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Abstract

Aims

1. Construction of High–Order DGFEMs for a class of second–order (quasilinear) PDEs;
2. Develop the a posteriori error analysis and adaptive mesh design of the DGFEM approximation of target functionals of the solution based on employing anisotropic $h$–/$hp$–refined meshes.
Measurement Problem: Given a user–defined tolerance $TOL > 0$, can we efficiently design $S_{h,p}$ such that

$$|J(u) - J(u_h)| \leq TOL.$$ 

*Fluid dynamics*: drag and lift coefficients.

*Other examples*: point value, flux, mean value, etc.

**Applications**

- Compressible (aerodynamic) flows.
Measurement Problem for Compressible Flows

- Measurement Problem: Given a user-defined tolerance $TOL > 0$, can we efficiently design $S_{h,p}$ such that

$$|J(u) - J(u_h)| \leq TOL.$$ 

- Fluid dynamics: drag and lift coefficients.
- Other examples: point value, flux, mean value, etc.

- Applications
  - Compressible (aerodynamic) flows.

P. Houston (Nottingham), R. Hartmann (DLR, Braunschweig)
Measurement Problem: Given a user-defined tolerance TOL > 0, can we efficiently design $S_{h,p}$ such that

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Applications

- Compressible (aerodynamic) flows.
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Adaptivity

Goal:

\[ |J(u) - J(u_h)| \leq \sum_{\kappa \in T_h} |\eta_\kappa(u_h)| \leq \text{Tol}. \]

Automatic refinement algorithm:

1. Start with initial (coarse) grid \( T_h^{(j=0)} \).
2. Compute the numerical solution \( u_h^{(j)} \) on \( T_h^{(j)} \).
3. Compute the local error indicators \( \eta_\kappa \).
Adaptivity

Goal:

\[ |J(u) - J(u_h)| \leq \sum_{\kappa \in T_h} |\eta_{\kappa}(u_h)| \leq \text{Tol}. \]

Automatic refinement algorithm:

0. Start with initial (coarse) grid \( T_h^{(j=0)} \).

1. Compute the numerical solution \( u_h^{(j)} \) on \( T_h^{(j)} \).

2. Compute the local error indicators \( \eta_{\kappa} \).
Adaptive Algorithm (\textit{hp–Adaptivity})

- Regularity estimation via truncated Legendre series expansions.

- If both $u$ and $z$ are deemed to be \textit{non-smooth}, apply anisotropic $h$–refinement.

- Else, perform anisotropic $p$–refinement
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Adaptive Algorithm \((h\text{-}Adaptivity)\)

- Element refinements.

\[
\mathcal{E}_1 \equiv \sum_{\kappa \in \mathcal{T}_{h,1}} |\eta_{\kappa}^{\text{new}}| \\
\mathcal{E}_2 \equiv \sum_{\kappa \in \mathcal{T}_{h,2}} |\eta_{\kappa}^{\text{new}}| \\
\mathcal{E}_3 \equiv \sum_{\kappa \in \mathcal{T}_{h,3}} |\eta_{\kappa}^{\text{new}}|
\]

- Solve local primal and dual problems on elemental patches.
- Boundary data extracted from global primal and dual solutions.
Adaptive Algorithm ($h$–Adaptivity)
Adaptive Algorithm ($h$–Adaptivity)

Algorithm 1
Select optimal refinement

$$\max_{i=1,2,3} \left( |\eta^\text{old}_i| - \mathcal{E}_i \right) / (\#\text{dofs} (\mathcal{T}_h,i) - \#\text{dofs} (\mathcal{T}_h,\kappa)).$$

Algorithm 2
- Prescribe an $h$–anisotropy parameter $\theta_h > 1$.
- When

$$\frac{\max_{i=1,2} (\mathcal{E}_i)}{\min_{i=1,2} (\mathcal{E}_i)} > \theta_h,$$

perform refinement in direction with minimal $\mathcal{E}_i$, $i = 1, 2$.
- else perform isotropic $h$–refinement.
Adaptive Algorithm ($h$–Adaptivity)

**Algorithm 1**
Select optimal refinement

\[
\max_{i=1,2,3} \left( |\eta^{old}_{\kappa}| - \mathcal{E}_i \right) / \left( \#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa}) \right).
\]

**Algorithm 2**
- Prescribe an $h$–anisotropy parameter $\theta_h > 1$.
- When

\[
\frac{\max_{i=1,2} (\mathcal{E}_i)}{\min_{i=1,2} (\mathcal{E}_i)} > \theta_h,
\]

perform refinement in direction with minimal $\mathcal{E}_i$, $i = 1, 2$.
- else perform isotropic $h$–refinement.
Ma = 0.5, Re = 5000, $\alpha = 2^\circ$ and adiabatic wall condition.

