AIM OF LECTURE 6

- Derive the Investment Opportunity Set
- Illustrate investment decisions of individuals

6.1 DERIVING THE INVESTMENT OPPORTUNITY SET

In the last lecture we found the portfolios that give the minimum variance for each level of target return. The logic is that investors would only choose among those portfolios if they have mean-variance preferences. We will now explore the tradeoff between risk and return (i.e. variance and mean) among efficient portfolios. This can be done by substituting the optimal portfolio \(w=g+h\mu\), see Lecture 5) into the objective function \(w'\Sigma w\), see Lecture 5), then we get the minimum variance as a function of the target level of return. That is, we get an equation relating portfolio variance to expected return for efficient portfolios.

Generally we could express the covariance between any two frontier portfolios, \(w_p\) and \(w_q\), as \(w_p'\Sigma w_q\), when we substitute for \(w_p = g+h\mu_p\) and \(w_q = g+h\mu_q\). This is what we are going to do in section 6.1.2. First, however, we should identify a particular portfolio that has the smallest variance of all portfolios, the so called Global Minimum-Variance Portfolio.

6.1.1 The Global Minimum-Variance Portfolio

The portfolio with the smallest variance is found by minimising \(w'\Sigma w\), but without the constraint on the target level of return: \(w'R \geq \mu\). We call this portfolio \(w_{GMV}\), and is found as \(w_{GMV} = \arg \min_w w'\Sigma w\), s.t. \(w'1=1\). This gives the same Lagrangean as in equation (2) in Lecture 5, but with \(\lambda=0\). The solution is then equation (12) in Lecture 5, but with \(\lambda=0\). Setting \(\lambda=0\) in equation (10) gives \(\mu=C=A\). So the expected return on the global minimum variance portfolio is

\[
\mu_{GMV} = \frac{A}{C} - \frac{R'\Sigma^{-1}1}{1'\Sigma^{-1}1}
\]  

(1)

Looking at the first line of equation (11) and using \(\mu_{GMV}=A/C\), we obtain

\[
w_{GMV} = \frac{\mu_{GMV}C}{BC-A^2} - \frac{A}{BC-A^2} + \frac{B - \mu_{GMV}A}{BC-A^2} - \frac{\Sigma^{-1}1}{C} \]

(2)
6.1.2 Covariances between Frontier Portfolios

The covariance between two frontier portfolios, \( w_p \) and \( w_q \), is \( w_p^\prime \Sigma w_q \), when we substitute for \( w_p = g + h\mu_p \) and \( w_q = g + h\mu_q \). First, write \( w_p = g + h(\mu_p - \mu_{GMV}) + h\mu_{GMV} \)

Similarly for \( w_q = w_{GMV} + h(\mu_q - \mu_{GMV}) \).

We then have

\[
\begin{align*}
    w_p^\prime \Sigma w_q &= \left[w_{GMV} + h(\mu_p - \mu_{GMV})\right]^\prime \Sigma \left[w_{GMV} + h(\mu_q - \mu_{GMV})\right] \\
    &= w_{GMV}^\prime \Sigma w_{GMV} + h(\mu_q - \mu_{GMV})\Sigma + (\mu_p - \mu_{GMV})h^\prime \Sigma \left[w_{GMV} + h(\mu_q - \mu_{GMV})\right] \\
    &= w_{GMV}^\prime \Sigma w_{GMV} + (\mu_q - \mu_{GMV})w_{GMV}^\prime \Sigma h + (\mu_p - \mu_{GMV})h^\prime \Sigma w_{GMV} \\
    &= (\mu_q - \mu_{GMV})(\mu_q - \mu_{GMV})h^\prime \Sigma h
\end{align*}
\]

The first term is the variance of the Global Minimum-Variance Portfolio, which is

\[
\begin{align*}
    w_{GMV}^\prime \Sigma w_{GMV} &= \frac{1}{C} [\Sigma^{-1}1]^\prime \Sigma^{-1}1 \\
    &= \left(\frac{1}{C}\right)^2 [\Sigma^{-1}1]^\prime \Sigma^{-1}1 \\
    &= \left(\frac{1}{C}\right)^2 [\Sigma^{-1}1]^\prime \Sigma^{-1}1 \\
    &= \left(\frac{1}{C}\right)^2I^\prime \Sigma^{-1}1 \\
    &= \frac{1}{C}
\end{align*}
\]

Remember that (Lecture 5):

\[
h = \frac{C \Sigma^{-1}R - A \Sigma^{-1}1}{BC - A^2}
\]

Then \( \Sigma h = (CR - A1)/(BC - A^2) \). Then

\[
\begin{align*}
    w_{GMV}^\prime \Sigma h &= \frac{Cw_{GMV}^\prime R - Aw_{GMV}^\prime 1}{BC - A^2} - \frac{C\mu_{GMV} - A}{BC - A^2} = 0
\end{align*}
\]

and

\[
\begin{align*}
    h^\prime \Sigma h &= \frac{Ch^\prime R - Ah^\prime 1}{BC - A^2} = \frac{C}{BC - A^2} R^\prime h
\end{align*}
\]
Remember that we identified two portfolios in the last lecture, \( g \) and \((g+h)\), which were components of all frontier portfolios. We also proved (Theorem 5.1), that the expected return on \( g \) and \((g+h)\) was zero and one, respectively. That is \( R'g=0 \) and \( R'(g+h)=1 \). But then,

\[
1 = R'(g+h) = R'g + R'h = 0 + R'h.
\]

So \( R'h=1 \) (this could also be verified by performing the multiplication using the definition of \( h \)).

