \delta_k + \frac{i}{k})^\theta, \quad (A.26) \]
\[ 1 + g = \beta(1 - \delta_h + \bar{A}_H (1 - x)), \quad (A.27) \]
\[ \frac{c}{k} + \frac{i}{k} = \frac{y}{k} = \bar{A}_G \left( \frac{l_G h}{k} \right)^{1-\alpha}, \quad (A.28) \]
\[ \frac{c \psi}{k} x = \frac{(1 - \alpha) \beta y}{(1 + \mu)} \frac{1}{k l_g}, \quad (A.29) \]
\[ \beta \theta (\alpha \frac{y}{k} + 1 - \delta_k) = [1 - \beta(1 - \theta)](1 + g)^\frac{1}{\psi}, \quad (A.30) \]
\[ 1 = x + l_G + l_H. \quad (A.31) \]

Equating the \(1 + g\) terms in the first equality of (A.26) and (A.27), and using equation (A.31), yields a linear relationship between \(l_G\) in terms of \(x\) as follows:

\[ (1 - \delta_h)(1 - \beta) = \bar{A}_H [l_G - (1 - \beta)(1 - x)]. \quad (A.32) \]

From equation (A.29) and the first part of equation (A.28), obtain \(\frac{y}{k}\) in terms of \(\frac{\psi}{k}\) and \(\frac{l_g}{l_g}\). Substituting this into (A.30), and then writing \(\frac{\psi}{k}\) in terms of \(g\) from (A.26), yields a further expression for \(g\) in terms of \(\frac{x}{l_g}\). Finally replace \(l_G\) by its representation in terms of \(x\), and \(g\) in terms of \(x\) from equation
(A.27), to get an equation solely in $x$:

\[
\begin{align*}
\frac{\theta(1-\beta)}{A_H} & - (1-\delta_k)(1-\alpha)(1-\delta_h + \tilde{A}_H(1-x))\psi - \frac{\theta(1-\delta_k)(1-\alpha)\beta}{1+\bar{\mu}} x \\
+ \frac{(1-\beta(1-\theta))(1-\alpha)}{1+\bar{\mu}} \beta^{\frac{1}{\bar{\mu}}}(1-\delta_H + \tilde{A}_H(1-x))\frac{1}{\beta} x & - \frac{(1-\beta + \beta(1-\alpha))(1-\beta)}{A_H} \psi \beta^{\frac{1}{\bar{\mu}}-1}(1-\delta_h + \tilde{A}_H(1-x))^{1+\frac{1}{\bar{\mu}}} = 0.
\end{align*}
\]

Once $x$ is solved from (A.33), $l_G$ can be solved from (A.32). The remaining endogenous variables are just functions of $l_G$ and $x$ and can be computed.

**B Appendix: Proofs**

**B.1 Proposition 3**

Divide (A.13) through by $k_{t+1}$

\[
\left( \frac{v_t}{k_{t+1}} \right) = E_t m_{t+1} \left\{ \left( \frac{v_{t+1}}{k_{t+2}} \right) \left( \frac{k_{t+2}}{k_{t+1}} \right) \right. + (d_{t+1} / k_{t+1}) \right\}.
\]

Noting that $[A_{Gl} F(k_t, l_{Gl} h_t) - (W_t/P_t) l_{Gl} h_t - \delta_t] = d_t$,

\[
\left( \frac{v_t}{k_{t+1}} \right) = E_t m_{t+1} \left\{ \left( \frac{v_{t+1}}{k_{t+2}} \right) \left( \frac{k_{t+2}}{k_{t+1}} \right) + \lambda_{t+1} A_{Gl+1} F_{2t+1} - \frac{i_{t+1}}{k_{t+1}} \right\}.
\]

Now use the adjustment cost equation (11) to rewrite the above as:

\[
\left( \frac{v_t}{k_{t+1}} \right) = E_t m_{t+1} \left\{ (A_{Gl+1} F_{2t+1} + 1 - \delta_k) + \left( \frac{v_{t+1}}{k_{t+2}} \right) \left( \frac{k_{t+2}}{k_{t+1}} \right) - \left( \frac{k_{t+2}}{k_{t+1}} \right)^{1/\theta} \right\}. 
\]

\[
(B.34)
\]
Using the definition of $q_t$ (21) rewrite this again as

$$
\left( \frac{n_t}{k_t+1} \right) = E_t m_{t+1} \left\{ (A_{Gt+1}F_{2t+1} + 1 - \delta_k) + \left( \frac{v_{t+1}}{k_{t+2}} \right) \left( \theta q_{t+1} \right)^{\theta/(1-\theta)} - (\theta q_{t+1})^{1/(1-\theta)} \right\} .
$$

(B.35)

Next verify that (B.34) collapses to (A.19) if $q_t = \frac{m}{k_{t+1}}$.

**B.2 Proposition 4**

Note that from equations (6) and (A.8), along the BGP,

$$
m_{t+1} = \frac{P_{t+1} \lambda_{t+1}}{P_t \lambda_t} = \frac{\beta}{1 + g}.
$$

(B.36)

Using (16), (A.19), and (B.36), and imposing the balanced growth condition, $\frac{n_t}{k_t} = \frac{i_{t+1}}{k_{t+1}}$ one obtains that

$$
\left( 1 - \delta_k + \frac{i_t}{k_t} \right)^{1-\theta} = \frac{\beta \theta}{1 + g} A_GF_1 + \frac{\beta}{1 + g} \left[ (1 - \theta) \left( 1 - \delta_k + \frac{i_t}{k_t} \right) + \theta(1 - \delta_k) \right].
$$

(B.37)

Use the adjustment cost function (11) to write

$$
\frac{i_t}{k_t} = (1 + g)^{1/\theta} - 1 + \delta,
$$

(B.38)

which after plugging into equation (B.37) yields the proposition result of equation (18). Also it is straightforward to verify from equation (A.20) the standard result as in such Lucas (1988) human capital models with leisure, that $1 + g = \beta[1 - \delta_h + A_H(1 - x)]$.

**B.3 Stock Return Equation**

Rewrite (22) as

$$
R_{mt+1} = \frac{q_{t+1} + (d_{t+1}/k_{t+1})(k_{t+1}/k_{t+2})}{q_t} \cdot \left( \frac{k_{t+2}}{k_{t+1}} \right).
$$

Noting that $d_{t+1} = (\alpha y_{t+1}/k_{t+1}) + 1 - \delta_k - (k_{t+2}/k_{t+1})^{1/\theta}$, and using
the above can be rewritten as:

\[ R_{mt+1} = q_{t+1} + \frac{\{ (\alpha y_{t+1}/k_{t+1}) + 1 - \delta_k - (\theta q_{t+1})^{\frac{1}{1-\theta}} \}}{q_t} \{ (\theta q_{t+1})^{\frac{\theta}{1-\theta}} \} \]

which after simplifying reduces to (23).

References


