Numerical simulation of potential and viscous ocean waves

Dr. Wei Bai
Postdoctoral Research Associate
Department of Computing and Mathematics
Manchester Metropolitan University

Potential waves

- Introduction
- Second-order time-domain method
- Boundary Element (BEM) simulation
- Numerical implementation
- Wave radiation induced by forced moving bodies
- Wave diffraction around fixed bodies
- Wave interaction with floating bodies
- Application of Finite Volume Method (FVM)
1. Introduction

- **Basic assumption**
  The fluid is incompressible, inviscid, and the motion irrotational.

- **Governing equation**
  \[ \nabla^2 \phi = 0 \]

- **Boundary conditions**
  - Free surface
    \[ \frac{DX}{Dr} = \nabla \phi, \quad \frac{D\phi}{Dr} = -g \hat{z} + \frac{1}{2} \nabla \phi \cdot \nabla \phi \]
  - Body surface
    \[ \frac{\partial \phi}{\partial n} = V_s \]
  - Other solid surfaces
    \[ \frac{\partial \phi}{\partial t} = 0 \]
  - Open boundary surface

- **Advantages and limitations**
  - It is efficient and powerful
  - Boundary element method can be adopted
  - The application is limited by the assumption

- **Difficulty**
  Nonlinear boundary conditions have to be satisfied on the instantaneous boundary surfaces not known *a priori*

- **Review of theories**
  - Frequency-domain method (linear and second-order)
  - Time-domain method
    - First-order and second-order methods
    - Fully nonlinear method

- **Numerical methods**
  - Boundary element method with different orders
  - Volume-discretised methods (FEM, FVM, FDM)
2. Second-order time-domain method

- Taylor series expansion is applied to the boundary conditions, and Stokes perturbation procedure is then used to establish corresponding problems at the first and the second-order of waves steepness.

\[
\frac{\partial \phi}{\partial t} \frac{\partial \eta}{\partial t} = f_1 \\
\frac{\partial \phi}{\partial t} + g \eta = g_1
\]

where

\[
f_1 = 0 \\
f_2 = -\frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \eta}{\partial y} - \eta \frac{\partial^2 \phi}{\partial t^2}
\]

\[
g_1 = 0 \\
g_2 = -\frac{1}{2} \nabla^2 \phi - \eta \frac{\partial^2 \phi}{\partial t^2}
\]

- The B-spline function based BEM is adopted, which is validated by the linear frequency-domain problem.

Case 1: Transient motion of a floating cylinder

Case 2: Second-order wave diffraction around a cylinder

Time history of vertical displacement on a freely floating cylinder with specified initial (a) displacement, (b) velocity: — present study; ●●●●, Newman 1985

Time history of wave force components to first and second order: ..., solution to first order; ----, second order component; ——, solution to second order
Case 3: Second-order wave interaction with a floating cylinder

Time history of body motion response components to first and second order: (a) surge, (b) heave, (c) pitch: \( \cdots \cdots \) solution to first order; \( \cdots \cdots \cdots \) second order component; \( \cdots \) solution to second order

- Summary

3. Boundary element (BEM) simulation

- Green’s second identity transforms Laplace equation into

\[
C(x_o) \phi(x_o) = \iint_S \left[ G(x, x_o) \frac{\partial \phi(x)}{\partial n} - \phi(x) \frac{\partial G(x, x_o)}{\partial n} \right] ds
\]

where the solid angle coefficient \( C(x_0) \) can be obtained by

\[
C(x_0) = -\iint_S \frac{\partial G(x, x_0)}{\partial n} ds
\]

