Element-based preconditioners for elasto-plastic problems in geotechnical engineering

C. E. Augarde¹, A. Ramage²,* † and J. Staudacher¹

¹School of Engineering, University of Durham, South Road, Durham DH1 3LE, U.K.
²Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, U.K.

SUMMARY
Iterative solvers are widely regarded as the most efficient way to solve the very large linear systems arising from finite element models. Their memory requirements are small compared to those for direct solvers. Consequently, there is a major interest in iterative methods and particularly the preconditioning necessary to achieve rapid convergence. In this paper we present new element-based preconditioners specifically designed for linear elasticity and elasto-plastic problems. The study presented here is restricted to simple associated plasticity but should find wide application in other plasticity models used in geotechnics.

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1. INTRODUCTION
At the heart of a non-linear finite element analysis for geotechnical engineering problems, in common with many other areas of structural engineering and elastostatics, lies the solution of algebraic non-linear equilibrium equations. These equations are usually solved with well-established incremental or iterative solution techniques (such as the modified Euler or Newton–Raphson methods), reducing the problem to a sequence of linear systems

\[ K \mathbf{u} = \mathbf{f} \] (1)
involving the structure stiffness matrix $K$ and load vector $f$ which must be solved for the nodal displacements $u$. Because of the size and sparsity of the matrices $K$ which occur in practice, solution of these systems consumes most of the computing resources required (in terms of both CPU time and memory) for a finite element analysis.

Efficient solution of large sparse linear systems is a vast and vibrant research area and the development of iterative solvers is now a mature research area in mathematics and computer science (see, for example [1–3]). The main focus of this research is on so-called Krylov subspace type iterative solution techniques, where convergence depends crucially on the eigenvalue spectrum of the coefficient matrix. Convergence can therefore be improved by introducing the concept of preconditioning: theoretically, this is equivalent to replacing $K$ by a preconditioned matrix $P^{-1}K$ whose eigenvalue spectrum facilitates faster iterative convergence. Considerable research has been carried out in recent years to find inexpensive ways to generate suitable preconditioners $P$ for a variety of problems with different types of $K$ [4].

Researchers in some areas of civil engineering (notably structural) have made use of iterative solution methods, mainly for linear elastic material models. For such problems, the coefficient matrix $K$ is symmetric and positive definite (SPD). This means that the preconditioned conjugate gradient (PCG) method (which has the advantage of being underpinned by well-established rigorous convergence theory) can be used [5, 6]. In particular, it can be shown that the number of CG iterations required for convergence to within a specified tolerance is proportional to the square root of the condition number $\kappa = \lambda_{\text{max}}/\lambda_{\text{min}}$, where $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ are the maximum and minimum eigenvalues of the coefficient matrix, respectively (see, for example, [7]). One preconditioning strategy is therefore to find a $P$ such that the condition number of $P^{-1}K$ is smaller than that of $K$, so that the number of iterations required for convergence will be reduced. However, implementation of the CG algorithm also requires that the action of the matrix $P^{-1}$ must be available at a reasonable cost. There is obviously a trade-off between these two conflicting aims. In this paper we focus for the most part on a study of how various preconditioners can reduce the number of PCG iterations required for convergence and leave an examination of their practical implementation in terms of savings in CPU time to Section 4, where we describe some numerical experiments using realistic geomechanical test problems.

Increasingly, development of preconditioners which reduce the computing resources required over direct solution methods and which are robust (that is, converge to an accurate solution) is based on the nature of the coefficient matrix $K$ for the specific class of problem being studied. This requires in-depth knowledge of the underlying physical model. In this paper, we are concerned with preconditioning of linear elastic and simple elasto-plastic problems. A brief summary of some relevant properties of finite element models of linear elasticity is therefore given in Section 2.1, and some examples of how these have been used to good advantage in constructing preconditioners are given in Section 2.2.

There are several examples in the literature of application of PCG methods to structural engineering problems [8–11] and multigrid solvers have also been studied in this context (see, for example, [12]). In this paper we focus on element-based methods, that is, methods in which connectivity information and the element stiffness matrices are stored but never combined together to form $K$ or $P$. We have previously presented a survey of the efficiency of element-based methods for linear elasticity problems in [13]; a short summary of some of the element-based techniques studied in that and other papers is given in Section 2.3. In Section 2.4, we combine these ideas with those in Section 2.2 to obtain a new element-based preconditioner for linear elastic problems and present some theoretical and numerical results which demonstrate its potential efficiency in terms of reducing iteration counts.
The work described above focuses on problems in linear elasticity. Because of the widespread use of elasto-plastic models for soils, the use of iterative solution methods in geotechnical finite element analyses is still at an early stage and there is little published research in comparison to structural engineering. Many geotechnical problems do not lead to SPD systems but to symmetric indefinite systems (such as Biot consolidation [14, 15]) or non-symmetric systems [16]. In these cases, PCG is not applicable and alternative methods such as MINRES [17] (for indefinite systems) and GMRES [18] or Bi-CGSTAB [19] (for non-symmetric systems) must be used. In recent years considerable advances have been made in the development of preconditioning techniques for coupled consolidation problems in geotechnics. It is recognized that iterative approaches are necessary for these problems even more than in the uncoupled case due to the increased numbers of degrees of freedom (resulting from the inclusion of pore pressures) [20]. Preconditioning for coupled problems has so far focussed on the robustness of solvers as a necessary precursor to a study of solver speed. Consolidation problems are solved via a time-stepping approach and there are several examples of studies which examine the link between time step size and ill-conditioning [21, 22]. Others have concentrated on simple preconditioners derived from the standard Jacobi or SSOR methods in conjunction with the symmetric quasi-minimal residual (QMR) method [23, 24].

In this paper, however, we confine our attention to simple plasticity models which lead to SPD systems. In Section 3 we investigate the effects of including simple associated plasticity models and describe potential preconditioning techniques for the elasto-plastic problem. Finally, in Section 4, we present the results of some numerical experiments on realistic geomechanical test problems.

2. ELASTIC PROBLEMS

In this section, we review some useful background material from the theory of finite elements and preconditioners and describe a new element-based preconditioner for linear elastic problems.

