HYDRODYNAMIC STABILITY THEORY

Problem sheet 2. Further linear stability.

Q1. Consider the two-dimensional Navier-Stokes equations for an incompressible fluid:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right], \]
\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re} \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]. \]

(1)

Here \( u = (u, v) \) are the components of the velocity vector in the \((x, y)\) directions, \( t \) is time, \( p \) is the pressure, and \( Re \) is a constant. Verify that \( u = (1 - y^2, 0) \) satisfies the equations exactly provided that

\[ p = p_B = \frac{-2x}{Re} + \text{constant}. \]

Consider small disturbances to this basic state and by writing

\[ (u, v, p) = (1 - y^2, 0, p_B) + \epsilon [\tilde{u}(x, y, t), \tilde{v}(x, y, t), \tilde{p}(x, y, t)], \]

where \( |\epsilon| << 1 \), obtain the linearised stability equations. Assume a normal mode form and give the corresponding stability equations in this case.

Q2. The Navier-Stokes equations in cylindrical polar coordinates \((r, \theta, z)\) are given by

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} + \frac{w}{r} \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re} \left[ \frac{\partial^2 u}{\partial r^2} - \frac{2}{r^2} \frac{\partial u}{\partial \theta} \right], \]
\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{w}{r} \frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{\partial p}{\partial \theta} + \frac{1}{Re} \left[ \frac{\partial^2 v}{\partial r^2} + \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right], \]
\[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + \frac{w}{r} \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re} \frac{\partial^2 w}{\partial z^2}, \]

(2)

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where
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \]

Here \( \mathbf{u} = (u, v, w) \) are the components of the velocity vector in the \((r, \theta, z)\) directions, \( p \) is the pressure, and \( Re \) is a constant.

Show that \( u = 0, v = V(r), w = 0, p = P(r) \) satisfies the equations provided
\[ \frac{dP}{dr} = \frac{V^2}{r}, \quad \frac{d}{dr} \left( \frac{d}{dr} + \frac{1}{r} \right) V = 0. \]

Hence deduce the possible forms for \( V(r) \). By writing disturbances to this steady state in the form
\[ (u, v, w, p) = (0, V(r), 0, P(r)) + \epsilon[\tilde{u}(r, \theta, z, t), \tilde{v}(r, \theta, z, t), \tilde{w}(r, \theta, z, t), \tilde{p}(r, \theta, z, t)], \]
where \( |\epsilon| << 1 \), obtain the linearised stability equations.

**Q3.** Derive the relationship between \( a \) and \( l \) that is required for
\[ f(x, y) = 2 \cos(lx\sqrt{3}) \cos(ly) + \cos(2ly) \]
to satisfy
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = -a^2 f. \]

Write \( f(x, y) \) as the sum of three cosine terms, and hence prove that
\[ f(x + \frac{4m\pi}{a\sqrt{3}}, y + \frac{4n\pi}{a}) = f(x, y) \quad (n, m = 0, \pm 1, \pm 2, \ldots). \]

By using a polar coordinate representation of \( f(x, y) \), show that there is symmetry under the transformation \( \theta \rightarrow \theta + \frac{\pi}{3} \), where \( \theta \) is the polar angle. What pattern would you expect to see in the fluid if this mode was realised in an experiment?

[Hint: You can use \( 2 \cos A \cos B = \cos(A - B) + \cos(A + B) \).]