- Using standard higher-order BEM, the discretised equation can be expressed in the matrix form

\[
\begin{bmatrix}
A^{(11)} & A^{(12)} \\
A^{(21)} & A^{(22)}
\end{bmatrix}
\begin{bmatrix}
X^{(1)} \\
X^{(2)}
\end{bmatrix}
= \begin{bmatrix}
B^{(1)} \\
B^{(2)}
\end{bmatrix}
\]
where

\[ X(1) = \{ \phi_1, \phi_2, \ldots, \phi_N \} \quad X^{(2)} = \left\{ \frac{\partial \phi}{\partial x_i}, \frac{\partial \phi}{\partial y_j}, \ldots, \frac{\partial \phi}{\partial x_j} \right\} \]

\[ A_{i,j}^{(1)} = C(x_i) + A_{i,j}^{(21)} \quad A_{i,j}^{(2)} = \sum_{m=1}^{M} \left[ \frac{\partial G(x_m, x_i)}{\partial N_j} N_j(\xi, \eta) \right] \]

\[ B_i^{(1)} = \sum_{i=1}^{N} \sum_{m=1}^{M} G(x_m, x_i) \frac{\partial (x_m)}{\partial N_i} N_i(\xi, \eta) \]

\[ B_i^{(2)} = -C(x_i) \psi(x_i) + B_i^{(1)} \]

- **Numerical integration**
  - Regular: standard Gauss-Legendre method
  - Singularity: triangular polar-coordinate transformation method

- **Wave force calculation**
  The pressure is expressible using the Bernoulli equation,

\[ p = -\rho \left( \frac{1}{2} \nabla \psi \cdot \nabla \psi + g \right) \]

Some auxiliary functions \( \psi \) are introduced, which also satisfy the Laplace equation and

- Free surface \( \psi_i = 0 \)
- Body surface \( \frac{\partial \psi}{\partial n} = n_i \)
- Other surfaces \( \frac{\partial \psi}{\partial n} = 0 \)

Finally, the forces and moments can be expressed as

\[ f_i = -\rho \iint_{S_x} \left( g + \frac{1}{2} \nabla \psi \cdot \nabla \psi \right) \frac{\partial \psi_i}{\partial N_i} \, ds - \rho \iint_{S_x} \left( n_i \cdot \nabla \psi_i - n_{ij} \cdot (\nabla \psi_i) \right) \, ds \]

\[ + \rho \int_{S_x} \left[ \nabla \psi_i \cdot \left( \frac{\partial \psi_i}{\partial \xi} \right) n \right] \left[ \nabla \psi_j \cdot \left( \frac{\partial \psi_j}{\partial \eta} \right) n \right] \, ds \]

\[ \times \left( \nabla \psi_i - n \times \frac{\partial \psi_i}{\partial \xi} \right) \cdot \left( \nabla \psi_j - n \times \frac{\partial \psi_j}{\partial \eta} \right) \, \, ds \]
4. Numerical implementation

- Mesh generation
- Time integration (Standard fourth-order Runge-Kutta procedure)
- Open boundary condition (artificial damping layer)
- Algebraic equation solver (Generalized Minimal RESidual, GMRES)
- Mesh regridding and interpolation
  The Laplace smooth technique is adopted to adjust slightly the horizontal position of nodes, and the interpolation can be employed based on the shape function.
- Intersection line

Validation: asymmetric initial impulse waves in a circular tank with a cylinder at its centre

A shaded plot of the wave profile (a=0.1, t=0.0, 6.0, 12.0, 18.0, 24.0, 30.0)
5. Wave radiation induced by forced moving bodies

- **Boundary condition on the body surface**,
  \[ V_n = \left( \ddot{\xi} - \omega \dot{\chi} (X - X_0) \right) a \]

- **Two coordinate systems are defined**: one is a space-fixed coordinate system, and the other is a body-fixed coordinate system. The transformation between them can be written as

\[
\begin{bmatrix}
  x' \\
  y' \\
  z'
\end{bmatrix} =
\begin{bmatrix}
  x + X_0 \\
  y + Y_0 \\
  z + Z_0
\end{bmatrix}
\begin{bmatrix}
  \cos \chi & - \sin \chi & 0 \\
  \sin \chi & \cos \chi & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

---

**Case 1: Wave radiation by a cylinder with heave motion**

Wave profile around the cylinder undergoing forced heave oscillation with \( a_3 = 0.02 \), \( w = 1.257 \)

**Case 2: Wave radiation by a cylinder with surge motion**

Wave profile around the cylinder undergoing forced surge oscillation with \( a_1 = 0.02 \), \( w = 1.257 \)
Case 3: Wave radiation by a cylinder with pitch motion