Drag coefficients:

$$J_{c_{dp}}(u) = \frac{2}{l \rho |\bar{v}|^2} \int_S p (n \cdot \psi_d) ds,$$
$$J_{c_{df}}(u) = \frac{2}{l \rho |\bar{v}|^2} \int_S (\tau n) \cdot \psi_d ds,$$

where

$$\psi_d = \begin{pmatrix} \cos(\alpha) & - \sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$J_{c_d}(u) \approx 0.056084.$$
NACA0012 Airfoil – Unstructured
NACA0012 Airfoil – Unstructured

![Graph showing error against Dofs for different refinement methods.](image-url)

- Iso h–Refinement
- Aniso h–Refinement

Error scale: $10^{-3}$ to $10^{-4}$

Dofs scale: $10^3$ to $10^5$
Mesh after 4 adaptive anisotropic refinements, with 3485 elements

Mesh after 8 adaptive anisotropic refinements, with 10401 elements
BTC0 (Comp. NS)

- $Ma = 0.5$.
- $Re = 5000$.
- $\alpha = 1^\circ$.
- Adiabatic wall condition.
- $J_{C_1}(u) \approx 0.002565$
BTC0 (Comp. NS)

```
plot error vs dofs

- Iso h-Refinement
- Aniso h-Refinement
```

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Mesh after 3 adaptive anisotropic refinements, with 2324 elements
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Anisotropic $p$–Refinement

Local Problems

\[ \mathcal{E}_1 \equiv |\eta_{\kappa}^{\text{new}}| \quad \mathcal{E}_2 \equiv |\eta_{\kappa}^{\text{new}}| \quad \mathcal{E}_3 \equiv |\eta_{\kappa}^{\text{new}}| \]
**Anisotropic $p$–Refinement**

**Local Problems**

\[ \mathcal{E}_1 \equiv |\eta^{\text{new}}_{\kappa}| \quad \mathcal{E}_2 \equiv |\eta^{\text{new}}_{\kappa}| \quad \mathcal{E}_3 \equiv |\eta^{\text{new}}_{\kappa}| \]

**Algorithm 1**

Select optimal refinement

\[
\max_{i=1,2,3} \left( \frac{|\eta^{\text{old}}_{\kappa}| - \mathcal{E}_i}{\text{#dofs}(\mathcal{T}_{h,i}) - \text{#dofs}(\mathcal{T}_{h,\kappa})} \right).
\]
Anisotropic $p$–Refinement

Local Problems

$E_1 \equiv |\eta^\text{new}_\kappa|$  $E_2 \equiv |\eta^\text{new}_\kappa|$  $E_3 \equiv |\eta^\text{new}_\kappa|$  

Algorithm 2

- Prescribe a $p$–anisotropy parameter $\theta_p > 1$
- When
  \[
  \frac{\max_{i=1,2}(E_i/(#\text{dofs}(T_{h,i}) - #\text{dofs}(T_{h,\kappa})))}{\min_{i=1,2}(E_i/(#\text{dofs}(T_{h,i}) - #\text{dofs}(T_{h,\kappa})))} > \theta_p,
  \]
  enrich in polynomial in the direction with minimal $E_i$, $i = 1, 2$.
- else perform isotropic $p$–refinement.
NACA0012 Airfoil – Unstructured

![Graph showing error vs. Dofs for different refinement types: Iso h-Refinement, Aniso h-Refinement, Iso hp-Refinement, Aniso h-/Iso p-Refinement. The y-axis represents error on a logarithmic scale, and the x-axis represents Dofs also on a logarithmic scale. The graph illustrates how different refinement strategies impact the error with increasing degrees of freedom.]
NACA0012 Airfoil – Unstructured

The diagram shows the convergence of different refinement strategies for a NACA0012 airfoil problem. The x-axis represents the number of degrees of freedom (Dofs) on a log scale, ranging from $10^5$ to $10^6$. The y-axis represents the error on a log scale, ranging from $10^{-4}$ to $10^{-3}$. The lines indicate the performance of various refinement methods:

- Red triangles: Iso h-Refinement
- Blue squares: Aniso h-Refinement
- Black diamonds: Iso hp-Refinement
- Magenta circles: Aniso h-/Iso p-Refinement
- Green stars: Aniso hp-Refinement

The graph demonstrates how the error decreases as the number of degrees of freedom increases, indicating improved accuracy with increased computational effort.
NACA0012 Airfoil – Unstructured

![Graph showing error vs. sqrt(Dofs) for different refinement methods.](image-url)
$hp$-mesh distribution after 6 adaptive (anisotropic $h$-/isotropic $p$-) refinements, with 2835 elements and 118520 degrees of freedom
$hp/p_x$–mesh distribution after 6 adaptive (anisotropic $h$–/anisotropic $p$–) refinements
hp/$\rho_y$–mesh distribution after 6 adaptive (anisotropic $h$–/anisotropic $p$–) refinements