There are two main theories in superconductivity:

i) Microscopic theory – describes why materials are superconducting

ii) Ginzburg-Landau Theory – describes the properties of superconductors in magnetic fields

Ginzburg-Landau theory

- This is a phenomenological theory, unlike the microscopic BCS theory. Based on a so-called phenomenological order parameter.
- Not strictly an ab initio theory, but essential for problems concerning superconductors in magnetic fields.
- G-L was the first theory to explain the difference between Type I and Type II superconductivity, and enable the calculation of two critical fields \( H_{c1} \) and \( H_{c2} \).

Outline of the Lecture

- Ginzburg-Landau theory
- The two G-L equations
- The basic phenomenology – Type I and Type II superconductors
- Time dependent G-L theory
- Ginzburg-Landau and Pinning
- Conclusions
Ginzburg-Landau Theory

Ginzburg and Landau (G-L) postulated a Helmholtz energy density for superconductors of the form:

\[ f = \alpha |\psi|^2 + \frac{1}{2} \beta |\psi|^4 + \frac{1}{2m} (-i\hbar \nabla - 2eA)\psi|^2 + \int \mathbf{H} dB \]

where \( \alpha \) and \( \beta \) are constants and \( \psi \) is the wavefunction. \( \alpha \) is of the form \( \alpha(T - T_C) \) which changes sign at \( T_C \).

The two Ginzburg-Landau Equations

- Functional differentiation w.r.t. \( \psi \) and \( \mathbf{A} \) gives the two G-L equations (coupled partial differential equations):
  
  G-L I \[ \frac{1}{2m} (-i\hbar \nabla - 2eA)^2 \Psi + \alpha \Psi + \beta |\Psi|^2 \Psi = 0 \]
  
  G-L II \[ J = -\frac{i\hbar e}{m} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) - \frac{4e^2}{m} A |\Psi|^2 \]

London Theory – Type I superconductors

- The London brothers -

\[ E = \mu_0 \lambda_L^2 \frac{\partial}{\partial t} J_s \]
\[ B = -\mu_0 \lambda_L^2 \nabla \wedge J_s \]

where \( \lambda_L^2 = \frac{m^*}{\mu_0 n_s e^2} \)

Substituting the second London equation into one of Maxwell’s equations we get the Meissner state:

\[ \nabla^2 B = \frac{1}{\lambda_L^2} B \]

Magnetisation of superconductors

The magnetisation \( (M) \) of a type I and type II superconductor, with the same \( H_c \) as a function of applied magnetic field \( (H_0) \). **Inset:** The phase diagrams of type I and type II superconductors.
The structure of the fluxon

- The order parameter, magnetic field ($B$) and supercurrent density ($J_C$) as a function of distance ($r$) from the centre of an isolated vortex ($\kappa \approx 8$). The order parameter squared is proportional to the density of superelectrons.

The Mixed state

- $\lambda$: depth of the current flow
- $d$: fluxon – fluxon spacing
- If $B$ increases, $d$ decreases.

The Mixed State in NbSe$_2$

- (a) The hexagonal Abrikosov lattice: (a) contour diagram of the order parameter from Kleiner et al.; the lines also represent contours of $B$ and streamlines of $J$. (b) Scanning-tunnelling-microscope image of the flux lattice in NbSe$_2$ (1 T, 1.8 K) from Hess et al. The vortex spacing is ~479Å.

The Mixed State in Nb

- Vortex lattice in niobium – the triangular layout can clearly be seen. (The normal regions are preferentially dark because of the ferromagnetic powder).
Flux Quantisation

- By considering the persistent current as a wave, phase coherence demands \( n: \text{integer} \) - leads to flux quantisation. \( k: \text{momentum of the superelectrons.} \)

G-L Theory – Type I and Type II superconductors

Ginzburg-Landau theory predicts that a superconductor should have two characteristic lengths:

- Coherence length
  \[ \xi = \frac{\hbar}{\sqrt{2m_e |\alpha|}} \]
- Penetration depth
  \[ \lambda = \frac{m_e \beta}{\sqrt{4e^2 \mu_0 |\alpha|}} \]

The Ginzburg-Landau parameter

This ratio, \( \kappa \), distinguishes Type-I superconductors, for which \( \kappa \leq 1/\sqrt{2} \), from Type-II superconductors which have higher \( \kappa \) values.

