Optimal Taxation in Overlapping Generations Economies

6.1 INTRODUCTION

In this chapter we analyse optimal taxation within an overlapping-generations (OLG) economy. The purpose is to establish to what extent fiscal policy would be different in dynastic versus OLG economies.¹ We will solve for the second best (full commitment) and the third best (partial or no commitment) optimal tax policy.

We find that the second best optimal capital income tax is zero in steady state if the social welfare function is weakly separable across generations (Koopmans (1960) form). This result is in contrast with previous studies of OLG economies (see Table 6.1 below) which have found the optimal capital income tax to be zero only in a special case: when individuals’ preferences are logarithmic in consumption and leisure.² The reason for the difference between previous studies and this

¹ This chapter also represents a bridge between chapter 5 which is concerned with endogenous taxation in a dynastic economy and chapter 7 which will analyse endogenous taxation in an OLG economy.

² One has therefore been tempted to believe that there has been a fundamental difference between the optimal tax programme in an OLG economy and in a dynastic economy, since it is a general result that the optimal capital income tax is zero in a dynastic economy with individual’s preferences being of the Koopmans form [Chamley (1986)].
chapter is that the previous studies have not found the second best optimal policy, but the third best, by letting the government only choosing prices that affect one generation at a time.

The intuition for the zero-capital income tax result in this chapter is closely related to the intuition behind the same result in a dynastic economy. In a dynastic economy it is optimal not to tax capital when individual’s utility is separable in consumption at different dates. In this chapter it is optimal not to tax capital when the utilities of individuals of different generations (i.e. of different "dates") enter in a separable way into the social welfare function.

In a dynastic economy it is optimal to tax capital heavily at the beginning of the optimisation period. This is generally not the case in the overlapping generations economy as long as the old generation enters into the social welfare function. When the old generation carries no weight confiscation of capital income is optimal.

The chapter also analyses third best policy, but under the more realistic assumption that the government chooses fiscal policy that affect two generations at the same time, e.g. present wage tax and present capital income tax or next period’s wage tax and next period’s capital income tax. This is also new to the literature.

Table 6.1 gives a taxonomy of previous studies of optimal taxation in overlapping generations economies.

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3 In steady state (when consumption levels are constant) the sums of elasticities of the utility function are the same, since the consumption levels in the past do not affect them. This implies [Atkinson and Stiglitz (1972)] consumption at different dates should be uniformly taxed, i.e. capital should be untaxed.
Table 6.1 - Optimal Taxation in OLG Economies

<table>
<thead>
<tr>
<th>Study</th>
<th>Consumers</th>
<th>Population growth</th>
<th>Timing concept</th>
<th>Taxes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diamond (1973)</td>
<td>differ in tastes</td>
<td>constant</td>
<td>recursive(^1)</td>
<td>linear on all commodities</td>
</tr>
<tr>
<td>Ordover and Phelps (1979)</td>
<td>differ in abilities</td>
<td>zero</td>
<td>recursive(^1)</td>
<td>non-linear on labour and capital</td>
</tr>
<tr>
<td>Atkinson and Sandmo (1980)</td>
<td>identical</td>
<td>constant</td>
<td>recursive(^1)</td>
<td>linear on labour, capital and consumption</td>
</tr>
<tr>
<td>Park (1991)</td>
<td>differ in tastes and time endowments</td>
<td>constant</td>
<td>recursive(^1)</td>
<td>linear on labour, capital and consumption</td>
</tr>
<tr>
<td>This chapter</td>
<td>differ in skills and tastes</td>
<td>constant</td>
<td>full commitment and recursive(^2)</td>
<td>linear on labour, capital and consumption</td>
</tr>
</tbody>
</table>

\(^1\) The period-\(t\) government controlling taxes that affect current generation only, e.g. the labour tax in period \(t\) and the capital income tax in period \(t+1\).

\(^2\) The period-\(t\) government controls either the taxes at time \(t\), or the taxes at time \(t+1\).

There are some common characteristics of these studies.\(^4\)

(i) They are characterised by perfect competition and constant returns-to-scale in (aggregate) production. Despite this it is not automatic that the First and Second Welfare Theorems apply in absence of distortionary taxes. It is well known that the OLG economies can give rise to dynamic inefficiency.

\(^4\) They are all, more or less, applications of the Diamond (1965) economy.
Optimal Taxation in OLG Economies

(because future generations’ preferences are not reflected in current prices). However, the possibility of dynamic inefficiency only arises if there is population growth.

(ii) Consumers are heterogeneous, not only with respect to age, but also in abilities (Ordover and Phelps) and tastes (Diamond). In Atkinson and Sandmo individuals only differ with respect to age.

(iii) Consumers live for two periods and have no bequest motives.

(iv) There is physical non-perishable capital, which initial level is greater that zero.

(v) Each government can only set taxes that affect only one generation at each point in time. For example setting present labour taxes and future capital income taxes. This rules out the possibility of finding the second-best structure of taxes (which would require optimisation over all future taxes), and rules out the possibility of analysing governments’ decisions that affect two generations simultaneously.

Diamond (1973) analyses optimal taxation in a very general Overlapping-Generations (OLG) economy, allowing for many consumption goods (privately and publicly produced) in each period and labour supply also in the second period. Individuals may differ in tastes, there are no bequest motives, there are no

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5 Cass (1972).

6 The Samuelson (1958) economy does not have capital, therefore individuals have no possibility in transferring consumption possibilities from the present to the future, other than (i) having a government doing so (transfer from young to old, pay-as-you-go pension system), or (ii) trading in fiat money.

7 Setting present labour taxes and present capital income taxes or future labour taxes and future capital income taxes would affect two generations simultaneously. This seems to be a more plausible formulation.
pure public goods), there is a constant population growth rate, and the government cares about all future generations, but can only tax the present generation.\(^8\)

The economy is studied under three regimes: (i) central planning (the government controls individuals’ quantities) which can be implemented through a decentralised economy with individual specific lump sum taxes, (ii) fully taxed economy (the government can tax all commodities), (iii) partially taxed economy (the government can tax only some commodities, or have to tax some commodities at the same rates).

However, Diamond’s focus is on the marginal products in public versus private production, rather than on the tax rules themselves. The main conclusions are that under all regimes, if the economy goes to a steady state, the marginal product of capital in public production equals the social discount rate, i.e. the Modified Golden Rule is obtained. Under regime (i) and (ii) this is also true for marginal product of capital in private production.

Ordover and Phelps (1979) use a simplified version of Diamond’s (1973) OLG economy.\(^9\) The difference is that in the latter framework there is only one consumption good available at each date, that individuals work only in their first period of their lives, and that there is no public production. Ordover and Phelps also abstract from population growth. In their economy the government has access to non-linear labour income and

\(^8\) The way in which Diamond solves the problem is by letting the present government controlling the prices affecting the present generation only. This implies that if for example the government has access to wage and interest taxes, then the present government chooses present wage tax and future interest tax.

\(^9\) They use the Diamond (1965) OLG economy which is the most common framework for tax analysis.
capital income taxes, and the government can differentiate the 
lump-sum transfers between the young generation and the old.

Ordover and Phelps focus on the tax rules, and their main 
results are that the marginal tax on the highest income earned 
is zero (both wage and interest income), and if each worker’s 
utility is everywhere weakly separable between consumption in 
period one and two and in leisure, then the capital income tax 
is always zero.

Atkinson and Sandmo (1980) examine optimal linear taxes 
when individuals of the same generation are identical and the 
population grows at a constant rate. The basic framework is 
otherwise close to Ordover and Phelps. Individuals consume in 
two periods but work only in the first and the government 
chooses taxes that affect only one generation.

Among their results are: (i) if generational lump sum taxes 
are allowed, the Modified Golden Rule holds in steady state and 
no distortionary taxes are used; (ii) if no lump-sum transfers are 
allowed and utility is additively separable and logarithmic, then, 
if the steady state obeys the Modified Golden Rule, the capital 
income tax is zero in that steady state; (iii) if on the other hand 
the modified golden rule cannot be reached, capital income may 
be taxed or subsidised in steady state.