2.1. Finite elements for linear elasticity

As a simple illustrative example of how the finite element method can be applied to a linear elastic problem, we consider a two-dimensional plane strain problem with linear elastic material behaviour, discretized on a domain $\Omega$ using elements whose nodes do not have rotational freedoms. In this paper we focus on the standard displacement formulation (see, for example, [25]) and do not consider alternative methods such as mixed formulations (see, for example, [26]), which may be more effective for particular types of incompressible problems. For the displacement finite element method, the $n \times n$ (where $n$ is the total number of degrees of freedom) structure stiffness matrix $K$ in (1) is found by evaluating

$$K = \int_{\Omega} B^T E \varepsilon_e B \, d\Omega$$

where

$$B = \begin{bmatrix}
\frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial x} & \cdots & \frac{\partial \phi_M}{\partial x} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_2}{\partial y} & \cdots & \frac{\partial \phi_M}{\partial y} \\
\frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_2}{\partial y} & \cdots & \frac{\partial \phi_M}{\partial y} & \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial x} & \cdots & \frac{\partial \phi_M}{\partial x}
\end{bmatrix}$$

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Here, the functions $\phi_1, \ldots, \phi_M$ are the finite element shape functions associated with the $M$ nodes in the mesh, and

$$E^{el} = \frac{E}{(1 - 2\nu)(1 + \nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & \frac{1}{2} - \nu \end{bmatrix}$$

is the elastic constitutive matrix where $E$ is Young’s modulus and $\nu$ is the Poisson ratio (see, for example, [25]). In practice, it is common to take the domain of integration in (2) to be the element volume and calculate individual element stiffness matrices $K_e$, each of which is of dimension $n_e \times n_e$ (where $n_e$ is the number of degrees of freedom on an element). The global SPD matrix $K$ can then be constructed using a Boolean connectivity matrix $C_e$ (of dimension $n_e \times n$) via the steps

$$K_e = C_e^T K_e C_e \quad \text{for } e = 1, \ldots, E, \quad K = \sum_{e=1}^{E} K_e$$

In the above equation, we have assumed that the vector of nodal displacements is ordered as

$$u = [u_1, u_2, \ldots, u_M, v_1, v_2, \ldots, v_M]^T,$$

where $u$ and $v$ are the displacement components in the $x$ and $y$ directions, respectively. This means that the resulting stiffness matrix has the block form

$$K = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix}$$

where $K_{xx}$ and $K_{yy}$ involve products of either $x$-derivatives, or $y$-derivatives, but not cross-products (involving both) and $K_{xy}$ involves the cross-products. Note that if the unknowns are numbered locally on each element in an analogous way to (6) this block structure also applies at an element level to each $K_e$. Furthermore, the idea extends in a natural way to three-dimensional problems. As will be seen in the next section, this structure can be taken advantage of in the construction of preconditioners.

### 2.2. Matrix reduction techniques

Many standard preconditioners can be applied only when $K$ in (1) is a Stieltjes matrix, that is, when $K = \{k_{ij}\}$ is SPD with $k_{ij} \leq 0$ for $i \neq j$. When $K$ is not a Stieltjes matrix (which is the case in most finite element analyses in geotechnics and structures), it is still possible to use these techniques by first approximating $K$ by a Stieltjes matrix, $S$ say, and basing a preconditioner on $S$. With this in mind, we define the following reduction techniques (using the terminology of [11]):

- **C-reduction** [1, 27]. A C-reduction of $K$ is carried out by lumping any positive off-diagonal entries in a row of $K$ onto the diagonal. The resulting matrix is a Stieltjes matrix.
- **D-reduction** [28, 29]. A D-reduction of $K$ is obtained by neglecting any connections between degrees of freedom of different types. This is equivalent to taking the block diagonal part of $K$ in (7), with the resulting matrix known as the separate displacement component of $K$.

Note that although C-reduction always produces a Stieltjes matrix, most of the time D-reduction does not. In order to obtain a Stieltjes version of the block diagonal D-reduced matrix, the two
reductions should be performed in sequence: we will henceforth refer to this combination as DC-reduction.

The use of these reductions for defining preconditioners for elastic problems has been studied by Gustafsson and Lindskog [30] and Saint-Georges et al. [11]. In the former work, which is the first of a trilogy of papers on the parallel solution of linear elasticity problems, some theoretical results are given on the use of D-reduction in preconditioning. The latter paper presents a comparison of C-reduction and DC-reduction based methods. The idea considered is to use either of the reductions to produce $S$ (a Stieltjes matrix approximation to $K$) then use any one of a number of standard methods to solve the preconditioner system (that is, to apply the action of $P^{-1} \equiv S^{-1}$). In [11] this is done via various incomplete Cholesky factorization techniques.

2.3. Common element-based preconditioners

As stated in Section 1, one popular idea for preconditioning finite element matrices is to use element-based methods which do not require the storage of any assembled matrices. One of the main attractions of these methods is that they allow relatively straightforward parallel implementation of iterative solvers (coupled with, for example, the element-based form of the CG algorithm in [31]). In this section we summarize the ideas behind some of the main element-based preconditioning methods which have been proposed previously. Most of these techniques were originally developed and analysed for Poisson problems. Full details of each method as applied to linear elastic problems are given in [13]: here, we simply highlight some of the important features in each case.

- **Diagonal scaling (DIAG):** Here, $P$ is the diagonal part of $K$. As the diagonal of the assembled matrix $K$ is the assembly of the diagonals of the element matrices, this can be implemented in an element-based way.

- **Element-by-element method (EBE):** This is the method of Hughes, Levit and Winget [32] based on performing a Cholesky factorization of regularized element matrices.

- **Element-based symmetric Gauss–Seidel (SGS):** Here, $P$ is a symmetric Gauss–Seidel type splitting applied on an element level (see, for example, [3, p. 378]).

- **Element matrix factorization (EMF):** This method, proposed by Gustafsson and Lindskog [33], involves performing a Cholesky factorization, $K_e = L_e L_e^T$ say, of each element matrix. For Poisson problems, this factorization always exists because the nullspace of $K_e$ is one dimensional. Writing $L_e = L_e + D_e$ where $L_e$ is strictly lower triangular and $D_e$ is diagonal with one zero diagonal entry, the factors $L_e$ and $D_e$ can be assembled via the procedure in (5) to form a global lower triangular matrix $L$ and diagonal matrix $D$, respectively. Note that a particular global and local numbering of unknowns is required to ensure that the first of these is true: nodes must be numbered locally on each element in order of increasing global node number to ensure that the assembled $L$ is indeed lower triangular in structure. The preconditioner $P$ is defined as

$$P(\eta) = \left[\left((1 + \eta h)^{-1}L + (1 + \eta h)D\right)\left((1 + \eta h)^{-1}L + (1 + \eta h)D\right)^T \right]$$

for some $\eta > 0$ where $h$ is the usual finite element mesh parameter (for example, the length of the longest edge of a triangular element). We will only consider the case $\eta = 0$, so the method becomes parameter free. Unfortunately, for linear elasticity problems this method can break down. For example, in two-dimensional the element matrices $K_e$ have a three-dimensional nullspace (corresponding to the three rigid-body modes) which results in difficulties in
computing the Cholesky factors in a numerically stable way. In addition, the unknowns cannot be easily numbered to ensure that the resulting $P$ in (8) is non-singular (a necessary condition for PCG).

- **Finite element preconditioning (FEP):** Presented by Kaasschieter [34], this method is very similar to EMF. Here, the decomposition $K_e = (L_e + D_e)D_e^+(L_e + D_e)^T$ is used where $L_e$ is strictly lower triangular, $D_e \geq 0$ is diagonal and $D_e^+$ is the pseudo-inverse of $D_e$. The resulting preconditioner is given by

$$P = (L + D)D^{-1}(L + D)^T$$

where $L$ and $D$ are the lower triangular and diagonal matrices arising from assembling the lower triangular and diagonal parts of $K_e$, respectively. Again, particular global and local numberings are required and difficulties associated with the positivity of $D$ arise when the method is applied to the linear elastic stiffness matrix.