Wave profile around the cylinder undergoing forced pitch oscillation with $\phi_5=\pi/60$, $w=1.257$

Case 4: Wave radiation by a cylinder with combined heave and pitch motion

Wave profile around the cylinder undergoing combined heave and pitch oscillation with $\phi_5=\pi/60$, $\phi_3=0.046$ and $w=1.257$

6. Wave diffraction around fixed bodies

- Domain decomposition method

- Iteration procedure
  1. Choose an initial Dirichlet condition
  2. Solve Laplace’s equation with Dirichlet condition
  3. Formulate a Neumann condition by taking an average of the obtained solutions
  4. Solve Laplace’s equation with Neumann condition
  5. Formulate a Dirichlet condition by taking an average of the obtained solutions
6. Calculate the maximum error on $\Gamma$ based on the potential obtained in $\Omega_1$ and $\Omega_2$
7. If error satisfies the prescribed stop criterion, then exit the loop
   - Singularity on interfaces
     - The top layers of elements at the interfaces are semi-discontinuous to avoid the singularity.
     - Algebraic equation solver (LU decomposition)

Case 1: Regular wave propagating in a wave tank

Comparison of the computer time at $\alpha=0.01$: (a) with and without domain decomposition for different meshes; (b) for different numbers of sub-domains
Case 2: Focused wave in a wave tank

Comparison of (a) Position of the focal points; (b) Time of focusing with the experimental data

Case 3: Regular wave diffraction around a vertical cylinder

Wave forces for different motion amplitudes of the wave maker at $w=2.0$

Wave run-ups for different motion amplitudes of the wave maker at $w=2.0$: (a) upstream side; (b) downstream side
Comparisons of wave forces on the cylinder with experimental data and other numerical results: (a) first-harmonic force; (b) second-harmonic force; (c) third-harmonic force; (d) fourth-harmonic force; (e) fifth-harmonic force; (f) sixth-harmonic force; (g) seventh-harmonic force

7. Wave interaction with floating bodies

- Decouple the wave-body interaction

  The wave forces can also be written as

  \[ f_i = -\rho \int S_b \{ \xi + \omega \{ X - X_i \} \} n/dt + Q_i \]

  The first term on the right hand side is proportional to acceleration and so can be regarded as the force due to added mass. We can finally get

  \[ \sum_{j=1}^{n_b} \left[ m_{ij} + c_{ij} \right] \delta \phi_j = Q_i + m_h \delta \phi_3 \]

  where \( c_{ij} = \rho \int \psi_i \phi_j dt \)

- Mesh regridding
Case 1: Wave diffraction at different $kr$ numbers

Wave profile around the cylinder at $a=0.02$ and $h=12T$ for three different $kr$ numbers: (a) $kr=0.6$; (b) $kr=1.0$; (c) $kr=1.6$

Case 2: Wave interaction with a truncated cylinder

Wave force on the fixed and floating truncated cylinders with draft $h=0.6d$ at $a=0.01$: (a) $Fx$; (b) $Fz$

Case 3: Wave interaction with a floating cylinder at different $kr$ numbers

Time history of motion on the floating truncated cylinder with draft $h=0.6d$ at $a=0.01$ for three different $kr$ numbers: (a) displacement in $x$ direction; (b) displacement in $z$ direction; (c) angle about $y$ direction

Wave profile around the floating truncated cylinder with the draft $h=0.6d$ at $a=0.01$ and $h=12T$ for three different $kr$ numbers: (a) $kr=0.6$; (b) $kr=1.0$; (c) $kr=1.6$. 
8. Application of Finite Volume Method (FVM)

- Semi Eulerian-Lagrangian approach

\[
\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} - \nabla \phi \cdot \nabla \eta
\]

\[
\frac{\partial \phi}{\partial t} = -g \eta - \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{\partial \eta}{\partial t} \frac{\partial \phi}{\partial z}
\]