Park (1991) uses the same framework as Atkinson and Sandmo 
but allows individuals to differ in preferences and in time 
endowments,\(^\text{10}\) and also performs the analysis with a

\(^{10}\) Park intends to allow individuals to differ in abilities, using \(h\) as notation 
for hours worked and \(l\) as notation for effective labour supply. \(l\) rightly 
enters the budget constrain, but \(h\) does not enter the utility function, instead 
\(l\) wrongly enters the utility. Thus, the paper does not allow for productivity 
differences. On the other hand Park allows individuals to differ in time 
endowments (i.e. total hours available for work and leisure) which is a 
completely different thing, since then individuals face the same wage rate.
consumption tax. Among the results are: (i) with identical logarithmic preferences (individuals differ only in time endowments) the optimum capital income tax is zero in steady state, (ii) with identical time endowments the optimum capital income tax is zero in steady state, (iii) if individuals differ in preferences and time endowments it may be positive, negative or zero.

To summarise: the above studies have found the optimal capital income tax to be zero in steady state if either: (i) individuals’ utilities are logarithmic in period-one and -two consumption and in leisure, or (ii) the government has a non-linear tax schedule and utilities are weakly separable between period-one and -two consumption and in leisure.

In light of these conclusions one may be tempted to think that there is a fundamental difference in optimal policy between OLG economies and dynastic economies, presumably because of the assumption of finitely lived households in OLG, (or equivalently absence of bequest motives) or because of the possibility of a dichotomy between the government’s and the individuals’ discount rates. However, as we shall see in this chapter, this conclusion is not correct. We will show that the optimality of the zero capital-income tax generally carries over to the OLG economy as well. The reason for the difference in the results between this chapter and the above mentioned studies is not in the basic assumptions on the economic structure but in that the latter have not found the second-best optimal taxes, since the government is assumed to control the

\[ \text{\footnotesize \ref{11} Recall from chapter 3 that the zero capital income tax has been derived in dynastic economies under less restrictive assumptions.} \]
taxes affecting the present generation only. If we want to find the second best we have to optimise with respect to all future taxes. The results will in general differ because of the time-inconsistency of optimal policy, but the second best gives us a useful benchmark at which we can evaluate the time-consistent policy (usually referred to as "third best").

Since the second best (open loop) formulation generally gives a time-inconsistent solution, we would be interested in the recursive formulation, when the government cannot commit to the future sequence of taxes. However, in the above mentioned studies the recursive formulation is such that the present government chooses taxes that affect the present generation only, i.e. the present labour income tax and the next period’s capital income tax. It is more plausible that a government chooses policy instruments that affect both generations at the same time, e.g. present labour income tax and present capital income tax, or future labour income tax and future capital income tax. We will address this issue as well in this chapter.

The chapter is structured as follows:

In section 6.2 a general OLG-economy is described. Individuals work only in the first period of their lives, and save for their retirement. Their savings constitute the capital stock. Individuals may have bequest motives, but only in terms of the gift given, not in terms of children’s utilities (otherwise we would usually have the standard dynastic economy).

In section 6.3 the optimal tax problem is solved when the government can choose all future taxes (full commitment). The government has access to a flat labour income tax and a flat capital income tax, and is able to pass on debt or assets to the next generation. It is proven that for social welfare being weakly separable across generations the optimal capital income tax is zero in steady state. It is also shown that the labour
income tax is generally positive in steady state.

In section 6.4 optimal taxation with no commitment is analysed. Capital income is not confiscated as long as the old generation enters into social welfare function.

Section 6.5 deals with optimal taxation with one-period commitment. If individuals have the same utility functions which are logarithmic in consumption and either logarithmic or linear in leisure then the capital income tax is always zero (not just in the steady state).

Section 6.6 analyses optimal public goods provision. Section 6.7 shows that Theorem 1 is robust to a number of extensions. Section 6.8 concludes the chapter.

6.2 THE ECONOMY

Individuals live for two periods. They consume both as young and as old, but work only when young. They have preferences over period-one consumption, period-one labour supply, period-two consumption, period-two bequests\(^{12}\) and period-one and period-two provision of public goods.

To account for population growth, let \(N_t\) be the size of the young generation at \(t\). Within each generation individuals differ (in productivity and possibly also in preferences), and the types, \(i\), are distributed according to \(F(i)\), which is continuous and stationary. This implies that while the population grows, the distribution within each generation is constant. We shall assume that the growth in \(N_t\) is constant and exogenous, \(N_t = (1+n)N_{t-1}\), and we normalise \(F\) such that \(\int dF(i) = 1\). A key difference between individuals within one generation is their productivity, which is represented by a skill parameter \(\gamma\).

\(^{12}\) Most of the analysis in this chapter, however, will abstract from bequests entirely.
In period one individual *i* born at *t* (with ability *γ* *i*) receives a bequest *m* *i* /(*1+n*), \(^13\) supplies labour *l* *i* on the market and consumes *c* *i* units of the only consumption good. He is paid *w* *i* per efficient unit of supplied labour, i.e. in proportion to *γ* *i* *l* *i*, and he saves *a* *i*+1 for the next period. Let *τ* _c_ denote the consumption tax rate, *τ* _l_ the wage income tax rate, and *τ* _k_ the capital income tax rate. In period two he receives after-tax return, *P* _i+1_, on his savings which is used for consumption *c* _i+1_ and leaving a bequest *m* _i+1_. The period-one and period-two provisions of the public good are denoted *G*_ *t* and *G*_ *t+1* respectively. It is convenient to structure the assumptions as follows:

### 6.2.1 Assumptions

**A1 Individual Preferences**

The utility function

\[
U^H\left(c^H_i, l^H_i, c^H_{i+1}, m^H_{i+1}, g_t, g_{t+1}\right) \tag{1}
\]

is assumed to be strictly concave in all arguments. *g*_ *t* and *g*_ *t+1* are per-capita public consumption. Any individual is assumed to choose his quantities of consumption, labour supply, savings and the bequest so as to maximise (1) subject to his budget constraints.

**A2 Individuals’ Constraints**

The individual budget constraints are

\[
(1 + τ_c^i) c^H_i + a^H_{i+1} = (1 - τ_c^i) w_i γ^H_i l^H_i + m^H_i/(1+n) \tag{2}
\]

**A3 Production**

\(^13\) The amount given by the parent is *m* *i*.
A large number of firms are operating under a constant-returns-to-scale technology. Therefore aggregate production, $Y_t$, can be calculated as if there was a representative firm employing the aggregate quantities of capital and labour, defined as $K_t \equiv N_t \int k_i \, dF(i)$ and $L_t \equiv N_t \int l_i \, dF(i)$ respectively.  

\[ F(K, L) = F_k(K, L)K + F_L(K, L)L \]

**A4 Government’s Constraint**

The government is allowed to borrow and lend freely at the market rate of interest and may take the aggregate expenditure requirement $G_t$, $t=0,...,\infty$, as (a) predetermined, (b) giving utility as a flow, (c) giving utility as a stock. Denote the aggregate government debt as $B_t$.

\[ B_{t+1} = \left[ 1 + (1 - \tau^k_t) r_t \right] B_t - \tau^k_t r_t K_t - \tau^L_t w_t L_t - \tau^C_t C_t + G_t \]

where aggregate consumption is $C_t \equiv N_t \int c_i \, dF(i)$

\[ + N_t \int \gamma c_i \, dF(i). \]

**A5 Government’s Objective**

The government seeks to maximise a Bergson-Samuelson welfare function including all future generations.

**A6 Population Growth**

The population growth rate, $n$, is exogenous (possibly zero)

\[ (1 + \tau^C_{t+1}) c^H_{t+1} + m^H_{t+1} = [1 - (1 - \tau^k_t) r_t] a^H_{t+1} \]
6.2.2 Discussion of the Assumptions

A1. Consumption in both periods but only labour supply in the first intends to capture the idea that individuals eventually retire from work, and have to save as young for future consumption as old. This follows Diamond (1965), and is the most common way of modelling OLG economies.

The key feature concerning the bequest motive is that the bequests themselves enter into the parents’ utility functions, not the children’s utilities. This breaks the links between marginal rates of substitution between different commodities at dates sufficiently distant apart (more than one period apart). This way of modelling bequests is not novel to the literature\textsuperscript{15} and provides a way of linking generations together in a non-dynastic way.

A2. The budget constraint (2) implies that capital markets are perfect: the individual can borrow and lend at the same market interest rate, without any constraints. The assumption corresponds to A2 in chapter 3. See chapter 3 for a discussion. As mentioned above labour is supplied in the first period only. This "forces" positive savings (savings for retirement).