### 2.4. Matrix reduction on an element level

In [13], we reviewed the above range of established element-based preconditioning methods for linear elastic problems and compared their performance on a variety of test problems. We also included some preliminary results using a new element-based method. In this section, we describe an improved version of this new method and provide some theoretical results indicating why it performs efficiently for elastic problems.

In the past, the reduction ideas in Section 2.2 have been applied to the fully assembled global finite element stiffness matrix $K$. Here, we propose applying these ideas on an element level, that is, we apply the reduction to each element stiffness matrix $K_e$ individually. We could then apply any of the standard preconditioners in Section 2.3 to these reduced element matrices. However, numerical evidence suggests that the most efficient option within this framework is to use the EMF method of Gustafsson and Lindskog [30] on the reduced matrices as a preconditioner. (Note that this represents an improvement on the method introduced in [13], where the FEP of Kaasschieter was used in this role.)

This combination of EMF with reduced element matrices leads us to propose three new element-based preconditioners. In each case, the EMF method of Gustafsson has been amended by the addition of a ‘pre-processing’ step which involves applying matrix reduction to the element matrices before the factorization step.

- **DC-EMF:** When DC reduction is applied to an element matrix, the reduced element matrix, $S_e$ say, is a Stieltjes matrix, so the element Cholesky factors needed for EMF are readily computable and lead to a non-singular $P$ (recall that this is not the case for the original matrices $K_e$).

- **C-EMF:** Applying C reduction alone also results in a Stieltjes matrix, so the same observations apply here.

- **D-EMF:** In this case $S_e$ is not a Stieltjes matrix, but as it is block diagonal with each Laplacian-like block having a one-dimensional nullspace, the existence of its Cholesky factors is again guaranteed.

We now examine the effect of applying these new methods to SPD linear systems. This will be done numerically in the next section, but first we make some theoretical observations based on
the condition number $\kappa = \lambda_{\max} / \lambda_{\min}$ of the preconditioned systems. The extreme eigenvalues of $P^{-1}K$ are given by the stationary values of a Rayleigh quotient as

$$\lambda_{\min} = \min_{x \neq 0} \frac{x^T K x}{x^T P x}, \quad \lambda_{\max} = \max_{x \neq 0} \frac{x^T K x}{x^T P x}$$  \hspace{1cm} (9)$$

[35, p. 18]. Without preconditioning, it is known that the linear elastic stiffness matrix arising from a uniform finite element discretization with mesh parameter $h$ has $\lambda_{\min}(K) = O(h^2)$ and $\lambda_{\max}(K) = O(1)$, giving $\kappa(K) = O(h^{-2})$ [7]. For a uniform two-dimensional mesh on a unit square with $N$ elements along each side (so that $h = 1/N$) this gives $\kappa(K) = O(N^2)$. We expect that the number of iterations $k$ required for PCG to converge to a specified tolerance will be proportional to the square root of $\kappa$, i.e. $k = O(N)$. Recall that the idea of applying preconditioning is to reduce $\kappa$ and hence $k$. An ‘ideal’ preconditioner in this sense is a method such that $\kappa(P^{-1}K) = O(1)$, so that the number of PCG iterations $k$ would be a constant independent of $h$.

For the element-based methods in Section 2.3, it can be shown that applying DIAG, EBE or SGS does not alter the asymptotic order of $\kappa$ (see, for example, [3, 36, 37], respectively). In contrast, EMF and FEP (at least applied to model Poisson problems) both satisfy $\lambda_{\min}(P^{-1}K) = O(1)$ and $\lambda_{\max}(P^{-1}K) = O(h^{-1})$, so that $\kappa(P^{-1}K) = O(h^{-1})$ (see [33, 34]), which translates into a slower growth in iteration counts with $N$. The question here is, is this improvement in performance retained when EMF is applied to the reduced element matrices as described above?

To help answer this question, we now define the concept of spectral equivalence and state two useful lemmas. We assume that $K$, $P$ and $S$ are all SPD matrices of the same dimension.

Definition 2.1 (Axelsson and Barker [7, p. 338])
A matrix $P$ is spectrally equivalent to $K$ if there exist positive constants $\alpha$ and $\beta$ independent of $h$ such that

$$\alpha \leq \frac{x^T K x}{x^T P x} \leq \beta \quad \forall x \neq 0$$

Note that from (9), spectral equivalence of $P$ and $K$ implies that the eigenvalues of $P^{-1}K$ are bounded by constants below and above.

Lemma 2.1
Let $K$ be the linear elastic stiffness matrix arising from the assembly of a set of local element matrices $K_e$, $e = 1, \ldots, E$, from a two-dimensional or three-dimensional solid problem. Let $S$ be the assembly of element matrices $S_e$ obtained by applying D-reduction or DC-reduction to each element matrix $K_e$. Then the matrices $K$ and $S$ are spectrally equivalent.

Proof 2.1

We emphasize that this lemma does not ensure spectral equivalence when applying C-reduction alone to $K_e$: this may explain the poorer performance of this method as seen in Section 2.5.

Lemma 2.2
Let $A$ be an SPD matrix arising from the assembly of a set of local finite element matrices $A_e$, $e = 1, \ldots, E$ and let $P$ be the EMF preconditioner applied to $A$ as defined in (8) (with $\eta = 0$). Then $\lambda_{\min}(P^{-1}A)$ is independent of $h$. 

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Proof 2.2
See Wathen [38].

We now apply these lemmas in combination to prove \( h \)-independence of the smallest eigenvalue of the DC-EMF or D-EMF preconditioned systems as \( h \to 0 \).

**Theorem 2.1**
Let \( \mathbf{K} \) be the linear elastic stiffness matrix and let \( \mathbf{P} \) be the DC-EMF or D-EMF preconditioner described above. Then the smallest eigenvalue of \( \mathbf{P}^{-1} \mathbf{K} \) is independent of \( h \).

**Proof 2.3**
For any SPD matrix \( \mathbf{S} \) of the appropriate dimension, we may write
\[
\lambda_{\min}(\mathbf{P}^{-1} \mathbf{K}) = \min_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T \mathbf{K} \mathbf{x}}{\mathbf{x}^T \mathbf{P} \mathbf{x}} = \min_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T \mathbf{S} \mathbf{x}}{\mathbf{x}^T \mathbf{P} \mathbf{x}} \\
\geq \min_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T \mathbf{K} \mathbf{x}}{\mathbf{x}^T \mathbf{S} \mathbf{x}} \cdot \min_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T \mathbf{S} \mathbf{x}}{\mathbf{x}^T \mathbf{P} \mathbf{x}} = \lambda_{\min}(\mathbf{S}^{-1} \mathbf{K}) \lambda_{\min}(\mathbf{P}^{-1} \mathbf{S})
\]

Suppose now that \( \mathbf{S} \) is the matrix assembled from the DC-reduced or D-reduced element matrices \( \mathbf{S}_e \) for \( e = 1, \ldots, E \). Then, from the spectral equivalence property in Lemma 2.1, \( \lambda_{\min}(\mathbf{S}^{-1} \mathbf{K}) \geq c_1 \) where \( c_1 \) is independent of \( h \). Similarly, using Lemma 2.2 we have that \( \lambda_{\min}(\mathbf{P}^{-1} \mathbf{S}) \geq c_2 \) where \( c_2 \) is also independent of \( h \). Hence, \( \lambda_{\min}(\mathbf{P}^{-1} \mathbf{K}) \geq c_1 c_2 \) and the proof is complete.