Thermodynamic Critical Field, \( H_C \)

- In Type I superconductors \( H_C \) is the critical field at which superconductivity is destroyed – \( \psi \) drops abruptly to zero in a first-order phase transition.
- At the thermodynamic critical field \( H_C \), the Gibbs free energies for superconducting and normal phases are equal. For the superconducting phase \( B = 0 \), and for the normal phase \( \psi = 0 \)

\[ H_c = \frac{|\alpha|}{\sqrt{\mu_0 \beta}} \]

Lower and upper critical fields

\[ H_{c1} \approx \frac{H_C}{\sqrt{2}} \ln \kappa \]

\[ H_c = \frac{|\alpha|}{\sqrt{\mu_0 \beta}} \]

\[ \mu_0 H_{c2} = \frac{\phi_0}{2\pi \xi^2} \]

\[ H_{c2} \approx \kappa H_C \approx \kappa^2 H_{c1} \]
Flux entering a superconductor

Time evolution of order parameter

This simulation is simply applying an instantaneous fields of $0.5H_{c2}$. 

Time evolution of order parameter
Time evolution of order parameter

Flux nucleation and entry into a superconductor

- Contour plot of a superconductor bounded by an insulating outer surface. The applied magnetic field was increased to above the initial vortex penetration field (i.e. to $Hp + 0.01Hc2$)

Other Ginzburg-Landau predictions

The magnetization of in the mixed state near $B_{c2}$ is given by:

$$
\mu_0 M = -\frac{(B_{c2} - B)}{(2\kappa^2 - 1)}\beta_A
$$

$$
\beta_A = \frac{\langle |\psi|^4 \rangle}{\langle |\psi|^2 \rangle^2}
$$

Surface barriers

$$
B_{C3} = 1.69B_{C2}
$$

Depairing current density:

$$
J_D = \frac{2H_{c2}}{3\sqrt{3}3\kappa^2\xi} = 0.385\frac{H_{c2}}{\kappa^2\xi}
$$
Josephson diffraction

\[ J_c \] computed for an \( \rho_N = 10 \rho_S \) junction in a 30\( \xi \)-wide \( \kappa = 5 \) superconductor.

Ginzburg-Landau predictions - restricted dimensionality behaviour

- **Behaviour of thin films** - A thin film has a much higher critical field (if the field lines are parallel to the film), than a bulk superconductor. This is predicted by Ginzburg-Landau theory.

- **Anisotropic Ginzburg-Landau theory** - It is possible to extend Ginzburg-Landau theory to anisotropic systems (e.g., high-\( T_c \) superconductors). This shows that \( H_c \) is isotropic, but \( H_{c2} \) is not.

- **Lawrence-Doniach theory** - 2D version of GL theory. Predicts that \( H_{c2} \) is higher for layered superconductors, and that at low \( T \)'s, fluxons are locked parallel to planes (a kind of ‘transverse Meissner effect’).

Flux Pinning using G-L theory

**Model**

- Pinning at surfaces\(^{25}\) (Dew-Hughes 1974)
- Pinning at surfaces\(^{27}\) (Dew-Hughes 1987)
- Flux shear past point pinning sites\(^{30}\) (Kramer, Brandt\(^{31}\) C\(_{66}\))
- Flux shear past point pinning sites\(^{30}\) (Kramer, Labusch\(^{31}\) C\(_{66}\))
- Maximum shear strength of lattice\(^{27}\) (Dew-Hughes 1987)
- Flux shear along grain boundaries in 2D\(^{105}\) (Pruymboom 1988)
- Collective pinning model\(^{30}\) (Larkin-Ovchinnikov)

**Force** \( F_p = J_c \cdot B \) per unit volume

\[
F_p = \frac{B_p^2}{4 \mu_0 \mu_k D} \left( 1 - \frac{b^2}{1 - \frac{a}{a_{c2}}} \right) \left( \frac{b}{3 \sqrt{3}} \right) \left( 1 - \frac{b}{D} \right) \left( 1 - \frac{b}{3 \sqrt{3}} \right)
\]

**Below \( H_c \), Type I superconductors are in the Meissner state:** current flows in a thin layer around the edge of the superconductor, and there is no magnetic flux in the bulk of the superconductor. (\( H_c \): Thermodynamic Critical Field.)