A3. Perfect competition, constant-returns-to-scale technology, and no production externalities. Analogously to the discussion in chapter 3, this is in order to focus on the efficiency-equity aspects of taxation, rather than an eventual corrective role.

A4. As already discussed in chapter 3 the treatment of

\[ N_t = (1+n)N_{t-1} \]
government expenditure does not affect the structure of the optimal tax formulas, while we can keep the sequence of $G_i$ as exogenous. However, in order to analyse the optimal provision of public goods we will endogenise $G_i$ later on in this chapter, by assuming that per capita public goods gives utility.

A5. That the government cares about all future generations is in line with previous literature. The restrictions on the welfare function will be imposed later (in the relevant sections).

A6. We allow for a constant population growth to account for the most general case. It is well known that population growth may cause dynamic inefficiency, thus giving a role for corrective taxation. However we will not focus on this aspect.

We shall later on make more explicit assumptions than these and then make a deeper discussion of the simplifications.

### 6.2.3 Economic Equilibrium

In this section it is described how individual and aggregate economic behaviour can be solved for, given any arbitrary sequences of tax rates and public expenditure. Define the gross return on capital as $R_t (1+r_t)$ and the after-tax prices as $P_t = 1 + (1-\tau_t^c) r_t$ and $\omega_t (1-\tau_t^l) w_t$. By profit maximisation the before-tax prices (the interest rate and the average wage rate) are given by $r = F_K$ and $w = F_L$.

**Individual Economic Behaviour**

Maximisation of (1) subject to (2)-(3) gives the individuals’ decision rules $\{c_i^u, l_i^u, c_{i+1}^t, m_{i+1}\}$ as functions of $\{\omega, \tau_t^c, \tau_{t+1}^c, P_{t+1}, m_t^u, g_t, g_{t+1}\}$ and indirect utilities.
For notational convenience denote consumer prices and public expenditure as a vector $q_t$,

$$q_t = \{P_{t-1}, \omega_t, \tau_{t-1}, g_t\}$$  \hspace{1cm} (8)

**Aggregate Economic Behaviour**

Because of the OLG structure of the economy it is not possible to have an aggregation theorem that gives a representative individual result (as in chapter 5).\(^{16}\) Bequests consists of the aggregate amount given by the old generation. Define the average bequests as $m_{t+1} = \int m_{t+1}^i dF(i)$, then the aggregate bequests depend on the distribution of $m^i$ in the previous period.

$$M_{t+1} = N_t \bar{m}_{t+1} = N_t \bar{m}_{t+1}(q_t, \tau_t, g_t, \{m_t^i\})$$  \hspace{1cm} (9)

Therefore, in general, the aggregate bequests is not a meaningful concept. It is only if we make assumptions such that the aggregate is independent of the distribution that equation (8) is useful.

Aggregate consumption (as well as all other quantities) may be rewritten in terms of per capita of the young generation. Define generation average consumptions as $\bar{c}_t = \int c_{t}^i dF(i)$ and $\bar{c}_t^{-1} = \int c_{t}^{-1} dF(i)$. Then

\(^{16}\)Recall from chapter 5 that when individuals have the same length of life (infinite), and differ only in capital endowments and in productivity, the aggregate economic behaviour coincides with a representative individual if the instantaneous utility function belongs to the additive or multiplicative HARA-family.
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Similarly for labour, let the average (in efficiency units) be \( \bar{l}_t \equiv \int F(l_i) \). Then

\[
\bar{l}_t = \frac{L_t}{N_t} - \bar{l}_t(q_t, v^c_t, g_t, \{m_t^{h-1}\})
\]

We, analogously, define per capita (old generation) debt as \( b_t \equiv B_t/N_t \), and per capita (old generation) capital as \( k_t \equiv K_t/N_t \), then, after some manipulation,

\[
b_{t+1} = P_t \frac{b_t - k_t}{1 + n} - F\left(\frac{k_t}{1 + n}, \bar{l}_t\right) + \omega_t v^c_t \frac{\bar{c}_t - \bar{c}_{t-1}^{h-1}}{1 + n} - \frac{k_t}{1 + n} + g_t
\]

Finally, the aggregate capital stock evolves according to

\[
k_{t+1} = \frac{k_t}{1 + n} + F\left(\frac{k_t}{1 + n}, \bar{l}_t\right) - \bar{c}_t - \frac{\bar{c}_{t-1}^{h-1}}{1 + n} - g_t
\]

Thus, the variables denoted by bars are average quantities within each generation and depend on the consumer prices, the amount of public goods and the distribution of bequests.

We will now turn to the optimal tax problem. In finding the second-best optimal taxes we have to optimise over all future taxes, thus solving the full-commitment formulation, assuming that the present government can "force" future governments to follow the optimal plan.
6.3 OPTIMAL INCOME TAXATION
- FULL COMMITMENT

6.3.1 The Optimal Tax Problem
In accordance with most of the previous literature we shall abstract from consumption taxation (setting $\tau_c=0$) and assume that individuals have no preferences over leaving bequests. In the previous literature the preferences of the social planner are utilitarian (sum of utilities) and thereby additively separable in generations and individuals. We shall keep our social welfare function more general than that, but give more structure than assumption A5. We shall assume

A5′ The social welfare function is Bergson-Samuelson over individuals within each generation

$$W_t=W(\{V_i\})$$

and of the Koopmans form over generations

$$J(W_t, W_{t+1}, W_{t+2},...) = U(W_t, J(W_{t+1}, W_{t+2},...))$$

The Koopmans form of utility was derived by Koopmans (1960). He postulates five properties for utility functions to obey, of which the key ones are separability and time independence. We shall explore these two conditions. Denote $\bar{W}=(W_t, W_{t+1}, W_{t+2},...)$. Consider the following programmes

$$W_t, W_{t+1}, W_{t+2},...$$

Then, using our notation, separability implies that if

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$17$ Recall that we mentioned in chapter 3 that Chamley (1986) found the optimal capital-income tax to be zero in steady state for an economy where a representative individual has a utility function of the Koopmans form.

$18$ The other properties are strong continuity and sensitivity of the utility function, and extreme programs (for scaling). These are of technical nature, and the reader is referred to Koopmans (1960) for a complete derivation.
then it must be true that
\[ J(W_t', \tilde{W}) \geq J(W_t, \tilde{W}') \]
and that if
\[ J(W_t, \tilde{W}) \geq J(W_t, \tilde{W}') \]
then it must be true that
\[ J(W_t, \tilde{W}) \geq J(W_t, \tilde{W}') \]

Koopmans shows that the separability condition (together with 
"strong continuity" and "sensitivity") implies a utility function 
of the form
\[ J(W_t) = U(W_t, J_1(t, \tilde{W})). \]

However, note that \( J_1 \) is a function different from \( J \). In order for 
\( J_1 \) to be the same as \( J \), time independence is needed. This is 
obtained by a fourth postulate:

For some \( W_t \) and for all \( \tilde{W} \)
\[ J(W_t, \tilde{W}) \geq J(W_t, \tilde{W}') \iff J(\tilde{W}) \geq J(\tilde{W}'). \]

The separability and time independence will in our application 
make social welfare of future generations separable from 
welfare of the present generation. It should be pointed out that 
all previous studies on optimal policy in overlapping-generations 
economies have assumed special cases of the 
Koopmans form (e.g. discounted sum of future generations’ 
utilities), therefore our treatment is more general than anyone 
else’s sofar. In fact, this is enough to obtain
**Theorem 1** Assume A1-A4, A5', A6, no bequest motives, no consumption taxes and that the government can enforce its optimal plan for the entire future. If the economy reaches a steady state, such that per capita quantities are constant, then the optimal capital income tax is zero in the steady state and the rate of interest obeys the modified Golden Rule.