Unfortunately, we cannot evaluate the \( h \)-dependence of the largest eigenvalue in such a straightforward way. We can, however, investigate its behaviour numerically. The values of \( \lambda_{\min} \), \( \lambda_{\max} \) and \( \kappa \) for various preconditioned systems for a set of structured linearly elastic problems (see Section 3.2, load step 1, \( v = 0.25 \)) are tabulated in Table I. These values were calculated using the \texttt{eigs} command in MATLAB [39]. The parameter \( N \) indicates the number of elements per side of the domain. Also displayed is the average growth factor \( \delta \) for each row. This is the average value (over each row) of the exponents \( \delta \) calculated when fitting a relationship of the form \( \lambda \approx c h^\delta \) (or \( \kappa \approx c h^\delta \)) to the displayed data, for some constant \( c \). (Note that in what follows we will use \( c \) to denote a generic constant, that is, a quantity independent of \( h \), which may take different values in different expressions.)

The values for DIAG show that \( \lambda_{\min}(\mathbf{P}^{-1} \mathbf{K}) \approx O(h^2) \) and \( \lambda_{\max}(\mathbf{P}^{-1} \mathbf{K}) \approx O(1) \), giving \( \kappa(\mathbf{P}^{-1} \mathbf{K}) \approx O(h^{-2}) \) as expected. This is also true for C-EMF, which does not appear to have altered the asymptotic behaviour: this is not surprising as C-EMF is not covered by Theorem 2.1. On the contrary, the \( h \)-independence of \( \lambda_{\min} \) predicted by Theorem 2.1 for D-EMF is clear, indicating an overall condition number dependence of the form \( \kappa(\mathbf{P}^{-1} \mathbf{K}) \approx O(h^{-1.8}) \). For DC-EMF, the \( h \)-independence of \( \lambda_{\min} \) is slightly less clear-cut. We believe this is because the grids used here are not large enough to indicate true asymptotic behaviour, and we still anticipate that the minimum eigenvalue will tend to a constant value as \( h \to 0 \), as per Theorem 2.1. In addition, our practical experience (see Section 2.5) suggests that the estimate \( \kappa(\mathbf{P}^{-1} \mathbf{K}) \approx O(h^{-0.7}) \) is over-optimistic for DC-EMF preconditioning: we expect the true asymptotic result to be closer to that of D-EMF.

In summary, although we have not obtained the \( O(h^{-1}) \) dependence of \( \kappa \) which can be achieved using EMF for Poisson problems, we still expect DC-EMF and D-EMF to offer an improvement in asymptotic terms over the other methods when applied to linear elastic problems.
Table I. Extreme eigenvalues of preconditioned linear elastic matrices.

<table>
<thead>
<tr>
<th>N</th>
<th>$\lambda_{\min}$</th>
<th>$\lambda_{\max}$</th>
<th>$\kappa$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.29e-2</td>
<td>3.30e-3</td>
<td>2.45e2</td>
<td>1.99e0</td>
</tr>
<tr>
<td>16</td>
<td>3.16e0</td>
<td>3.24e0</td>
<td>9.82e2</td>
<td>-1.23e-2</td>
</tr>
<tr>
<td>32</td>
<td>3.27e0</td>
<td>3.27e0</td>
<td>3.94e3</td>
<td>-2.00e0</td>
</tr>
<tr>
<td>64</td>
<td>2.07e-4</td>
<td>3.27e0</td>
<td>1.58e4</td>
<td>-2.00e0</td>
</tr>
<tr>
<td>128</td>
<td>5.16e-5</td>
<td>3.27e0</td>
<td>6.34e4</td>
<td>-2.00e0</td>
</tr>
</tbody>
</table>

2.5. Numerical results: elasticity

In this section we present the results of some numerical experiments which compare the performance of the preconditioners described in Sections 2.3 and 2.4. The tests model the linear elastic response of a square plane strain grid to in-plane loading, using a set of structured meshes of linear strain triangular elements. The number of elements per side of the square domain, N, was varied from N = 8 to 128, giving linear systems of dimension n = 578 (N = 8) to n = 33,282 (N = 128).

The numbers of iterations $k$ required to satisfy the convergence criterion

$$\|f - Ku\|_2 / \|f\|_2 \leq 10^{-6}$$

in each case (where $\| \cdot \|_2$ represents the Euclidean norm of a vector) using the starting vector $u = 0$ are given in Table II. All iteration counts quoted are for the solution of the linear system arising at the first load step. As the response is elastic, this is fully representative of the systems constructed at other load steps. Each problem was solved with four sets of material properties, namely, the Poisson ratio $\nu = 0, 0.25, 0.4$ and $0.49$. The calculations in this section were carried out using MATLAB [39].

As stated above, for the unpreconditioned problem we expect the number of PCG iterations required for convergence to be proportional to $N$, or in other words, plotting $N$ versus the iteration counts in bilogarithmic axes should produce a straight line with slope 1. If applying preconditioning reduces $\kappa$, then we expect that for the preconditioned systems

$$k = cN^{\delta}$$

for some constant $c$ and exponent $\delta < 1$, giving a straight line of slope less than one. The information from Table II is displayed graphically in this way in Figure 1, where $\log_2(k)$ is plotted against $\log_2(N)$. A dashed line with slope 1 is included for comparison purposes. As expected, the line in Figure 1 corresponding to DIAG (which does not affect the asymptotic order of the condition number of $K$) has slope 1. This is also true for EBE, SGS and C-EMF, indicating that, in the asymptotic
Table II. Iteration counts for linear elastic test problems.