- The 2\textsuperscript{nd}-order Central Difference scheme is used, and the following algebraic equation can be developed at node \( P \)

\[
A_e \phi_e + A_w \phi_w + A_n \phi_n + A_s \phi_s + A_t \phi_t + A_y \phi_y = 0
\]

\[
A_{E,W} = \frac{S_{E,W}}{\Delta x_{E,W}} \quad A_{N,S} = \frac{S_{N,S}}{\Delta y_{N,S}} \quad A_{T,B} = \frac{S_{T,B}}{\Delta y_{T,B}}
\]

\[
A_r = -(A_e + A_w + A_n + A_s + A_t + A_y)
\]

Case 1: Regular wave propagation

Case 2: Focused wave generation

- Possible future works
Summary

- Advantages of BEM
  - Accuracy
  - Robusticity
- Weakness of BEM
  - Efficiency
- Possible future works
  - Parallel computing
  - Multiple structures (for platforms or energy devices)
  - Breaking waves

Viscous flows

- Introduction
- SIMPLE algorithm based on 2D unstructured mesh
- Numerical applications
- Artificial Compressibility Method (ACM)
- Cartesian cut cell approach
- 2D viscous flow with free surface
- 3D free surface flows
- Simulation of 3D turbulence flows
1. Introduction

- Navier-Stokes equations
  \[ \frac{\partial u_j}{\partial x_j} = 0 \]
  \[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_i \partial x_j} + g_i \]

- Discretization methods (FDM, FEM, FVM)
- Numerical grids (structured, block-structured, unstructured)
- Pressure-velocity coupling
  SIMPLE, projection method, artificial compressibility method
- Free surface
  - Eulerian method (VOF)
  - Lagrangian method
  - Arbitrary Lagrangian-Eulerian (ALE) method

2. SIMPLE algorithm based on 2D unstructured mesh

- Mesh generation using Delaunay triangulation technique

Examples for the mesh generation
- Discretization of the convective fluxes $C_j$
  
  A second-order upwind difference scheme with the deferred correction is used.

\[
C_j = F_j \phi_j^{(2)} + F_j \left( \phi_j^{(2)} - \phi_j^{(1)} \right)^{-1}
\]

- The colocated arrangement of nodes
  
  The cell face velocities are calculated by the momentum interpolation

\[
u_j = \frac{1}{2} \left[ \left( \frac{V}{a_{j_1}} \right)_n + \left( \frac{V}{a_{j_2}} \right)_P \right] \left( \frac{p_n - p_P}{|a|} \right) - \frac{1}{2} \left( \frac{V p_n + V p_P}{|a|} \right) \frac{d_j}{|S|} S_j
\]

- Discretization of the diffusive fluxes $D_j$

\[
D_j = D_j^u + D_j^v = - \left( \phi_{j_n} - \phi_{j_P} \frac{d_j}{|P|} \right) S_j - \frac{1}{2} \left( \phi_{j_n} - \phi_{j_P} \frac{N_j}{|N|} \right) S_j
\]

- The unsteady term: an implicit three-level scheme of second-order accuracy

- Calculation of the pressure correction

\[
a_j^p p_n = \sum_{j=1}^{N_h} a_j^p p_{h_j} + b_j^p
\]

where

\[
a_j^p = \left( \frac{V}{a_{j_1}} \right)_n + \left( \frac{V}{a_{j_2}} \right)_P \frac{a_j}{|S|}, \quad a_j = \sum_{j=1}^{N_h} a_j^p, \quad b_j^p = - \sum_{j=1}^{N_h} F_j
\]

- Correct the pressure and the velocity

\[
p_n = p_n^e + p_n^p, \quad u_{j_n} = u_{j_n}^e - a_{j_1} \frac{V}{a_{j_1}} p_{h_j}
\]

- Validation: Flow in a skewed lid-driven cavity

Comparison of velocity $u$ and $v$ on the centerlines with other’s results
3. Numerical applications

Case 1: Flow past a rotating cylinder

Patterns of instantaneous streamlines for Re=200, \( \alpha = 2.07 \): (a) 2.0, (b) 4.0, (c) 6.0, (d) 10.0, (e) 15.0, (f) 20.0, (g) 25.0, (h) 30.0

