HYDRODYNAMIC STABILITY THEORY


Q1. Consider a flow in a region $y \in [0, \pi]$, governed by the following equation:

\[
\frac{\partial U}{\partial t} = f(U) + \frac{1}{R} \frac{\partial^2 U}{\partial y^2}
\]

where $R > 0$ is a real parameter and $U = 0$ at $y = 0, \pi$.

(i) For $f(U) = U + U^3$, write down the linearized stability equation for a perturbation to the trivial basic state $U_B = 0$. Hence determine the values of $R$ at which bifurcation points can be found. If the lowest value of $R$ for which a bifurcation occurs is denoted by $R = R_c$, find $R_c$.

(ii) For a perturbation about the minimum critical value $R = R_c + \epsilon R_1$, $(0 < \epsilon \ll 1)$ and an expansion in the form

\[
U = U_B + \epsilon^{1/2} A(\tau) \sin(y) + \epsilon U_2(y, \tau) + \epsilon^{3/2} U_3(y, \tau) + \cdots,
\]

where $\tau = \epsilon t$, derive an amplitude equation for the perturbation amplitude $A(\tau)$. Classify the bifurcation type and draw a bifurcation diagram.

Q2. Consider a flow governed by the nonlinear equation

\[
U_t - U^2 = U_{xx},
\]

where $U$ satisfies the boundary conditions $U_x = 0$ on $x = 0, 1$, and initial condition $U(x, t = 0) = U_0(x)$.

Introduce the Fourier cosine series

\[
U = \frac{1}{2} \sum_{n=\infty}^{n=\infty} a_n(t) e^{in\pi x},
\]

where $a_n = a_{-n}$ are real functions of the time coordinate $t$.

Show that

\[
\frac{da_n}{dt} = -n^2 \pi^2 a_n + \frac{1}{2} \sum_{m=\infty}^{m=\infty} a_m a_{n-m},
\]

and hence that

\[
\frac{da_0}{dt} \geq \frac{1}{2} a_0^2.
\]

Show that, if $a_0(t = 0) > 0$ then

\[
a_0(t) \geq \frac{2}{t_0 - t}.
\]
where $t_0$ is a positive constant. Hence conclude that if $\int_0^1 U_0(x)dx > 0$ the flow approaches a singularity as $t \to t_0$.

Q3. Attempt question B6 of the 2007 exam. This can be found at http://www.intranet.man.ac.uk/past-papers/EPS/Mathematics/Sem2-2007.shtml. A photocopy will be handed out in class.