Proof: We solve the open-loop problem (present government can "force" future governments to implement the present government’s optimal policy). The government chooses a sequence $q_t=\{\omega_t, P_t, \ldots\}, t=0, \ldots, \infty$, subject to the aggregate constraints (12) and (13) and subject to individuals’ optimal behaviour. We may write the Lagrangean as

$$\mathcal{Q} = J(W_0, W_1, \ldots) + \sum_{t=0}^{\infty} \mu_t [B(q_t, b_t, k_t, P_t) - b_{t+1}]$$

$$+ \sum_{t=0}^{\infty} \lambda_t [K(q_t, k_t, P_t, \omega_{t-1}) - k_{t+1}]$$

where $B(q_t, b_t, k_t, P_t)$ and $K(q_t, k_t, P_t, \omega_{t-1})$ are the right-hand-sides of (12) and (13) respectively. The first-order conditions are

$$\frac{\partial J}{\partial W_t} \frac{\partial W_t}{\partial \omega_t} + \mu_t \left[ \bar{l}_t + (\omega_t - w) \frac{\partial \bar{l}_t}{\partial \omega_t} \right]$$

$$+ \lambda_t \left[ w \frac{\partial \bar{\epsilon}_t}{\partial \omega_t} - \frac{\partial \bar{\epsilon}_t}{\partial \omega_t} \right] - \frac{\lambda_{t+1} \left[ \frac{\partial \bar{\epsilon}_{t+1}}{\partial \omega_t} \right]}{1+n} = 0$$

(15)
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\[
\frac{\partial J}{\partial W_t} \frac{\partial W_t}{\partial P_{t+1}} + \mu_t(w_t - w_{t+1}) \frac{\partial \bar{I}_t}{\partial P_{t+1}} + \frac{\mu_{t+1}}{1+n} (b_{t+1} + k_{t+1}) \\
+ \lambda_t \left[ w_t \frac{\partial \bar{c}_t'}{\partial P_{t+1}} - \frac{\partial \bar{c}_t'}{\partial P_{t+1}} \right] - \frac{\lambda_{t+1}}{1+n} \left[ \frac{\partial \bar{c}_t'}{\partial P_{t+1}} \right] = 0 \\
- \mu_t + \frac{\mu_{t+1}}{1+n} P_{t+1} = 0
\]  

(16)

\[
\frac{\mu_{t+1}}{1+n} (P_{t+1} - R_{t+1}) - \lambda_t + \frac{\lambda_{t+1}}{1+n} R_{t+1} = 0
\]  

(17)

Since

\[
\frac{\partial J}{\partial W_t} - \prod_{s=0}^{t} \frac{\partial U(W_s, J_s, \bar{W})}{\partial J_s} \frac{\partial U(W_t, J_{t-1}, \bar{W})}{\partial W_t}
\]  

(19)

we may write

\[
\frac{\partial J}{\partial W_{t+1}} - \frac{\partial U(W_{t+1}, J_{t+1}, \bar{W})}{\partial W_{t+1}} = \delta_t \frac{\partial U(W_t, J_{t-1}, \bar{W})}{\partial W_t}
\]  

(20)

where

\[
\delta_t = \frac{\partial U(W_t, J_{t-1}, \bar{W})}{\partial J}
\]  

(21)

We may write (19) as

\[
\frac{\partial J}{\partial W_t} = \prod_{s=0}^{t} \delta_s \frac{\partial U(W_t, J_{t-1}, \bar{W})}{\partial W_t}
\]  

(22)
Furthermore define \( \tilde{\mu}_t = \prod_{s=0}^{t-1} \delta_s^{-1} \mu_s \), \( \tilde{\lambda}_t = \prod_{s=0}^{t-1} \delta_s^{-1} \lambda_s \), then multiplying by \( \prod_{s=0}^{t-1} \delta_s^{-1} \) we obtain\(^{19}\)

\[
\frac{\partial U}{\partial W_t} \frac{\partial W_t}{\partial \omega_t} + \tilde{\mu}_t \left[ \bar{l}_t + (\omega_t - w_t) \frac{\partial \bar{l}_t}{\partial \omega_t} \right] + \tilde{\lambda}_t \left[ w_t \frac{\partial c_t'}{\partial \omega_t} - \frac{\partial c_t'}{\partial \omega_t} \right] - \frac{\delta_{t+1} \tilde{\lambda}_{t+1}}{1+n} \left[ \frac{\partial c_{t+1}'}{\partial \omega_t} \right] = 0
\tag{23}
\]

\[
\frac{\partial U}{\partial W_t} \frac{\partial W_{t+1}}{\partial P_{t+1}} + \tilde{\mu}_t (\omega_t - w_t) \frac{\partial \bar{l}_t}{\partial P_{t+1}} + \frac{\delta_{t+1} \tilde{\mu}_{t+1}}{1+n} (b_{t+1} + k_{t+1}) - \frac{\partial \bar{l}_t}{\partial P_{t+1}} - \frac{\delta_{t+1} \tilde{\lambda}_{t+1}}{1+n} \left[ \frac{\partial c_{t+1}'}{\partial P_{t+1}} \right] = 0
\tag{24}
\]

\[
- \tilde{\mu}_t + \frac{\delta_{t+1} \tilde{\lambda}_{t+1}}{1+n} P_{t+1} = 0
\tag{25}
\]

\[
\frac{\delta_{t+1} \tilde{\lambda}_{t+1}}{1+n} (P_{t+1} - R_{t+1}) - \tilde{\lambda}_t + \frac{\delta_{t+1} \tilde{\lambda}_{t+1}}{1+n} R_{t+1} = 0
\tag{26}
\]

In steady state (23) and (24) are constant over time, and thereby \( \tilde{\mu}_t \) and \( \tilde{\lambda}_t \) are constants. Then (25) implies \( 1+n=\delta P \), which together with (26) gives

\[
(\tilde{\lambda} - \tilde{\mu}) \frac{R-P}{P} = 0
\tag{27}
\]

Finally we need to prove that (27) can be fulfilled if and only if \( P=R \), i.e. we need to prove that \( \tilde{\lambda} - \tilde{\mu} \neq 0 \). Substituting (25) into (26) and rearranging gives

\(^{19}\) Note that we have \( \prod_{s=0}^{t-1} \delta_s^{-1} \mu_{s+1} = \delta_{s+1} \prod_{s=0}^{t-1} \delta_s^{-1} \mu_{s+1} = \delta_{s+1} \tilde{\mu}_{s+1} \).
Endogenous Taxation in a Dynamic Economy

\[
\frac{\delta_{t+1} \tilde{\lambda}_{t+1}}{1+n} = \frac{\tilde{\lambda}_t}{R_{t+1}} - \frac{\tilde{\mu}_t}{R_{t+1}} \left( 1 - \frac{R_{t+1}}{P_{t+1}} \right)
\]

(28)

Substituting (28) into (23) gives

\[
\frac{\partial U}{\partial W_t} \frac{\partial W_t}{\partial \omega_t} + \tilde{\mu}_t \left[ \tilde{I}_t + (\omega_t - w_t) \frac{\partial \tilde{I}_t}{\partial \omega_t} \right]
\]

\[
+ \tilde{\lambda}_t \left[ w_t \frac{\partial \tilde{I}_t}{\partial \omega_t} - \tilde{c}_t' \frac{\partial \tilde{c}_t'}{\partial \omega_t} \right] - \left[ \frac{\tilde{\lambda}_t}{R_{t+1}} - \frac{\tilde{\mu}_t}{R_{t+1}} \left( 1 - \frac{R_{t+1}}{P_{t+1}} \right) \right] \frac{\partial \tilde{c}_t'}{\partial \omega_t} = 0
\]

(23')

The individuals’ intertemporal budget constraints are

\[
c_t^{\mu} + \frac{c_{t+1}^{\mu}}{P_{t+1}} = \omega_t \gamma t_t^{\mu}
\]

(29)

Differentiating (29) with respect to \( \omega_t \) and aggregate to obtain

\[
\frac{\partial \tilde{c}_{t+1}}{\partial \omega_t} = P_{t+1} \left[ \tilde{I}_t + \omega_t \frac{\partial \tilde{I}_t}{\partial \omega_t} - \frac{\partial \tilde{c}_t'}{\partial \omega_t} \right]
\]

(30)

Substituting (30) into (23'), and rearranging, gives

\[
\frac{\partial U}{\partial W_t} \frac{\partial W_t}{\partial \omega_t} = (\tilde{\mu}_t - \tilde{\lambda}_t) \left[ w_t \frac{\partial \tilde{I}_t}{\partial \omega_t} - \frac{\partial \tilde{c}_t'}{\partial \omega_t} - P_{t+1} \left( \tilde{I}_t + \omega_t \frac{\partial \tilde{I}_t}{\partial \omega_t} - \frac{\partial \tilde{c}_t'}{\partial \omega_t} \right) \right]
\]

This proves \( \tilde{\lambda} - \tilde{\mu} > 0 \), since individuals’ indirect utilities are increasing in after-tax wage, so the first term is positive. Finally (25) gives

\[
\frac{1+r}{1+n} = \delta^{-1}
\]

(31)

which is the modified Golden Rule. QED
Under surprising generality we have derived the zero capital-income-tax result. Thus, we need not have infinitely lived individuals (or bequest motives) for this rule to be optimal, nor do we need separability in consumption and leisure, nor do we need intertemporal separability of the individual utility function, nor do we need non-linear tax schedules.