<table>
<thead>
<tr>
<th>Method</th>
<th>v = 0.0</th>
<th>v = 0.25</th>
<th>v = 0.4</th>
<th>v = 0.49</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
</tr>
<tr>
<td>DIAG</td>
<td>20</td>
<td>47</td>
<td>103</td>
<td>215</td>
</tr>
<tr>
<td>EBE</td>
<td>13</td>
<td>24</td>
<td>47</td>
<td>94</td>
</tr>
<tr>
<td>SGS</td>
<td>14</td>
<td>25</td>
<td>51</td>
<td>103</td>
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<td>EMF</td>
<td>31</td>
<td>77</td>
<td>282</td>
<td>1836</td>
</tr>
<tr>
<td>C-EMF</td>
<td>32</td>
<td>61</td>
<td>146</td>
<td>391</td>
</tr>
<tr>
<td>D-EMF</td>
<td>19</td>
<td>29</td>
<td>44</td>
<td>68</td>
</tr>
<tr>
<td>DC-EMF</td>
<td>21</td>
<td>28</td>
<td>41</td>
<td>60</td>
</tr>
</tbody>
</table>

For all featured values of $v$, C-EMF consistently requires more iterations than DIAG, which in turn requires more than EBE and SGS, with these two methods behaving very similarly. The performance of (unreduced) EMF is erratic, reflecting the fact that this method has difficulties for element matrices of higher dimensional nullspace. (Very similar (but slightly worse) behaviour was exhibited by FEP, so these results have been omitted here.) The reduced methods D-EMF and DC-EMF perform well: numerical calculation gives an exponent of $\delta \simeq 0.6$ in (11). This is slightly better than suggested by the results in Table I regarding the reduction in condition number for
3. ELASTO-PLASTIC PROBLEMS

We have established that the element-based preconditioner DC-EMF in Section 2.4 is competitive in terms of iteration counts for linear elastic problems. In this section, we investigate the effects of these methods (recall that $k \simeq \sqrt{k}$). As DC-EMF has a slightly smaller constant of proportionality and involves a trivial amount of extra work over D-EMF, we recommend DC-EMF as the method of choice for elastic problems. We note, however, that this method suffers from degradation in performance with increasing the Poisson ratio in the same way as all of the other methods.

Figure 1. Iteration counts for linear elastic problems: (a) $\nu = 0$; (b) $\nu = 0.25$; (c) $\nu = 0.4$; and (d) $\nu = 0.49$.  

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of adding plasticity to the problem and perform some numerical experiments to test iterative convergence on some simple elasto-plastic problems.

3.1. Plasticity

In the majority of geotechnical engineering problems, elasticity alone is inadequate to model the response of soil to loading. Soil behaviour is dominated by irrecoverable deformations, therefore, a suitable continuum framework in which to work is elasto-plasticity. Many elasto-plastic models for soil have been presented and a survey of the most significant ones is given in [25]. Here, we confine ourselves to one of the simplest elasto-plastic formulations, which is suitable for the modelling of 'undrained' soil behaviour (that is, where incompressibility is assumed). Since most non-linear models for soil nevertheless assume elastic behaviour at very small strains, the work described above remains relevant to what follows.

An elasto-plastic constitutive model requires the specification of a yield function $F$, a plastic potential function $\phi$ and a hardening/softening rule. Representing the stresses and strains (normal and shear) in vector form as

$$\mathbf{\sigma} = [\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}]^T, \quad \mathbf{\varepsilon} = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}]^T$$

a stress strain relationship of the form

$$\mathbf{\sigma} = \mathbf{E}^{cp} \mathbf{\varepsilon}$$

can be then obtained with

$$\mathbf{E}^{cp} = \mathbf{E}^{el} - \mathbf{E}^{pl}$$

which $\mathbf{E}^{el}$ (see (4)) represents the purely elastic behaviour and $\mathbf{E}^{pl}$ represents plasticity. Under the assumptions of perfect plasticity (zero hardening/softening) and associated plastic flow (identical yield and plastic potential functions), the matrix

$$\mathbf{E}^{pl} = \frac{\partial F}{\partial \sigma} \frac{\partial F^T}{\partial \sigma} \mathbf{E}^{el}$$

is symmetric and, in linear algebra terminology, $\mathbf{E}^{cp}$ is simply a rank-one update of $\mathbf{E}^{el}$. As an example of this type, we consider the von Mises model, which has a non-dimensionalized yield function given by

$$F(\mathbf{\sigma}) = \left(\frac{\sigma_x - \sigma_y}{s_u} + \frac{\sigma_y - \sigma_z}{s_u} + \frac{\sigma_z - \sigma_x}{s_u}\right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right) - 8s_u^2$$

where $s_u$ represents the undrained shear strength of the material. Introducing the stress invariant $p = (\sigma_x + \sigma_y + \sigma_z)/3$, it can be shown that

$$\mathbf{E}^{pl} = \frac{3G}{4s_u^2} \bar{\mathbf{\sigma}}\bar{\mathbf{\sigma}}^T, \quad \bar{\mathbf{\sigma}} = [\sigma_x - p, \sigma_y - p, \sigma_z - p, \tau_{xy}, \tau_{xz}, \tau_{yz}]^T$$

where $\bar{\mathbf{\sigma}}$ is the vector of deviatoric stresses. The key question now is, how does this type of update affect the performance of element-based preconditioning methods? At the onset of plasticity,
Table III. Extreme eigenvalues of preconditioned elasto-plastic matrices.

<table>
<thead>
<tr>
<th></th>
<th>(N)</th>
<th>(8)</th>
<th>(16)</th>
<th>(32)</th>
<th>(64)</th>
<th>(128)</th>
<th>(\delta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIAG</td>
<td>(\lambda_{\text{min}})</td>
<td>1.18e–2</td>
<td>3.01e–3</td>
<td>7.55e–4</td>
<td>1.87e–4</td>
<td>4.67e–5</td>
<td>2.00e0</td>
</tr>
<tr>
<td></td>
<td>(\lambda_{\text{max}})</td>
<td>3.17e0</td>
<td>3.29e0</td>
<td>3.56e0</td>
<td>3.89e0</td>
<td>4.13e0</td>
<td>3.69e0</td>
</tr>
<tr>
<td></td>
<td>(\kappa)</td>
<td>2.69e2</td>
<td>1.09e3</td>
<td>4.72e3</td>
<td>2.08e4</td>
<td>8.84e4</td>
<td>5.00e5</td>
</tr>
<tr>
<td>DC-EMF</td>
<td>(\lambda_{\text{min}})</td>
<td>9.67e–2</td>
<td>4.32e–2</td>
<td>1.85e–2</td>
<td>1.04e–2</td>
<td>4.53e–3</td>
<td>1.10e0</td>
</tr>
<tr>
<td></td>
<td>(\lambda_{\text{max}})</td>
<td>3.21e0</td>
<td>5.00e0</td>
<td>7.44e0</td>
<td>1.28e1</td>
<td>2.10e1</td>
<td>6.84e1</td>
</tr>
<tr>
<td></td>
<td>(\kappa)</td>
<td>3.32e1</td>
<td>1.16e2</td>
<td>4.02e2</td>
<td>1.23e3</td>
<td>4.64e3</td>
<td>1.78e0</td>
</tr>
<tr>
<td>D-EMF</td>
<td>(\lambda_{\text{min}})</td>
<td>1.34e–1</td>
<td>8.31e–2</td>
<td>2.73e–2</td>
<td>1.51e–2</td>
<td>6.48e–3</td>
<td>1.09e0</td>
</tr>
<tr>
<td></td>
<td>(\lambda_{\text{max}})</td>
<td>7.48e0</td>
<td>2.85e1</td>
<td>9.74e1</td>
<td>2.65e2</td>
<td>8.78e2</td>
<td>1.72e0</td>
</tr>
<tr>
<td></td>
<td>(\kappa)</td>
<td>5.58e1</td>
<td>3.43e2</td>
<td>3.57e3</td>
<td>1.75e4</td>
<td>1.35e5</td>
<td>2.81e0</td>
</tr>
<tr>
<td>C-EMF</td>
<td>(\lambda_{\text{min}})</td>
<td>5.33e–3</td>
<td>1.25e–3</td>
<td>2.96e–4</td>
<td>6.96e–5</td>
<td>1.71e–5</td>
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</tr>
<tr>
<td></td>
<td>(\lambda_{\text{max}})</td>
<td>1.01e0</td>
<td>1.02e0</td>
<td>1.01e0</td>
<td>1.09e0</td>
<td>1.54e0</td>
<td>1.52e–1</td>
</tr>
<tr>
<td></td>
<td>(\kappa)</td>
<td>1.89e2</td>
<td>8.16e2</td>
<td>3.41e3</td>
<td>1.57e4</td>
<td>9.01e4</td>
<td>2.22e0</td>
</tr>
</tbody>
</table>