Substituting for \( R'h=1 \) in (7) gives \( h'\Sigma h=C/(BC-A^2) \), which substituted into (3) (also using (6)) gives

\[
\frac{w_p' \Sigma w_q}{w_q' \Sigma w_q} = \frac{1}{C} + \frac{(\mu_p - \mu_{GMV})(\mu_q - \mu_{GMV})}{BC - A^2} \tag{8}
\]

Equation (8) gives us the covariance or return between two frontier portfolios, \( w_p \) and \( w_q \).

To obtain the variance of any frontier portfolio we may use equation (8) and replace \( w_q \) with \( w_p \). We then have

\[
\frac{w_p' \Sigma w_p}{w_p' \Sigma w_p} = \frac{1}{C} + \frac{(\mu_p - \mu_{GMV})^2}{BC - A^2} \tag{9}
\]

Equation (9) gives all efficient combinations between mean and variance. Investors with mean variance preferences would only be interested in those efficient combinations. For those investors equation (9) is the investment-opportunity set (when only risky assets are available). This is the topic of the next section.

6.1.3 The Investment-Opportunity Set with Risky Assets

Equation (9) is the investment-opportunity set for investors with mean-variance preferences. It is like a budget constraint, telling how much expected return an investor has to give up as a result of reducing the risk exposure (reducing the variance). It is not a linear budget constraint (that we are used to from Microeconomics), but a quadratic one.

We can graph the relationship as follows. There are two levels of return giving the same variance, and the variance reaches its minimum (=1/C) when the level of return equals \( A/C \). The function must therefore look as in Figure 6.1, below.

**Figure 6.1**

[Figure posted separately on DUO]

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1 This is done as follows: \( R'h = \frac{CR\Sigma^2 R - AA' \Sigma^{-1}}{BC - A^2} = \frac{CB - AA}{BC - A^2} = 1 \).
6.2 INDIVIDUALS’ INVESTMENT DECISIONS

If individuals have mean-variance preferences, then they have indifference curves in the mean-variance space:

**Figure 6.2**

[Figure posted separately on DUO]

We can then combine the indifference curves with the portfolio frontier (the investment opportunity set). The tangency point gives us the individual’s optimal choice, as in Figure 6.3.

**Figure 6.3**

[Figure posted separately on DUO]

If there is only one individual in the economy, this tangency point is the market portfolio (the market portfolio a portfolio with weights in all assets proportional to their market value).

**Exercise 6.1** How can we be sure that the tangency point is the market portfolio?

We may draw indifference curves of three different individuals, say I, II, and III. They will hold three different portfolios: \( w_I \), \( w_{II} \), and \( w_{III} \), with mean and variance: \((\mu_I, \sigma_I)\), \((\mu_{II}, \sigma_{II})\), and \((\mu_{III}, \sigma_{III})\), respectively; see Figure 6.4 below.

Because of the spanning property these three portfolios may be represented as a different linear combination of only two portfolios. For example pick portfolios \( w_{GMV} \) and \( w_{II} \). Then we can find \( \alpha_I \), \( \alpha_{II} \), and \( \alpha_{III} \), such that:

\[
\begin{align*}
    w_I &= \alpha_I w_{GMV} + (1-\alpha_I)w_{II}, \\
    w_{II} &= \alpha_{II} w_{GMV} + (1-\alpha_{II})w_{II}, \\
    w_{III} &= \alpha_{III} w_{GMV} + (1-\alpha_{III})w_{II}. 
\end{align*}
\]

Obviously, \( \alpha_{II} = 0 \). Furthermore \( \alpha_{III} < 0 \), i.e. individual III must go short in the Global-Minimum-Variance portfolio in order to reach higher return than offered by \( w_{II} \).

This is true for any number of individuals. We call such a result **two-fund separation**.

**Exercise 6.2** Can we say anything about the market portfolio in the three-individual case?

**Exercise 6.3** Show that no individual will (solely) invest in the Global-Minimum-Variance Portfolio.
6.3 STUDY SUGGESTIONS

Make sure you understood the shape of the investment opportunity set, and why this is a budget constraint for an individual with mean-variance preferences.

In this lecture we have followed the treatment in the text-books Huang & Litzenberger and Ingersoll. The main text-book, Copeland and Weston, does not derive the shape of the investment opportunity set, but instead tries to argue its shape by examples involving two risky assets (it also begin with an n-asset problem on p.173, but the analysis is incomplete and rather "messy"). The analysis of Huang & Litzenberger and Ingersoll is much more "cleaner," and if you feel comfortable with the treatment of this Lecture you may skip pp.155-165 and 173-180 in C&W.

Read pp.166-169 in C&W to understand an individual’s portfolio choice.
Make an attempt to solve Exercise 6.1-6.3.

6.4 NEXT TIME

Next time we will use the investment-opportunity set derived in this Lecture to analyse portfolio selection in general equilibrium. This will give us the Capital Asset Pricing Model (CAPM).

REFERENCES