Case 2: Flow past a rotary oscillating cylinder

Patterns of instantaneous streamlines for Re=200, \( \omega = 2 \), \( f_s = 0.2 \): (a) 50.0, (b) 51.0, (c) 52.0, (d) 53.0, (e) 54.0, (f) 55.0
The variation of the lift and drag coefficients with $f_s$ for $Re=200$, $\alpha=2$: (a) $f_s=0.2$, (b) 0.5, (c) 1.5

Case 3: Mean drag force on deepwater riser clusters

The time-averaged mean flow field in near wake of cylinder

Mean velocity along the wake centreline

Drag coefficient of the downstream cylinder
4. Artificial Compressibility Method (ACM)

- **Advantages of ACM**
  - Fully implicit scheme
  - Suitable for discontinuous problems

- **The ACM uses the incompressibility constraint**, 
  \[ \frac{1}{\beta} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

- Using FVM, the integral form is
  \[ \frac{\partial}{\partial t} \int \Omega \left( \nabla \cdot F + \frac{\partial (\rho u u) + \rho g_n}{\partial s} \right) ds = 0 \]

  \[ \mathbf{Q} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} \beta u \\ u^2 + p/\rho \\ uv \\ v^2 + p/\rho \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} \beta v \\ uv \\ \rho \end{bmatrix} \]

- Introducing a pseudo-time derivative, in conjunction with the Newton’s expansion method, the final discretized equation is expressed as
  \[ I_n \Delta \left[ A^{t_{n+1}} + \frac{\partial \mathbf{R}(\mathbf{Q}^{t_{n+1}})}{\partial \mathbf{Q}} \right] \Delta (\mathbf{Q}^{t_{n+1}})^n = \left[ I_n \left( \mathbf{Q}^{t_{n+1}} - (\mathbf{Q} \mathbf{A})^n + \mathbf{R}(\mathbf{Q}^{t_{n+1}}) \right) \right] \]

- **Evaluation of numerical fluxes**
  A 1D Riemann problem is assumed at each cell edge, then the inviscid fluxes are
  \[ \mathbf{r} = \frac{s_l}{s_l - s_R} \mathbf{r}(q^L) + \frac{s_R}{s_l - s_R} \mathbf{r}(q^R) - \frac{s_l s_R}{s_l - s_R} (q^L - q^R) \]  
  (HLL solver)

  \[ \mathbf{r} = \frac{1}{2} \left[ \mathbf{r}(q^L) + \mathbf{r}(q^R) - \mathbf{D} (q^L - q^R) \right] \]  
  (Roe’s solver)

  Where
  \[ \mathbf{D} = \mathbf{R} |\Delta L| \]

- **An Approximate LU factorization (ALU) method** is used to solve the system of equations,
  \[ (\mathbf{D} + \mathbf{L}) \cdot \mathbf{D}^{-1} \cdot (\mathbf{D} + \mathbf{U}) \cdot \Delta \mathbf{Q} = \mathbf{RHS} \]

- **Weakness of ACM**
  - Complex to understand and employ
  - A parameter \( \beta \) to be determined
Case 1: Lid-driven flow in a box

Comparison of velocity u and v on the centerlines with other’s results

Case 2: Sloshing wave in a rectangular tank

Time history of wave elevation at the center of tank

5. Cartesian cut cell approach

- Advantages of cut cell approach
  - Suitable for moving complex bodies
  - Based on Cartesian structured mesh

- Principles of cut cell approach

- 3D Examples
Limitation of current 2D cut cell program
- Origin has to be located at (0, 0)
- Background grid has to be uniform
- Body boundary can not reach the edge of background grid

2D Examples

6. 2D viscous flow with free surface

Treatment of different types of cells
- For fluid cells (0)
- For cut cells (1)
- For merged cells (3, 4)
- For solid cells (2) and small cells (5)
- The SIMPLE method based on the structured colocated mesh is used to solve the fluid field at each time step.
- Semi Eulerian-Lagrangian approach is used to update the free water surface.