individual element matrices will be changed in this way as they turn plastic, but the overall effect of this on the global matrix \(K\) is unclear.

Table III shows the extreme eigenvalues of the preconditioned elasto-plastic matrices from the same sequence of structured test problems studied in Table I. This time, however, 100 load steps have been carried out so \(K\) has a plastic component and is no longer purely elastic (see Section 3.2 for details). With the assumptions of perfect plasticity and associated flow used here, \(E^p\) in (12) is SPD, so all of the preconditioners studied are guaranteed to exist for this elasto-plastic problem. Unfortunately, the spectral equivalence result in Lemma 2.1 no longer holds, so we cannot expect to retain the improvement in condition number seen in Table I for DC-EMF and D-EMF. This is verified in Table III, where it can be seen that the asymptotic \(h\)-dependence of both methods has worsened with the addition of plasticity to the model. We note, however, that this deterioration is worse for D-EMF, reinforcing our choice of DC-EMF as the preferred method.

3.2. Numerical results: plasticity

We now present some numerical results comparing the performance of the following three element-based methods on a sequence of elasto-plastic problems:

- **Diagonal scaling (DIAG):** This is the standard reference method as before.
- **EMF with DC-reduction (DC-EMF):** From Section 2, this is our method of choice for elastic problems.
- **DC-EMF applied to the elastic part (ELAS-DC-EMF):** As observed above, adding the plastic part to each element matrix requires a simple update of the elastic constitutive matrix (4). This suggests the possibility that the main features of the element matrices may be retained by a preconditioner based on the elastic part only (a similar idea is suggested in [20] for a preconditioner based on global Crout factorizations). This approach has the obvious computational advantage that \(E^e\) does not change from load step to load step, so the preconditioner \(P\) need only be calculated once at the beginning of each simulation. We denote this approach, that is using DC-EMF applied to the elastic part of \(K_e\), by ELAS-DC-EMF.
The set of test problems is based on the same sequence of structured grids of linear strain triangles described in Section 2.5, varying $N$ as before. This time, however, the underlying physical problem is a plane strain rigid footing modelled by prescribing vertically downwards displacements on selected surface nodes (see Figure 2(a)). We compare the performance of the PCG method with DIAG, DC-EMF and ELAS-DC-EMF preconditioning. The numbers of iterations $k$ required for convergence are tabulated in Table IV, and the same results are displayed graphically in Figure 3 (although the results for ELAS-DC-EMF are omitted from the graphs they are almost identical to those for DC-EMF). Again convergence test (10) and a zero initial vector were used, with two representative Poisson ratios of $\nu = 0.25$ and 0.49. In each case, a prescribed displacement is applied over a number of equal incremental steps: here iteration counts are given for the linear systems constructed at four different load steps, denoted by $s = 1, 25, 50, 100$. All systems are purely elastic at step one, with the amount of plasticity present increasing at each load step. A representative measure of the percentage plasticity in the mesh at each step is tabulated in Table V. Specifically, this is the ratio of the cumulative area associated with yielding Gauss points in the mesh to the total number of Gauss points, where three-point Gauss quadrature has been used to evaluate the element matrices. This quantity varies with $N$ but, with the exception of the coarsest mesh, the values are broadly comparable.

The results in Table IV confirm the main observation in Section 3.1, namely that the asymptotic improvement seen with DC-EMF for purely elastic problems is not retained. Specifically, when plasticity is added, DC-EMF is asymptotically equivalent to DIAG. However, the former method clearly requires fewer iterations per system, so is still worth pursuing in terms of the search for an efficient preconditioner. We also make the important observation that the iteration counts for ELAS-DC-EMF and DC-EMF are remarkably similar. This leads us to conclude that the idea of preconditioning via the elastic part is indeed a sensible one due to the potential gain in efficiency caused by a reduction in the number of iterations required. This idea is explored further in Section 4, where we present a comparison of methods in terms of CPU times as well as iteration counts, as the total runtime is of course the key factor in the overall algorithm performance.

Figure 2. Detail of footing test problem: (a) due to symmetry considerations, only the shaded region is meshed; and (b) typical unstructured mesh of linear strain triangles as used in Section 4.
Table IV. Iteration counts for structured footing problems.

<table>
<thead>
<tr>
<th>s</th>
<th>N</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>DIAG</td>
<td>60</td>
<td>113</td>
<td>254</td>
<td>494</td>
<td>958</td>
</tr>
<tr>
<td></td>
<td>DC-EMF</td>
<td>25</td>
<td>28</td>
<td>40</td>
<td>59</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>ELAS-DC-EMF</td>
<td>25</td>
<td>28</td>
<td>40</td>
<td>59</td>
<td>85</td>
</tr>
<tr>
<td>25</td>
<td>DIAG</td>
<td>58</td>
<td>114</td>
<td>255</td>
<td>509</td>
<td>1030</td>
</tr>
<tr>
<td></td>
<td>DC-EMF</td>
<td>24</td>
<td>27</td>
<td>49</td>
<td>90</td>
<td>137</td>
</tr>
<tr>
<td></td>
<td>ELAS-DC-EMF</td>
<td>24</td>
<td>28</td>
<td>49</td>
<td>89</td>
<td>141</td>
</tr>
<tr>
<td>50</td>
<td>DIAG</td>
<td>60</td>
<td>115</td>
<td>265</td>
<td>535</td>
<td>1095</td>
</tr>
<tr>
<td></td>
<td>DC-EMF</td>
<td>24</td>
<td>30</td>
<td>65</td>
<td>118</td>
<td>209</td>
</tr>
<tr>
<td></td>
<td>ELAS-DC-EMF</td>
<td>24</td>
<td>30</td>
<td>67</td>
<td>119</td>
<td>211</td>
</tr>
<tr>
<td>100</td>
<td>DIAG</td>
<td>60</td>
<td>119</td>
<td>278</td>
<td>580</td>
<td>1182</td>
</tr>
<tr>
<td></td>
<td>DC-EMF</td>
<td>24</td>
<td>36</td>
<td>75</td>
<td>132</td>
<td>263</td>
</tr>
<tr>
<td></td>
<td>ELAS-DC-EMF</td>
<td>25</td>
<td>37</td>
<td>75</td>
<td>133</td>
<td>263</td>
</tr>
</tbody>
</table>

