Data Oscillation and Convergence of Adaptive FEM

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Data Oscillation and Convergence of Adaptive FEM.
Outline

Introduction
    Model Problem
    Mesh Adaptivity

Old Framework
    Old Framework

New Framework
    Introduction
    Error Reduction
    Main Result
Model Problem

Let’s:

- \( \Omega \) a polygonal domain bounded in \( \mathbb{R}^2 \)
- \( f \) a given function in \( L^2(\Omega) \),
- \( A \) is a piecewise constant positive symmetric matrix on \( \Omega \),

Problem: seek \( u \) the solution of the problem

\[
\begin{aligned}
-\text{div}(A \nabla u) &= f & \text{in } \Omega, \\
        u &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]
Weak Form

Problem: seek \( u \in H^1_0(\Omega) \) such that

\[
a(u, v) = (f, v)_{0,\Omega} \quad \forall v \in H^1_0(\Omega),
\]

where

\[
a(u, v) := \int_{\Omega} (A \nabla u) \cdot \nabla v \, dx,
\]

\[
\|v\|_{\Omega} := \left( \int_{\Omega} (A \nabla v) \cdot \nabla v \, dx \right)^{1/2},
\]

\[
(f, v)_{0,\Omega} := \int_{\Omega} f v \, dx.
\]
Meshes and Discrete Problems(I)

Define:

- $\mathcal{T}_H$ conforming triangulation of $\Omega$
  $\mathcal{T}_H$ resolves the jumps of $A$,
- $S_H$ is the set of the sides of the triangles of $\mathcal{T}_H$,
- $V_H$ space of piecewise linear functions over $\mathcal{T}_H$.

$$V_H \subset H_0^1(\Omega) \cap C^0(\Omega) \quad \text{(conforming)}$$
Meshes and Discrete Problems(II)

Problem: seek $u_H \in V_H$ such that

$$a(u_H, v_H) = (f, v_H)_{0, \Omega} \quad \forall v_H \in V_H.$$
A Posteriori Error Estimate

\[ \eta_H^2 := \sum_{S \in S_H} \eta_{S,H}^2, \]

\[ \eta_{S,H}^2 := \| H_S^{1/2} J_S \|^2_{0,S} + \| Hf \|^2_{0,\Omega_S}, \]

\[ J_S := [A \nabla u_H]_S \cdot \nu. \]

Properties:

1. \[ \| u - u_H \|_{\Omega}^2 \leq C_1 \eta_H^2 \] (Reliability)
2. \[ C_2 \eta_{S,H}^2 - C_3 \| H(f - f_H) \|^2_{0,\Omega_S} \leq \| u - u_H \|_{\Omega_S}^2 \] (Local Efficiency)

All constant \( C_i \) are independent of \( H \)
Goal

Given a tolerance $\varepsilon > 0$, compute an approximated solution $u_H$:

$$\|\|u - u_H\|\|_\Omega \leq \varepsilon.$$
Mesh Adaptivity (Algorithm A)

1. Solve the problem for $u_H$
2. If $\|u - u_H\|_\Omega > \varepsilon$ Then
3. Mark the elements to be refined
4. Refine the mesh
5. Go To 1
6. End
Regularity: $u \in H^{1+\beta}(\Omega) \cap H_0^1(\Omega)$

$$||u - u_H|| \leq C H_{max}^{1+\beta} |u|_{1+\beta}.$$ 

Remarks:
- strategy: reduce $H_{max}$,
- refine everywhere soon or later,
- mesh adaptivity doesn’t fit in this framework.
Definitions

Oscillations:

$$\text{osc}(f, \mathcal{T}_H) := \left( \sum_{\tau \in \mathcal{T}_H} \|H_{\tau}(f - f_H)\|_2^2 \right)^{1/2}.$$ 

Marking Strategy: for a given $0 < \theta < 1$,

$$\left( \sum_{S \in \mathcal{S}_H} \eta_{S,H}^2 \right)^{1/2} \geq \theta \eta_H.$$  \hspace{1cm} (1)

Refinement by newest-vertex bisection.
Refined Mesh

Define:

- $\mathcal{T}_h$ conforming triangulation of $\Omega$
  $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$
- $V_h$ space of piecewise linear functions over $\mathcal{T}_h$

$$V_h \subset H^1_0(\Omega) \cap C^0(\Omega) \quad \text{(conforming)}$$

$$V_H \subset V_h$$
**Error Reduction (I)**

**Theorem**

Let $\mathcal{T}_H$ be a triangulation of $\Omega$, $\hat{\mathcal{T}}_H$ and $\hat{S}_H$ be as defined in Marking strategy 1. Let $\mathcal{T}_h$ be the refinement of $\mathcal{T}_H$. Then there exist constants $\mu > 0$ and $0 < \alpha < 1$, such that for any $\varepsilon > 0$ if

$$\text{osc}(f, \mathcal{T}_H) \leq \mu \varepsilon,$$

then either $\|u - u_H\| \leq \varepsilon$ or the solution $u_h$ on $\mathcal{T}_h$ satisfies

$$\|u - u_h\| \leq \alpha \|u - u_H\|.$$
Error Reduction (II)

$\mu > 0$ and $0 < \alpha < 1$ depends on:

- the minimum angle of the mesh $\mathcal{T}_H$,
- the value of $\theta$ in the Marking strategy 1,
- the continuity constant of the bilinear form $a(\cdot, \cdot)$,
- the coercivity constant of the bilinear form $a(\cdot, \cdot)$,
Remarks

- there is a condition on the initial mesh \( \text{osc}(f, I_H) \leq \mu \varepsilon \),
- if all the conditions are satisfied, the prescribed tolerance \( \varepsilon \) may be met in finite steps,
- the mesh size \( H_{\text{max}} \) may not tend to 0,
Second Marking Strategy

Marking Strategy: for a given $0 < \hat{\theta} < 1$, enlarge $\hat{\mathcal{T}}_H$ such that:

$$\text{osc}(f, \hat{\mathcal{T}}_H) \geq \hat{\theta} \text{osc}(f, \mathcal{T}_H).$$

(2)
Oscillation Reduction

Theorem

Let $\mathcal{T}_H$ be a triangulation of $\Omega$, $\hat{\mathcal{T}}_H$ and $\hat{S}_H$ be as defined in Marking strategy 2. Let $\mathcal{T}_h$ be the refinement of $\mathcal{T}_H$. Then there exists constant $0 < \hat{\alpha} < 1$, such that

$$\text{osc}(f, \mathcal{T}_h) \leq \hat{\alpha} \text{osc}(f, \mathcal{T}_H).$$
Mesh Adaptivity (Algorithm B)

1. Solve the problem for $u_H$
2. If $\|u - u_H\|_\Omega > \varepsilon$ Then
3. Marking strategy 1
4. Marking strategy 2
5. Refine the mesh
6. Go To 1
7. End
Main Result

**Theorem**

Let $u_k$ be a sequence of finite element solutions produced by Algorithm B. There exist positive constants $C_0$, $\beta < 1$, depending on the initial mesh and the data of the problem, such that

$$\|u - u_k\|_\Omega \leq C_0 \beta^k.$$
Remarks

- the error may not decay at each single step,
- the condition on the initial mesh is only sufficient!