**In Type II superconductors,** between the lower critical field (\( H_{c1} \)) and the upper critical field (\( H_{c2} \)), magnetic flux penetrates into the sample, giving a "mixed" state.
Reversible Magnetization Loop

- The reversible response of a superconductor

\[ M(H) = \frac{\langle H_{c2} - H \rangle}{(2\kappa^2 - 1)\beta_4} \]

\[ \mu_0 H_{c2} \]

Magnetization Characteristic

\[ 100\xi \times 80\xi, \kappa = 5 \]

M vs. H for a Superconductor Coated With a Normal Metal, \( \kappa = 5 \)

- \( H = 0.05 H_{c2} \)
- The material is in the Meissner state

Magnetization Loop

- \( H = 0.10 H_{c2} \)
- The material is in the Meissner state

Magnetization Loop

- \( H = 0.15 H_{c2} \)
- The material is in the mixed state
Magnetization Loop

- $H = 0.40 \, H_c^2$
- Note the nucleation of fluxons at the superconductor-normal boundary

Magnetization Loop

- $H = 0.70 \, H_c^2$
- In the reversible region, one can determine $\kappa$

Magnetization Loop

- $H = 0.90 \, H_c^2$
- The core of the fluxons overlap and the average value of the order parameter drops

Magnetization Loop

- $H = 1.00 \, H_c^2$
- Eventually the superconductivity is destroyed
**Magnetization Loop**

- \( H = 0.50 \, H_{c2} \)
- Note the Abrikosov flux-line-lattice with hexagonal symmetry

**Time-dependent Ginzburg-Landau equations**

\[
\frac{1}{\xi_0^2} \left( |\Delta|^2 - \left( 1 - \frac{T}{T_c} \right) \right) \Delta + \left( \frac{\mathbf{V}}{i} - \frac{2e}{\hbar} \mathbf{A} \right)^2 \Delta + \frac{1}{\mathcal{D}} \left( \frac{\partial}{\partial t} + i \frac{2e\varphi}{\hbar} \right) \Delta = 0
\]

\[
\mathbf{J}_e = \frac{1}{2e\mu_o \lambda_0^2} \text{Re} \left( \Delta^* \left( \frac{\hbar}{i} \mathbf{V} - 2e\mathbf{A} \right) \Delta - \sigma \left( \nabla \varphi + \frac{\partial \mathbf{A}}{\partial t} \right) \right)
\]

- These equations were postulated by Schmid (1966), and then derived using microscopic theory in the gapless case by Gor'kov and Eliashberg (1968)

**Uncertainty/Complexity in G-L theory**

- Thermodynamics is seductive but very complex.
- There are number of different approaches to deriving the G-L equations (DeGennes, Campbell, Clem, Pippard, Jackson and Landau and Lifshitz)
- There is confusion in the literature about the length scales on which G-L applies
- TDGL for real systems has only been possible in the last 3 - 5 years
Model for a polycrystalline superconductor

- A collection of truncated octahedra
- Avoids the problematic straight channels of a simple cubic system.

Obtaining bulk $J_c$

- The Bean critical state model is used to obtain $J_c$.
- A symmetrical straight-line Bean profile is fitted to the calculated $B$-field data.
- The $E$-field is calculated from the ramp rate by Maxwell’s equations.

Branching test runs – up, fast

$E_{\text{max}} = -5.33 \times 10^{-3} H_c \rho / \kappa \xi$

Diffusive resistivity $\approx 31 \rho_S$

Layout of granular system

Superconductor
Weak superconductor
Normal metal
Edges matched by periodic boundary conditions

Show the 2-field movie!
Time dependant Ginzburg-Landau theory provides the framework for understanding flux pinning.

Flux motion in low fields (0.43 $H_{c2}$) along grain boundaries.

Order Parameter at 0.43 $B_{c2}$
- The motion of flux through the system takes place predominantly along the grain boundaries.

Normal Current at 0.430 $H_{c2}$
- The normal current movie shows that dissipation above $J_c$ occurs principally in the grain boundaries.
Flux motion in high fields into the interior of grain boundaries

Order Parameter at 0.942 $H_{c2}$

Conclusion

- Ginzburg-Landau theory is a triumph of physical intuition. It is the central theory for understanding the properties of superconductors in magnetic fields.

- Bibliography/electronic version of the talk will be available at: http://www.dur.ac.uk/superconductivity.durham/