6.4 OPTIMAL INCOME TAXATION
- NO COMMITMENT

We shall assume a simplified social welfare function:

\[ A5'' \text{ The social welfare function is Bergson-Samuelson over individuals within each generation} \]
\[ W_t = W(V_t) \]
and a discounted sum over generations
\[ J(W_t, W_{t+1}, W_{t+2}, \ldots) = \sum_{t=0}^{\infty} \delta^t U_t \]

where
\[ U_t = W_t + \theta W_{t-1} \]

So instantaneous welfare \( U_t \) for each generation is a sum of own Bergson-Samuelson welfare \( W_t \) and the Bergson-Samuelson welfare of the old generation \( W_{t-1} \). It is assumed that the young members of this generation discounts the welfare of the old by \( \theta < 1 \). The way the \( J \) is specified allows for two-sided altruism, but may, as usual, be rewritten into a one-sided altruism specification
\[ J(W_t, W_{t+1}, W_{t+2}, \ldots) = \theta W_{t-1} + (1 + \theta \delta) \sum_{t=0}^{\infty} \delta^t W_t \]

It is convenient to eliminate one state \( k \). At \( t \) the capital stock is the old generation’s savings minus public debt. The old generation’s savings (assets) is a function of past after-tax wage \( \omega_{t-1} \), and expected future after-tax return on savings \( P'_t \). That is
\[ a_t = k_t + b_t = a(\omega_{t-1}, P'_t) \]
Therefore the relevant state at $t$ is $(\omega_{t-1}, b_t)$. Economic policy at $t$ is a pair $(x_t, P_t)$, such that $\omega_t = x_t \omega_{t-1}$. In equilibrium economic policy $(x_t, P_t)$ is a function $\Xi$ of the state, i.e.

$$(x_t, P_t) = \Xi(\omega_{t-1}, b_t)$$

Individuals of generation $t-1$ rationally predict $P_t$ as a policy function $P^e_t = P^e(\omega_{t-1}, b_t)$. In that way we may write the capital stock in equilibrium as

$$k_t = a(\omega_{t-1}, P^e(\omega_{t-1}, b_t)) - b_t$$

Similarly, the generation of $t$ will base their decisions on present after-tax wage rate and expected future after-tax return on savings. Analogously $P^e_{t+1} = P^e(\omega_t, b_{t+1})$, so anything that changes government debt at $t+1$ must change individuals’ (and the government’s) expectations about $P^e_{t+1}$. Since $c_t$ and $\bar{l}_t$, are functions of expected $P^e_{t+1}$, we may write equation (12) as

$$b_{t-1} = P_t \frac{a(\omega_{t-1}, b_t)}{1+n} - F \left( \frac{a(\omega_{t-1}, b_t) - b_t}{1+n}, l(\omega_t, P^e_{t+1}) \right)$$

$$+ \omega_t l(\omega_t, P^e_{t+1}) - \frac{a(\omega_{t-1}, b_t) - b_t}{1+n} + g_t$$

and

$$P^e_t = P^e(\omega_{t-1}, b_t), \quad P^e_{t+1} = P^e(\omega_t, b_{t+1})$$

Denote

$$d = 1 - (\omega_t - \omega_{t-1}) \frac{\partial \bar{l}_t}{\partial P^e_{t+1}} \frac{\partial P^e_{t+1}}{\partial b_{t-1}}$$
Differentiating with respect to the policy variables we obtain

$$d \frac{\partial b_{t+1}}{\partial x_t} = \left[ \tilde{l}_t + (\omega_t - w_t) \left( \frac{\partial \tilde{l}_t}{\partial \omega_t} + \frac{\partial \tilde{l}_t}{\partial P_{t+1}^e} \frac{\partial P_{t+1}^e}{\partial \omega_t} \right) \right] \omega_{t-1}$$

(35)

$$d \frac{\partial b_{t+1}}{\partial P_t} = \frac{a_t}{1+n}$$

(36)

Differentiating with respect to the states

$$d \frac{\partial b_{t+1}}{\partial b_t} = \frac{P_t - R_t}{1+n} \left[ \frac{\partial a_t}{\partial \omega_{t-1}} \frac{\partial P_t^e}{\partial b_t} + \frac{R_t}{1+n} \right]$$

(37)

$$d \frac{\partial b_{t+1}}{\partial \omega_{t-1}} = \frac{P_t - R_t}{1+n} \left[ \frac{\partial a_t}{\partial \omega_{t-1}} \frac{\partial P_t^e}{\partial \omega_{t-1}} \right]$$

$$+ \left[ \tilde{l}_t + (\omega_t - w_t) \left( \frac{\partial \tilde{l}_t}{\partial \omega_t} + \frac{\partial \tilde{l}_t}{\partial P_{t+1}^e} \frac{\partial P_{t+1}^e}{\partial \omega_t} \right) \right] x_t$$

(38)

Similarly for welfare: $W_t = W(\omega, P_{t+1}) = W(x, \omega, P(x, \omega, b_{t+1}))$. We also have $W_t^{e_{t-1}} = W_t^{e_{t-1}}(\omega_{t-1}, P_t^e) = W_t^{e_{t-1}}(\omega_{t-1}, P_t^e(\omega_{t-1}, b_t))$.

**Lemma 1** At date $t$ marginal change in instantaneous welfare with respect to the after-tax net return to savings is the same as marginal change in expected instantaneous welfare with respect to expected change in after-tax net return to savings, i.e.

$$\frac{\partial W_{t-1}}{\partial P_t} = \frac{\partial W_t^{e_{t-1}}}{\partial P_t^e}$$

(39)
Proof: Let $V^{t-1}$ be the indirect utility of an arbitrary individual of the old generation. Then $\partial V^{t-1}/\partial P_t = a_t^{i} \partial \nu^{t-1}/\partial c^{i-1}$. Next, $\partial V^{t-1}/\partial P^e_t = [\partial \nu^{t-1}/\partial c^{i-1}] [\partial c^{i-1}/\partial P^e_t] + [\partial \nu^{t-1}/\partial c^{i-1}] [\partial P^e_t/\partial P^e_t] + [\partial \nu^{t-1}/\partial c^{i-1}] [\partial P^e_t/\partial c^{i-1}]$. Differentiating the individual’s budget constraint gives $\partial c^{i-1}/\partial P^e_t + P_t^{1-1} \partial c^{i-1}/\partial P^e_t - \omega_{t-1} \partial P^e_t/\partial P^e_t - P_t^{2-1} c^{i-1} = 0$, this together with the individual’s optimality conditions gives $\partial V^{t-1}/\partial P^e_t = [\partial \nu^{t-1}/\partial c^{i-1}] c^{i-1} P_t^{2-1} = [\partial \nu^{t-1}/\partial c^{i-1}] a_t^{i}$. QED

We write the recursive problem as

$$J(o_{t-1}, b_t) = \max_{x_t, P_t^e} \{ W_t(x_t o_{t-1}, P_t^e) + \theta W_{t-1}(o_{t-1}, P_{t-1}) + \delta J(x_t o_{t-1}, b_{t-1}) \}$$

s.t. $\dot{b}_{t+1} = B(z_t)$

where the derivatives of $B(\cdot)$ are given by (36)-(37), and $P_t^e$ and $P_{t+1}^e$ are given by (33). The necessary conditions are

$$\left[ \frac{\partial W_t}{\partial o_t} + \frac{\partial W_t}{\partial P_t^e} \frac{\partial P_{t+1}^e}{\partial o_t} + \delta \frac{\partial J_{t+1}}{\partial o_t} \right] o_{t-1}$$

$$+ \left[ \frac{\partial W_t}{\partial P_t^e} \frac{\partial P_{t+1}^e}{\partial x_t} + \delta \frac{\partial J_{t+1}}{\partial x_t} \right] x_t = 0$$

$$\theta \frac{\partial W_{t-1}}{\partial P_t} + \left[ \frac{\partial W_t}{\partial P_t^e} \frac{\partial P_{t+1}^e}{\partial b_{t+1}} + \delta \frac{\partial J_{t+1}}{\partial b_{t+1}} \right] b_{t+1} = 0$$
Optimal Taxation in OLG Economies