4. SOLVING PRACTICAL PROBLEMS

So far we have restricted our attention to artificially constructed MATLAB test problems on structured domains. In reality, we want a preconditioner which is going to be efficient for real-life simulations of genuine geotechnical engineering problems. With this in mind, we have used the f90 geotechnical finite element code OXFEM, developed at Oxford University, to model the same rigid footing problem as before (see Figure 2(a)) but on some more realistic unstructured meshes of linear strain triangles. OXFEM uses a modified Euler approach to solve the finite element equations, where a load or prescribed displacement is applied over a number of equal incremental steps. In the following tests, all analyses involve application of the prescribed footing displacement over one load stage comprising 100 load steps. We use five unstructured grids of quadratic triangles (denoted by A–E): the smallest is illustrated in Figure 2(b). Again we focus on \( v = 0.25 \) and 0.49 as the values of the Poisson ratio, and take ‘snapshots’ at load steps 1, 25, 50 and 100. The numbers...
Figure 3. Iteration counts for structured footing problems: (a) $\nu = 0.25$, $s = 1$; (b) $\nu = 0.25$, $s = 25$; (c) $\nu = 0.25$, $s = 50$; (d) $\nu = 0.25$, $s = 100$; (e) $\nu = 0.49$, $s = 1$; (f) $\nu = 0.49$, $s = 25$; (g) $\nu = 0.49$, $s = 50$; and (h) $\nu = 0.49$, $s = 100$.

Table V. Percentage plasticity for structured footing problems.

<table>
<thead>
<tr>
<th>$N$</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
</tr>
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<tbody>
<tr>
<td>$s$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>25</td>
<td>1.0</td>
<td>1.3</td>
<td>1.4</td>
<td>1.3</td>
<td>1.2</td>
</tr>
<tr>
<td>50</td>
<td>3.1</td>
<td>4.4</td>
<td>5.0</td>
<td>4.1</td>
<td>3.9</td>
</tr>
<tr>
<td>100</td>
<td>4.2</td>
<td>10.2</td>
<td>10.4</td>
<td>8.8</td>
<td>8.2</td>
</tr>
<tr>
<td>$\nu = 0.25$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3.1</td>
<td>4.2</td>
<td>4.5</td>
<td>3.3</td>
<td>3.0</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>9.1</td>
<td>10.0</td>
<td>8.3</td>
<td>7.7</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>12.0</td>
<td>12.8</td>
<td>10.0</td>
<td>9.5</td>
<td></td>
</tr>
</tbody>
</table>

of unknown $n$ and the percentage plasticity for each problem (calculated as before) are shown in Table VI. In this more realistic setting, we tabulate not only iteration counts but also execution times (in CPU seconds), in order to make objective comparisons between the methods.

The number of iterations $k$ required to satisfy convergence criterion (10) using DIAG, DC-EMF and ELAS-DC-EMF preconditioning, with initial vector $u = 0$, is given in Table VII. These results are also presented graphically in Figure 4, where the $x$-axis represents $\log_2(n)$ and the $y$-axis represents $\log_2(k)$. The plots show that for all three methods, the iteration counts approximately

Table VI. Percentage plasticity for unstructured footing problems.

<table>
<thead>
<tr>
<th>Problem</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>4006</td>
<td>9818</td>
<td>14854</td>
<td>22518</td>
<td>29990</td>
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<table>
<thead>
<tr>
<th>s</th>
<th>v = 0.25</th>
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</thead>
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<tr>
<td>1</td>
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<tr>
<td>25</td>
<td>1.4 1.4 1.3 1.2 1.2</td>
</tr>
<tr>
<td>50</td>
<td>4.9 4.2 4.6 4.4 4.5</td>
</tr>
<tr>
<td>100</td>
<td>9.5 8.3 9.1 9.1 9.1</td>
</tr>
</tbody>
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<table>
<thead>
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<th>s</th>
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<tr>
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<td></td>
</tr>
<tr>
<td>9.0 7.7 8.5 8.5 8.5</td>
<td></td>
</tr>
<tr>
<td>10.4 8.8 10.0 10.9 10.7</td>
<td></td>
</tr>
</tbody>
</table>

Table VII. Iteration counts for unstructured footing problems.

<table>
<thead>
<tr>
<th>Problem</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>v = 0.25</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DIAG</td>
<td>k</td>
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<td>445</td>
<td>550</td>
<td>683</td>
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<tr>
<td>t_k</td>
<td>0.86</td>
<td>3.28</td>
<td>6.22</td>
<td>12.01</td>
<td>17.31</td>
</tr>
<tr>
<td>1 DC-EMF</td>
<td>k</td>
<td>55</td>
<td>73</td>
<td>80</td>
<td>98</td>
</tr>
<tr>
<td>t_k</td>
<td>0.31</td>
<td>1.03</td>
<td>1.75</td>
<td>3.33</td>
<td>4.60</td>
</tr>
<tr>
<td>ELAS-DC-EMF</td>
<td>k</td>
<td>55</td>
<td>73</td>
<td>80</td>
<td>98</td>
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<tr>
<td>t_k</td>
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<td>1.02</td>
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<td>6.31</td>
<td>12.30</td>
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<td>k</td>
<td>66</td>
<td>95</td>
<td>106</td>
<td>125</td>
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<tr>
<td>t_k</td>
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<td>1.36</td>
<td>2.32</td>
<td>4.19</td>
<td>6.22</td>
</tr>
<tr>
<td>ELAS-DC-EMF</td>
<td>k</td>
<td>67</td>
<td>96</td>
<td>101</td>
<td>128</td>
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<tr>
<td>t_k</td>
<td>0.39</td>
<td>1.37</td>
<td>2.30</td>
<td>4.25</td>
<td>6.41</td>
</tr>
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<td>k</td>
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<td>600</td>
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<td>139</td>
<td>144</td>
<td>177</td>
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<td>t_k</td>
<td>0.53</td>
<td>1.98</td>
<td>3.53</td>
<td>6.23</td>
<td>8.27</td>
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<td>ELAS-DC-EMF</td>
<td>k</td>
<td>92</td>
<td>141</td>
<td>146</td>
<td>179</td>
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<tr>
<td>t_k</td>
<td>0.53</td>
<td>2.03</td>
<td>3.25</td>
<td>5.94</td>
<td>8.64</td>
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<tr>
<td>DIAG</td>
<td>k</td>
<td>341</td>
<td>528</td>
<td>647</td>
<td>840</td>
</tr>
<tr>
<td>t_k</td>
<td>1.05</td>
<td>3.97</td>
<td>7.41</td>
<td>14.62</td>
<td>21.59</td>
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<tr>
<td>100 DC-EMF</td>
<td>k</td>
<td>103</td>
<td>171</td>
<td>171</td>
<td>214</td>
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<tr>
<td>t_k</td>
<td>0.59</td>
<td>2.45</td>
<td>3.73</td>
<td>7.09</td>
<td>9.90</td>
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satisfy a relationship of the type $k = c\sqrt{n}$, that is, each bilogarithmic plot has slope $\simeq 0.5$ (the dashed line has slope 0.5 for comparison). Note that this is consistent with the structured results in Section 3.2, where $n = N^2$. The results for DC-EMF and ELAS-DC-EMF are again almost identical. Table VII also tabulates $t_k$, the total time in CPU seconds taken for each set of $k$ iterations. Figure 5(a) shows a plot of $\bar{t}_k$, the time in CPU seconds required for one iteration of each method, against $n$ (using bilogarithmic axes): this verifies that for all three methods $\bar{t}_k = cn$.