Case 1: Current flow past a circular cylinder

Hydrodynamic parameters of current flow over a fixed circular cylinder

<table>
<thead>
<tr>
<th>Method</th>
<th>( C_D )</th>
<th>( C_L )</th>
<th>( C_T )</th>
<th>( C_F )</th>
<th>( C_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lemaire and Piquet (1994)</td>
<td>1.40</td>
<td>1.46</td>
<td>0.75</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td>Bevi et al. (1990)</td>
<td>---</td>
<td>---</td>
<td>1.35</td>
<td>---</td>
<td>0.80</td>
</tr>
<tr>
<td>Perla et al. (2000)</td>
<td>---</td>
<td>---</td>
<td>1.31</td>
<td>---</td>
<td>0.18</td>
</tr>
<tr>
<td>Duan and Dang (1994)</td>
<td>---</td>
<td>---</td>
<td>1.13</td>
<td>---</td>
<td>0.18</td>
</tr>
<tr>
<td>Chen et al. (1994)</td>
<td>1.37</td>
<td>1.39</td>
<td>1.33</td>
<td>0.72</td>
<td>0.20</td>
</tr>
<tr>
<td>Chen and Antonia (1992)</td>
<td>1.53</td>
<td>1.43</td>
<td>1.48</td>
<td>0.63</td>
<td>0.18</td>
</tr>
<tr>
<td>Parent et al. (2000)</td>
<td>---</td>
<td>---</td>
<td>1.37</td>
<td>---</td>
<td>0.18</td>
</tr>
<tr>
<td>Marzouk et al. (2001)</td>
<td>---</td>
<td>---</td>
<td>1.30</td>
<td>---</td>
<td>0.18</td>
</tr>
<tr>
<td>Wu and Hu (2008)</td>
<td>1.30</td>
<td>1.32</td>
<td>1.36</td>
<td>0.56</td>
<td>0.34</td>
</tr>
</tbody>
</table>

Comparison of wave profile with analytical solution

Case 2: Sloshing wave in a rectangular tank

Comparison of wave profile with analytical solution
Case 3: Wave generation in a wave tank

Comparison of wave elevation with experiment data

Comparison of wave profile with analytical solution for the second-order stokes wave

Case 4: Wave travelling over a submerged bar

Time history of wave elevation at six different points and its comparison with experimental data
Case 5: Wave generation by a submerged ball undergoing force vertical oscillation

Wave profile for the submerged ball undergoing forced vertical oscillation at $T=2s$ with two different amplitudes.

Case 6: Wave generation by a submerged ball undergoing force horizontal oscillation

Wave profile for the submerged ball undergoing forced horizontal oscillation at $T=2s$ with two different amplitudes.

7. 3D free surface flows

Case 1: Lid-driven flow in a box

Comparison of velocity $u$ and $v$ on the centerlines with other's results.

Case 2: Sloshing wave 1 (initial a wave shape)

Comparison of wave elevation at the center of tank with analytical solution.
Case 3: Sloshing wave 2 (initial a straight line)

Case 4: 3D sloshing wave

Wave profile of a 3D sloshing wave

Comparison of wave profile with analytical solution

Case 5: Regular wave generation

Wave profile at \( w = 2.0 \): (a) \( t = 7.5T \); (b) \( t = 15T \)

- Weakness of this program
  - Introduce more numerical errors for large \( a/h \)
  - Based on the structured mesh without the consideration of structures
8. Simulation of 3D turbulence flows

- 2D cut cell mesh generation is adopted in the horizontal plane, and this mesh is directly extended in the vertical direction.

- The Smagorinsky LES turbulence model is added in the program.

- Calculations of flow past a flat plane and a circular cylinder will be performed soon to verify the program.

- Future works.

Interested research topics

- Improve current numerical methods (BEM, FVM)
  - Accuracy, efficiency, robusticity
- Two-phase flow model and Meshless methods
  - Parallel, GPU
- Offshore engineering
  - TLP, WIO, VIV
- Coastal engineering
  - Impact, overtopping, scour
- Renewable energy devices
  - Wind turbine, Pelamis, Manchester Bobber, Oyster
Thank you!