Substituting (37) into (44) gives

\[
\frac{\partial J_t}{\partial \omega_{t-1}} = \theta \frac{\partial W_{t-1}}{\partial \omega_{t-1}} + \left[ \frac{\partial W_t}{\partial \omega_t} + \frac{\partial W_t}{\partial P_{t-1}^e} \frac{\partial P_{t-1}^e}{\partial \omega_t} + \delta \frac{\partial J_{t-1}}{\partial \omega_t} \right] x_t 
\]

(43)

\[
+ \left[ \frac{\partial W_t}{\partial P_{t-1}^e} \frac{\partial P_{t-1}^e}{\partial b_{t-1}} + \delta \frac{\partial J_{t-1}}{\partial b_{t-1}} \right] \frac{\partial b_{t-1}}{\partial \omega_{t-1}}
\]

Next, (36) in (42) gives

\[
\frac{\partial J_t}{\partial b_t} = \theta \frac{\partial W_{t-1}}{\partial P_t^e} \frac{\partial P_t^e}{\partial b_t} + \left[ \frac{\partial W_t}{\partial P_t} \frac{\partial P_t^e}{\partial b_{t-1}} + \delta \frac{\partial J_{t-1}}{\partial b_{t-1}} \right] \frac{\partial b_{t-1}}{\partial b_t}
\]

(44)

Substituting (37) into (44) gives

\[
\frac{\partial J_t}{\partial b_t} = \frac{\partial W_t}{\partial P_t} \frac{\partial P_t^e}{\partial b_t} \delta J_{t-1} \frac{S_t}{d(1+n)} + \theta \frac{\partial W_{t-1}}{\partial P_t^e} \frac{\partial P_t^e}{\partial b_t}
\]

(45)

where

\[
S_t = R_t + (P_t - R_t) \frac{\partial a_t}{\partial P_t^e} \frac{\partial P_t^e}{\partial b_t}
\]

(46)

Next, (36) in (42) gives

\[
\theta \frac{\partial W_{t-1}}{\partial P_t} + \left[ \frac{\partial W_t}{\partial P_t} \frac{\partial P_t^e}{\partial b_{t-1}} + \delta \frac{\partial J_{t-1}}{\partial b_{t-1}} \right] \frac{a_t}{1+n} \frac{1}{d} = 0
\]

(47)

together with (45)

\[
\frac{\partial J_t}{\partial b_t} = \theta \frac{\partial W_{t-1}}{\partial P_t} \left[ \frac{\partial P_t^e}{\partial b_t} - \frac{S_t}{a_t} \right]
\]

(48)

shifting forward
Substituting into (45)

\[
\frac{\partial J^t_{t-1}}{\partial b^t_{t-1}} = \theta \frac{\partial W^t_t}{\partial P^t_{t-1}} \left[ \frac{\partial P^e_t}{\partial b^t_{t-1}} - \frac{S^t_{t+1}}{a^t_{t+1}} \right]
\]

and combining with (48) gives

\[
\frac{\partial J^t_{t+1}}{\partial b^t_t} - \frac{\partial W^t_t}{\partial P^t_{t+1}} \left[ (1 + \theta \delta) \frac{\partial P^e_{t+1}}{\partial b^t_{t+1}} - \theta \delta \frac{S^t_{t+1}}{a^t_{t+1}} \right] \frac{S^t_t}{d(1+n)} + \theta \frac{\partial W^t_{t+1}}{\partial P^t_t} \frac{\partial P^e_t}{\partial b^t_t}
\]

and combining with (48) gives

\[
\frac{\partial W^t_{t+1}}{\partial P^t_t} - \frac{\partial W^t_t}{\partial P^t_{t+1}} \left[ 1 - Q^t_{t+1} \frac{\partial P^e_{t+1}}{\partial b^t_{t+1}} \right] \frac{\delta R^t_{t+1}}{d(1+n) a^t_{t+1}} \frac{a^t_t}{a^t_{t+1}}
\]

where

\[
Q^t_{t+1} = \frac{R^t_{t+1} - P^t_{t+1}}{R^t_{t+1}} \frac{\partial a^t_{t+1}}{\partial P^t_{t+1}} + \frac{1 + \theta \delta}{\theta \delta} \frac{a^t_{t+1}}{R^t_{t+1}}
\]

In steady state \( J_p = \frac{\partial J}{\partial b_t} = \partial J^t_{t+1}/\partial b^t_{t+1} \), and \( J_\omega = \frac{\partial J}{\partial \omega_{t+1}} = \partial J_{t+1}/\partial \omega_t \).

**Theorem 2** Assume A1-A4, A5", A6, no bequest motives, no consumption taxes and that the government cannot commit to future taxes. If the economy reaches a steady state and if the individuals’ expectations about the future capital-income tax are independent of the level of public debt, then the steady state is characterised by the modified Golden Rule. Otherwise not necessarily.

Proof: If \( \frac{\partial P^e_t}{\partial b^t_{t+1}} = 0 \), we have by equation (34) that \( d=1 \), then (51) in steady state implies the modified Golden Rule.

QED
It is clear from (51) that the modified Golden Rule not necessarily is obtained. For example suppose preferences are such that individuals’ savings and labour supplies are independent of the return on savings (i.e. if utility is logarithmic) then (34) implies $d=1$, and (52) implies $Q=[(1+θδ)a]/[θδR] > 0$. It is plausible that the equilibrium capital income tax is positively related to the level of outstanding public debt, so that $∂P_t^e/∂b_{t+1} < 0$. Then (51) in steady state implies that $δR/(1+n)<1$, i.e. the rate of return on capital is less than the modified Golden Rule, and presumably the capital stock is greater. This even raises the possibility that the steady state could be dynamically inefficient.

We have assumed that we have an ”interior” solution to the optimal tax problem. For this to be true it is necessary that the young generation care about the old, i.e. that $θ>0$. Otherwise we would have a corner solution where $P$ is arbitrarily low. If the lowest value of $P$ ”allowed" is unity (i.e. interest income may be confiscated but not savings in itself), it is clear that this value would always be attained. Then, setting $P_t=1$ in (32) and in individuals’ utilities we have an optimal tax problem in only $ω_t$. Also obviously $∂P_t^e/∂b_{t+1}=0$. Then we have

**Theorem 3** Assume A1-A4, A5", A6, no bequest motives, no consumption taxes and that the government cannot commit to future taxes. If the economy reaches a steady state and if the old generation carry no weight ($θ=0$), and the maximum allowable interest tax is 100%, then the steady state is characterised by the modified Golden Rule.

Equation (40) is still the functional equation, but with only $x_i$ as control. The modified versions of equations (41), (43) and (44), when the derivation of $b_{t+1}$ obtained from (32), are
respectively. By (55) the theorem is proven. QED

### 6.5 OPTIMAL INCOME TAXATION

#### - ONE-PERIOD COMMITMENT

#### 6.5.1 General Formulation

With one period commitment policy at time $t+1$ is committed to at time $t$. Therefore at $t$, $\omega_{t}$, $\omega_{t-1}$, and $P_{t}$ are given. The state equation for public debt is still given by (32), and we treat $\omega_{t-1}$ as a state distinct from $\omega_{t}$. The controls of the government are $x_{t}$ and $y_{t}$, such that

\[
\omega_{t+1} = x_{t} \omega_{t}
\]

\[
P_{t+1} = y_{t} P_{t}
\]