The above data corroborate the fact that DC-EMF (with or without including the plastic part of the matrix in constructing the preconditioner) exhibits the same behaviour asymptotically as diagonal scaling. However, looking at the values of $t_k$ in Table VII confirms that in practice the factorization methods require less time overall for iteration. This is of course not the full story, as we have so far taken no account of any set-up costs of the preconditioner. For DIAG, these are negligible: we consider diagonal scaling to be an integral part of the original coefficient matrix.

For DC-EMF and ELAS-DC-EMF, the element matrix reduction must be carried out, the resulting element matrices factorized and the preconditioner formed. Representative CPU times (in seconds)
for this set-up phase, denoted by $t_s$, are given in Table VIII and plotted in Figure 5(b). We deduce that this cost satisfies $t_s = cn^2$. Note that for DC-EMF this procedure must be carried out at each load step, while for ELAS-DC-EMF it is done only once at the beginning of each simulation (as the elastic part of the matrix is unchanged by the loading). This difference is crucial for overall execution time, as seen below.

In order to obtain a meaningful assessment of the preconditioners’ performance in practice, we conclude our study by presenting results for a set of full analyses. We use the same five unstructured footing problems, but run a full simulation over 100 loading steps. The total times $t$ and total PCG iteration counts $k$ for each case are shown in Table IX. Plots of $\log_2(n)$ against $\log_2(t)$ for $v = 0.25$ and 0.49 are shown in Figures 5(c) and (d), respectively. Combining the asymptotic results above, we would expect the total time for DIAG to satisfy $t = cn^{1.5} (=kt_k)$, whereas the equivalent result for DC-EMF should be $t = cn^2$, as the overall cost will eventually be dominated by the set-up phase time $t_s$. This is verified in the figures, where dashed lines with slopes 1.5 and 2 have been included for reference: as $n$ increases, the slope of the DC-EMF line is increasing. Theoretically, ELAS-DC-EMF also has a set-up time issue, but in practice as the set-up is done only once (for the first load step) its effect is not important in an asymptotic sense. This is again verified by the Figures 5(c) and (d), where the line corresponding to ELAS-DC-EMF also has slope 1.5 (like DIAG). The preconditioner appears also to be effective even when there is significant yielding in the problem, and the footing is close to failure. Figure 6 shows the load–displacement curves for the footing for the smallest unstructured footing problem (A) and two values of the Poisson ratio. In both cases the ‘snapshots’ which we have taken at $s = 1$, 25, 50, 100 are at significantly different points on the curve, with step $s = 100$ clearly at failure. From these results, we conclude that ELAS-DC-EMF is a promising method for preconditioning these problems.
Figure 5. CPU times for unstructured footing problems: (a) average time per iteration; (b) reduction/factorization times; (c) total simulation times, ν = 0.25; and (d) total simulation times, ν = 0.49.

Table VIII. Representative CPU times for element matrix reduction and factorization.

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<th>B</th>
<th>C</th>
<th>D</th>
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<tr>
<td>n</td>
<td>4006</td>
<td>9818</td>
<td>14854</td>
<td>22518</td>
<td>29990</td>
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<tr>
<td>(t_s)</td>
<td>0.84</td>
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<td>8.48</td>
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Table IX. Overall results for unstructured footing problems.

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<th>C</th>
<th>D</th>
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<tr>
<td>DIAG $k$</td>
<td>20</td>
<td>30</td>
<td>36</td>
<td>44</td>
<td>50</td>
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<tr>
<td>$\nu = 0.25$</td>
<td>61.45</td>
<td>226.39</td>
<td>422.38</td>
<td>792.11</td>
<td>1176.17</td>
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<td>DC-EMF $k$</td>
<td>5660</td>
<td>8624</td>
<td>9049</td>
<td>11479</td>
<td>11812</td>
</tr>
<tr>
<td>$\nu = 0.49$</td>
<td>108.27</td>
<td>482.93</td>
<td>983.66</td>
<td>2138.27</td>
<td>3575.36</td>
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<tr>
<td>ELAS-DC-EMF $k$</td>
<td>5663</td>
<td>8703</td>
<td>9334</td>
<td>11649</td>
<td>12013</td>
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<tr>
<td>$\nu = 0.49$</td>
<td>33.54</td>
<td>128.73</td>
<td>214.76</td>
<td>405.08</td>
<td>562.13</td>
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Figure 6. Load–displacement curves for the footing in problem (A) at $\nu = 0.25$ and 0.49.

5. SUMMARY

In this paper we have proposed some new element-based preconditioners for elasto-plastic problems in geotechnical engineering. These are based on a combination of the ideas of matrix reduction with
existing element-based methods for Poisson problems. In Section 2.4 we presented a combination of theoretical and numerical results which suggest that, for purely elastic problems, two of the methods (D-EMF and DC-EMF) offer an improvement in terms of asymptotic behaviour over traditional methods such as diagonal scaling (DIAG). This is supported by the numerical experiments in Section 2.5. We then investigated the effects of using these same methods on a simple elasto-plastic model. This impaired the asymptotic performance of the new methods, but we showed in Section 3.2 that they still retained an advantage over DIAG in terms of iteration counts. Finally, in Section 4 we solved some more realistic unstructured mesh problems, performing full analyses as well as focussing on individual load steps. Our overall conclusion is that applying the DC-EMF method to the elastic part of the matrix (so that the preconditioner does not change with loading) provides a potentially competitive new element-based preconditioner.

We have confined this study to a simple yield criterion, and have shown that the elastic parameters in the underlying problem are of greater significance than the onset of plasticity. While it is yet to be proven, it would seem likely that this is the case for most other simple yield criteria. This is a topic for future research. We note, however, that extending the study to materials with non-associated flow rules is complicated by the fact that to deal with the non-symmetric systems produced, different iterative solution techniques would be required which are less amenable to the type of analysis carried out here.

REFERENCES


39. MATLAB, Mathworks Inc.: http://www.mathworks.com/


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