We write the recursive problem as

\[
\frac{\partial W_{t}}{\partial \omega_{t}} + \delta \frac{\partial J_{t+1}}{\partial \omega_{t}} + \delta \frac{\partial J_{t+1}}{\partial b_{t+1}} \left[ \bar{l}_{t} + (\omega_{t} - w_{t}) \frac{\partial \bar{l}_{t}}{\partial \omega_{t}} \right] = 0
\]

\[
\frac{\partial J_{t}}{\partial \omega_{t-1}} = \left[ \frac{\partial W_{t}}{\partial \omega_{t}} + \delta \frac{\partial J_{t+1}}{\partial \omega_{t}} \right] x_{t}
\]

\[
+ \delta \frac{\partial J_{t-1}}{\partial b_{t}} \left[ \frac{1-R_{t}}{1+n} \right] \frac{\partial a_{t}}{\partial \omega_{t-1}} + \left( \bar{l}_{t} + (\omega_{t} - w_{t}) \frac{\partial \bar{l}_{t}}{\partial \omega_{t}} \right) x_{t}
\]

\[
\frac{\partial J_{t}}{\partial b_{t}} = \frac{\partial J_{t+1}}{\partial b_{t+1}} \frac{R_{t}}{1+n}
\]
\[ J(\omega_t, P_t, b_t, \omega_{t-1}) = \max_{x_t, y_t} \left\{ W_t(\omega_t, P_t, x_t, y_t) + \theta W_{t-1}(\omega_{t-1}, P_{t-1}) + \delta J(\omega_{t-1}, P_{t-1}, b_{t-1}, \omega_t) \right\} \] (57)

The necessary conditions are

\[ \delta \frac{\partial J_{t+1}}{\partial \omega_t} = 0 \] (58)

\[ \frac{\partial W_t}{\partial P_{t+1}} + \delta \frac{\partial J_{t+1}}{\partial P_{t+1}} + \delta \frac{\partial J_{t+1}}{\partial b_{t+1}} \frac{\partial b_{t+1}}{\partial y_t} = 0 \] (59)

\[ \frac{\partial J_t}{\partial \omega_t} = \frac{\partial W_t}{\partial \omega_t} + \delta \frac{\partial J_{t+1}}{\partial \omega_{t+1}} x_t + \delta \frac{\partial J_{t+1}}{\partial b_{t+1}} \frac{\partial b_{t+1}}{\partial \omega_t} + \delta \frac{\partial J_{t+1}}{\partial \omega_t} \] (60)

\[ \frac{\partial J_t}{\partial P_t} = \frac{\partial W_t}{\partial P_t} y_t + \delta \frac{\partial J_{t+1}}{\partial P_{t+1}} + \delta \frac{\partial J_{t+1}}{\partial b_{t+1}} y_t + \delta \frac{\partial J_{t+1}}{\partial P_t} \] (61)

\[ \frac{\partial J_t}{\partial b_t} = \delta \frac{\partial J_{t+1}}{\partial b_{t+1}} \frac{\partial b_{t+1}}{\partial b_t} \] (62)

\[ \frac{\partial J_t}{\partial \omega_{t-1}} = \theta \frac{\partial W_{t-1}}{\partial \omega_{t-1}} + \delta \frac{\partial J_{t+1}}{\partial b_{t+1}} \frac{\partial b_{t+1}}{\partial \omega_{t-1}} \] (63)

Substituting for the derivatives of (32) into (59) and (60) we obtain

\[ \frac{\partial W_t}{\partial P_{t+1}} + \delta \frac{\partial J_{t+1}}{\partial P_{t+1}} + \delta \frac{\partial J_{t+1}}{\partial b_{t+1}} (\omega_t - w_t) \frac{\partial b_{t+1}}{\partial P_{t+1}} = 0 \] (64)
Endogenous Taxation in a Dynamic Economy

Using (64) in (61) (and the derivatives of (32)) we obtain

\[
\frac{\partial J_t}{\partial P_t} - \theta \frac{\partial W_{t-1}}{\partial P_t} + \delta \frac{\partial J_{t-1}}{\partial P_{t-1}} \left[ \frac{a_t}{1 - n} + \frac{P_t - R_t}{1 + n} \frac{\partial a_t}{\partial P_t} \right]
\]  

(66)

Similarly for (62) and (63)

\[
\frac{\partial J_t}{\partial \omega_{t-1}} = \theta \frac{\partial W_{t-1}}{\partial \omega_{t-1}} + \delta \frac{\partial J_{t-1}}{\partial \omega_{t-1}} \left[ \frac{a_{t-1}}{1 - n} + \frac{P_{t-1} - R_{t-1}}{1 + n} \frac{\partial a_{t-1}}{\partial P_{t-1}} \right]
\]  

(67)

\[
\frac{\partial J_{t-1}}{\partial \omega_t} = \theta \frac{\partial W_t}{\partial \omega_t} + \frac{\partial J_{t-1}}{\partial \omega_t} \left[ \frac{P_{t-1} - R_{t-1}}{1 + n} \frac{\partial a_{t-1}}{\partial P_{t-1}} \right]
\]  

(68)

Substituting (67) into (66) and (68) and shifting forward gives

\[
\frac{\partial J_{t+1}}{\partial P_{t+1}} = \theta \frac{\partial W_t}{\partial P_{t+1}} + \frac{\partial J_{t+1}}{\partial P_{t+1}} \left[ \frac{a_{t+1}}{R_{t+1}} + \frac{P_{t+1} - R_{t+1}}{R_{t+1}} \frac{\partial a_{t+1}}{\partial P_{t+1}} \right]
\]  

(69)

\[
\frac{\partial J_{t+1}}{\partial \omega_{t+1}} = \theta \frac{\partial W_t}{\partial \omega_{t+1}} + \frac{\partial J_{t+1}}{\partial \omega_{t+1}} \left[ \frac{P_{t+1} - R_{t+1}}{R_{t+1}} \frac{\partial a_{t+1}}{\partial \omega_{t+1}} \right]
\]  

(70)

Using (69) and (70) in (64) we have

\[
\frac{\partial W_t}{\partial P_{t-1}} = -\delta \frac{\partial J_{t-1}}{\partial P_{t-1}} \left[ \frac{a_{t-1}}{R_{t-1}} + (\omega_t - w_t) \frac{\partial l_t}{\partial P_{t-1}} + \frac{P_{t-1} - R_{t-1}}{R_{t-1}} \frac{\partial a_t}{\partial P_{t-1}} \right]
\]  

(71)

Using (69) and (70) in (65)

\[
\frac{\partial W_t}{\partial \omega_t} = -\delta \frac{\partial J_{t-1}}{\partial \omega_t} \left[ l_t + (\omega_t - w_t) \frac{\partial l_t}{\partial \omega_t} + \frac{P_{t-1} - R_{t-1}}{R_{t-1}} \frac{\partial a_t}{\partial \omega_t} \right]
\]  

(72)
Then we have

**Theorem 4** Assume A1-A4, A5”, A6, no bequest motives, no consumption taxes and that the government takes present taxes as given and can only commit to taxes one period ahead. If the economy reaches a steady state then the steady state is characterised by the modified Golden Rule. The capital-income tax is not necessarily zero.

Proof: Follows by (66). QED

### 6.5.2 Logarithmic Utility

Further results may be obtained if we restrict ourselves to a special class of preferences:

*Logarithmic in consumption and leisure*

\[ u_i = \ln c_i + \beta \ln c_i^{i+1} + \eta^{-1} \ln (L - l_i) \]

or

*Logarithmic in consumption and linear in leisure*

\[ u_i = \ln c_i + \beta \ln c_i^{i+1} - \eta^{-1} l_i^{i+1} \]

Both these utility functions give rise to identically the same choice if \( L \) is normalised such that \( L = 1 + (1 + \beta)\eta \). Then we have

\[ l_i^{i+1} = (1 + \beta)\eta, \quad a_i^{i+1} = \beta \gamma \omega \]

and

\[ V^\omega = \text{constant} + (1 + \beta) \ln \omega + \beta \ln P_{i+1} \]

Therefore

\[
\frac{\partial W_t}{\partial \omega_t} = \frac{1 + \beta}{\omega_t} \int \frac{\partial W_t}{\partial V^\mu} dF(i) \quad \frac{\partial W_t}{\partial P_{i+1}} = \frac{\beta}{P_{i+1}} \int \frac{\partial W_t}{\partial V^\mu} dF(i) \quad (73)
\]
Theorem 5 Assume A1-A4, A5′, A6, no bequest motives, no consumption taxes and that the government takes present taxes as given and can only commit to taxes one period ahead. If the individuals’ preferences are logarithmic in consumption and leisure, or logarithmic in consumption and linear in leisure, then the capital-income tax is always zero.

Proof: Since for this class of preferences \( \frac{\partial l_t^u/\partial \omega_t}{\partial P_t} = \frac{\partial a_t}{\partial P_{t+1}} = 0 \) equation (71) becomes

\[
\frac{\partial W_t}{\partial P_{t+1}} + \delta \frac{\partial J_{t+1} a_{t+1}}{\partial b_{t+1} R_{t+1}} = 0
\]  

(75)

Using (75) in (72) we obtain

\[
\frac{\partial w_t}{\partial \omega_t} - \frac{R_{t+1}}{a_{t+1}} \frac{\partial a_t}{\partial \omega_t} = \frac{R_{t+1}}{a_{t+1}} \frac{P_{t+1} - R_{t+1}}{a_{t+1}} \frac{\partial a_t}{\partial \omega_t}
\]  

(76)

which together with (73) and (74) gives

\[
\frac{1 + \beta}{\beta} \frac{P_{t+1}}{\omega_t} = \frac{R_{t+1}}{\beta \eta \omega_t} + \frac{P_{t+1} - R_{t+1}}{\omega_t}
\]  

(77)

Since average labour supply is \( \bar{t} = (1+\beta) \eta \) (77) can be fulfilled if and only if \( P_{t+1} = R_{t+1} \), i.e. capital income is untaxed.

QED

6.6 RELAXING THE ASSUMPTIONS
Optimal Taxation in OLG Economies

Bequests and Consumption Taxation

Suppose individuals’ characteristics are perfectly inheritable. Suppose monetary bequest motives. Perfect inheritability implies that a child with characteristic $i$ will inherit $m_{t+1}^i$, which has been bequeathed by a person with the same characteristic $i$. This implies that we may write the law of motion for bequests as a set of first-order difference equations

$$m_{t+1}^i = m_t^i(p_{t+1}, \omega_t, m_t^i, \tau_{t+1}, g_t, g_{t+1}), \forall i \quad (78)$$

Associate $\psi^i$ with each constraint (78), then the extra term in the Lagrangean (14) is

$$\sum_{i=0}^\omega \int \psi^i \left[ m_{t+1}^i(p_{t+1}, \omega_t, m_t^i, \tau_{t+1}, g_t, g_{t+1}) - m_t^i \right]dF(i) \quad (79)$$

The extra term (79) does not change the key steps in the proof of Theorem 1, therefore the zero capital income tax still holds in steady state even if monetary bequests are included. It is obvious that it is true also when the consumption tax is included. The assumption underlying (78) and thereby also (79) is perfect inheritability. If this was not true and we would have randomization of characteristics, the specification of the problem becomes much harder. Essentially we would not have a steady state of the kind we assumed in the proofs of the Theorems. A way around the specification problem is to assume preferences such that, whatever randomization, the aggregate bequests are independent on the distribution, so that we would have [from equation (9)]

$$M_{t+1} = N_t \bar{m}_{t+1}^i = N_t \bar{m}_{t+1}^i\left( q_t, \tau_{t+1}, g_t, \bar{m}^i \right) \quad (80)$$

and that the aggregate behaviour of consumption and labour supply (and thereby savings) is dependent only on aggregate bequests. Then, again, the steps in the proof of Theorem 1
remain unchanged and the optimum capital income tax is zero in the steady state. When we include monetary bequest motives in the endogenous tax problem we will rely on the second route, i.e. when the aggregate is only a function of the aggregate, so we need not assume that the individuals’ characteristics are perfectly inheritable to render the problem tractable.

6.7 SUMMARY AND CONCLUSIONS

In this chapter we solved for the optimal tax rules in an overlapping generations economy, assuming different degrees of commitment: a) full commitment (second best), b) no commitment, c) partial commitment (next periods’ taxes are chosen today). No other study has previously solved cases a), b) or c), but instead a hybrid between b) and c): the present government chooses present wage tax and next period’s capital income tax. These studies have found the zero capital income tax only in steady state in the logarithmic utility case.

The main results are summarised in Table 6.2 below.

We next explain in detail the results and we compare them with the results from chapter 3, where we studied optimal taxation with a one-individual dynastic economy with same timing between the government and the individual; from chapter 4, where we analysed the same economy as in chapter 3 but when policy can be revised less often; and from chapter 5, where we studied endogenous taxation in a dynastic economy.

Table 6.2 - Optimal Policy
In this chapter it was shown that in case \( a \) (second best) the optimal capital income tax is zero in steady state if the social welfare function is weakly separable across generations. That steady state also obeys the Modified Golden Rule. Thus, this result coincides with the zero capital-income tax results of chapter 3, 4, and 5.

The intuition for the zero-capital income tax of this chapter is very similar to the one for the dynastic economy. In a dynastic economy it is optimal to tax consumption at different dates uniformly in steady state, if individuals’ utilities are separable in consumption of different dates (like the Koopmans form). Thus, the transfer of consumption possibilities from one date to the other should be non-distorted. Consequently we should not tax the price of transferring consumption from one date to another, i.e. interest should be untaxed. In an overlapping generations economy we have that when the welfare function is separable across generations (like the Koopmans form) it is optimal in steady state not to distort the price of transferring consumption possibilities across generations. The price of transferring consumption possibilities

<table>
<thead>
<tr>
<th>Capital-Income Tax</th>
<th>Steady State</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a ) Full Commitment</td>
<td>Zero in steady state*</td>
</tr>
<tr>
<td>(second best)</td>
<td>Modified Golden Rule</td>
</tr>
<tr>
<td>( b ) No Commitment</td>
<td>Generally not zero</td>
</tr>
<tr>
<td>(third best)</td>
<td>Not Modified Golden Rule**</td>
</tr>
<tr>
<td>( c ) One-Period Commitment</td>
<td>Generally not zero*</td>
</tr>
<tr>
<td>(third best)</td>
<td>Modified Golden Rule</td>
</tr>
</tbody>
</table>

* Always zero if utility is logarithmic and social welfare is additive across generations.
** Unless the old generation carry no weight in the social welfare function.
from one generation to the other is the interest on public debt. Thus we should in this case exempt interest from taxation.

The reason why this result has not been found before is that previous studies have not solved the second-best problem (full commitment).

In case \( b \) the optimal capital-income tax was generally not zero, and the steady state was generally not the Modified Golden Rule. The Modified Golden Rule was obtained as a special case if either (i) the level of public debt does not affect the individuals’ expectations about the capital income tax (i.e. if the government’s feedback rule for the capital income tax is independent of the level of public debt) or (ii) the old generation carries no weight in social welfare and the maximum allowable capital income tax is 100%.

The results in case \( b \) differ from what we obtained in chapters 3 and 4. In chapters 3 and 4 it turned out that in steady state there were no tax distortions if the government could not commit to any of the future tax rates. The reason for the difference is that in chapters 3 and 4 the motivation for taxation was funding of public expenditure. Then the government could build up funds enough to cover all future expenditure, and thereby reaching a zero-tax equilibrium. In this chapter, however, the motivation for taxation is not only funding of public expenditure, but also redistribution. Therefore, even if the government would build up funds enough to cover public expenditure, it may still find it optimal to tax capital if individuals differ in their asset holdings. It is not sure that these asset holdings would be confiscated, because if the government cares about the old generation it cannot, if it chooses to confiscate, compensate the old generation in the future by giving them a zero-tax equilibrium, simply because they will not be around at that date. If the government does not care
about the old, confiscation is always optimal, as established in
Theorem 3.

In case c) the capital income tax was generally not zero, but the
steady state was always the Modified Golden Rule. However,
if the individual utilities are logarithmic in consumption and
leisure, the capital income tax was always zero, even out of
steady state.

In case c) we have partial precommitment. The government
was choosing the labour- and capital taxes one period in
advance. In assuming partial precommitment this analysis is
more closely related to chapter 4, where we solved the one-
individual optimal-tax problem assuming that the government
could revise the tax rates less often than the individual could
change her consumption decision. In chapter 4 we found the
zero-tax equilibrium. This is not necessarily true in this chapter,
simply because taxation is motivated by distributional concern
as well as for funding of public expenditure.

We found that the optimal capital-income tax was zero
already out of the steady state when individuals have
logarithmic utility and the government has an objective function
which is the discounted sum of Bergson-Samuelson welfare
functions. Some authors, Atkinson and Sandmo (1980) and Park
(1991), have found the zero capital-income tax result only at the
steady state when individuals have logarithmic utility. It should
be pointed out that they solved a different case: the case when
the government chooses the present labour tax and the future
capital income tax.
REFERENCES
(Chapter 6)


